1. In class, we defined a linear transformation $L(p)$ of $\mathbb{R}^n$ associated to a permutation $p$ in the symmetric group $S_n$. Recall that if $e_j$ is the $j^{th}$ standard basis vector of $\mathbb{R}^n$, then $L(p)e_j = e_{p(j)}$. Prove that $\det(L(p)) = \pm 1$.

Proof. Because $L(p)e_j = e_{p(j)}$, it is clear that $L(p)$ takes an orthonormal basis of $\mathbb{R}^n$ to another orthonormal basis. Consequently, $L(p)$ is orthogonal, i.e., $^tL(p)L(p) = 1$. (This is a fact from linear algebra; in your textbook you can find a review of it in Lemma 5.1.10). Taking determinants gives $\det(L(p))^2 = 1$, so $\det(L(p)) = \pm 1$.

Another proof that $\det(L(p)) = \pm 1$ can be given by proving that one can go from $L(p)$ to the identity matrix $1_n$ by swapping columns a finite number of times. (For example, proceed by induction on $n$.) As swapping columns changes the determinant of a matrix by $-1$, this proves that $L(p)$ has determinant $\pm 1$. We omit the details.

2. Prove that if $n \geq 3$, then the center of the symmetric group $S_n$ is the trivial subgroup $\{1\}$.

Here is one way that you can proceed: Suppose $p$ is in the center. One wants to prove that $p(j) = j$ for all $j$. To see that $p(1) = 1$, consider the fact that $p$ commutes with the permutation $g = (1234 \cdots n)$. Argue similarly for other $j \neq 1$.

Proof. As $pg = gp$, one has $p(g(1)) = g(p(1))$. But $g(1) = 1$, so $p(1) = g(p(1))$. But the only fixed point of the permutation $g$ is 1, so $p(1) = g(p(1))$ implies $p(1) = 1$.

Similarly, taking $g = (12 \cdots (j-1)(j+1) \cdots n)$ and considering $pg = gp$ proves that $p(j) = j$. As $p(j) = j$ for all $j$, $p$ is the identity permutation.


Proof. One is given $a^3b = ba^3$ with $a$ of order 7. Consequently, $a^{-1}b = a^6b = a^3ba^3 = (ba^3)a^3 = ba^6 = ba^{-1}$.

Rearranging gives $ab = ba$.

4. Chapter 2, exercise 4.3.

Proof. One always has $ab$ is conjugate to $ba$, as $ba = a^{-1}(ab)a$. Because conjugate group elements have the same order, $ab$ has the same order as $ba$. 

5. Chapter 2, exercise 5.1.

Proof. First suppose $G$ is cyclic, with generator $x$. This means that $G = \{x^n : n \in \mathbb{Z}\}$. Then if $\varphi : G \to G'$ is surjective,

$$G' = \varphi(G) = \{\varphi(x^n) : n \in \mathbb{Z}\} = \{y^n : n \in \mathbb{Z}\}$$

where $y = \varphi(x)$. Thus, $G'$ is cyclic.

Now suppose that $G$ is abelian, and $x, y \in G'$. Because $\varphi : G \to G'$ is surjective, there exists $a, b \in G$ so that $x = \varphi(a)$ and $y = \varphi(b)$. Therefore

$$xy = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = yx$$

where we have used $ab = ba$ because $G$ is abelian. This completes the proof.

6. Chapter 2, exercise 5.5.

Proof. First note that a matrix $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is in $\text{GL}_n(\mathbb{R})$ if and only if $a$ and $d$ are invertible, because $\det(h) = \det(a)\det(d)$.

Now, to check that $H$ is a subgroup, one needs to check that $H$ is closed under multiplication, has the identity, and is closed under inverses. It is clear that $H$ contains the identity. To see that $H$ is closed under multiplication, one multiplies

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ 0 & d_1d_2 \end{pmatrix}$$

(1)

and observes that the product is still in $H$. To see that $H$ is closed under inverses, one checks that the inverse of $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is $\begin{pmatrix} a^{-1} & -a^{-1}b^{-1} \\ 0 & d^{-1} \end{pmatrix}$, which is still in $H$.

The equation (1) implies that the map $H \to \text{GL}_n(\mathbb{R})$ sending $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to $a$ is a homomorphism. Its kernel consists of the set of matrices $\left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : d \in \text{GL}_{n-r}(\mathbb{R}) \right\}$.