Mathematics 100A Homework 8
Due: Wednesday 25 November 2020

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Please feel free to reach out to me or the TA if you have any questions.

1. Chapter 6, exercise 5.1

Proof. Let $r_1$ denote reflection across $\ell_1$ and $r_2$ reflection across $\ell_2$. Then $r_2r_1$ is rotation by the angle $\pm\frac{2\pi}{n}$, depending on how $\ell_1$ and $\ell_2$ are oriented with respect to each other. Assume without loss of generality then that $r_2r_1 = \rho_2\frac{2\pi}{n} = x$. Let $G = \langle r_1, r_2 \rangle$ denote the group generated by $r_1$ and $r_2$ inside of $O_2$. Then $G = \langle r_1, r_2 \rangle = \langle r_1, r_2 r_1 \rangle = \langle r_1, x \rangle \approx D_n$, as desired.

2. Chapter 6, exercise 5.4

Proof. We have $L = Za + Zb = Za' + Zb'$. Write $(a', b')$ for the $2 \times 2$ matrix with columns the vectors $a'$ and $b'$, and similarly write $(a, b)$ for the $2 \times 2$ matrix with columns the vectors $a$ and $b$. Because the vectors are linearly independent, these matrices are invertible.

Now, because $a', b' \in Za + Zb$, we can express $a'$ and $b'$ as integer combinations of the vectors $a, b$. In matrix notation, $(a', b') = (a, b)P$ for a matrix $P$ with integer entries. Similarly, $(a, b) = (a', b')Q$ for a matrix $Q$ with integer entries. Combining gives $(a, b) = (a, b)PQ$. But $(a, b)$ is invertible so $1 = PQ$. Taking determinants $1 = \det(P)\det(Q)$ is a product of two integers, because $P, Q$ have integer entries. It follows that $\det(P) = \pm 1$.

3. Chapter 6, exercise 5.10

Proof. Let $\varphi$ denote the homomorphism $M_2 \to O_2$. Then $\varphi(f)$, $\varphi(g)$ are each rotations, so $fgf^{-1}g^{-1}$ is in the kernel of $\varphi$, so is a translation.

We must verify that $fgf^{-1}g^{-1}$ is not the identity, or equivalently, that $f$ and $g$ do not commute. To see this, suppose $f$ is rotation about a point $P$ by an angle $\theta$ and $g$ is rotation about a point $Q$ by an angle $\phi$. We check that $f(g(P)) \neq g(f(P))$. Indeed, $g(f(P)) = g(P)$ because $f$ fixes $P$. Now, because $P$ and $Q$ are distinct and $\phi$ is not a multiple of $2\pi$, $g(P) \neq P$. It follows that $f(g(P)) \neq g(P)$ because $P$ is the only fixed point of $f$. This completes the proof.