1. Chapter 2, exercise 11.1

Proof. One has \((x, y)^n = (x^n, y^n)\) is equal to 1 if and only if \(x^n = 1\) and \(y^n = 1\). Because the order of \(x\) is \(r\) and the order of \(y\) is \(s\), \((x, y)^n = 1\) if and only if \(r|n\) and \(s|n\). Thus the order of \((x, y)\) is the least common multiple of \(r\) and \(s\). \(\square\)

2. Chapter 2, exercise 11.4

Proof. There are three parts.

(a) Yes, \(G = \mathbb{R}^x = \{\pm 1\} \times \mathbb{R}_{>0}^x = H \times K\). Indeed, \(G = HK, H \cap K = \{1\}\) and \(H\) commutes with \(K\).

(b) No, \(G\) is not a product \(H \times K\). Indeed, \(H\) does not commute with \(K\) as subgroups of \(G\) as \(\left( \begin{array}{cc} t_1 & x \\ t_2 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & t^{-1}_1 \\ x & t^{-1}_2 \end{array} \right) = \left( \begin{array}{cc} 1 & t_1 x/t_2 \\ t^{-1}_2 & 1 \end{array} \right)\).

(c) Yes, \(G = C^x = S^1 \times \mathbb{R}_{>0}^x = H \times K\). Indeed, \(G = HK, H \cap K = \{1\}\) and \(H\) commutes with \(K\). \(\square\)

3. Chapter 2, exercise 11.5

Proof. Set \(z = (z_1, z_2)\) and \(g = (g_1, g_2)\) arbitrary elements of \(G_1 \times G_2\). One has \(zg = (z_1, z_2)(g_1, g_2) = (z_1g_1, z_2g_2)\) and \(gz = (g_1, g_2)(z_1, z_2) = (g_1z_1, g_2z_2)\). Thus \(zg = gz\) if and only if \(z_1g_1 = g_1z_1\) and \(z_2g_2 = g_2z_2\). Thus, \(z\) is in the center of \(G_1 \times G_2\) if and only if \(z_1\) is in the center of \(G_1\) and \(z_2\) is in the center of \(G_2\). Consequently, the center of \(G_1 \times G_2\) is \(Z_1 \times Z_2\), where \(Z_j\) is the center of \(G_j\). \(\square\)

4. Chapter 2, exercise 11.6

Proof. Let \(H\) and \(K\) be the normal subgroups of \(G\) of orders 3 and 5, respectively. First note that \(H \cap K = \{1\}\). Indeed, if \(x \in H \cap K\), then the order of \(x\) divides both 3 and 5 and so must be 1. We claim that \(H\) and \(K\) commute. This follows just as in the proof of the proof of Proposition 2.11.4(d) in the book: If \(h \in H\) and \(k \in K\) then \(hkh^{-1}k^{-1}\) is in both \(H\) and \(K\) by the normality of \(H\) and \(K\) and so is equal to 1. In other words, \(HK \approx H \times K\) is a subgroup of \(G\). Because \(H\) and \(K\) have orders that are prime, we can select \(x \in H\) with order 3 and \(y \in K\) with order 5. Now the result follows from the first exercise applied to \(xy\). \(\square\)