Lecture 6

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Last time:

Thm: If m, n are positive integers, $(m,n) = 1$, and $d \mid mn$, then $\exists$ unique pos integers d_1, d_2 so that $d = d_1 d_2$, $d_1 \mid m$ and $d_2 \mid n$. Conversely, if $d_1 \mid m$, $d_2 \mid n$, then $d_1 d_2 \mid mn$.

Ex: $m = 2 \cdot 17$, $n = 3^2$

divisors of m: $1, 2, 17, 2 \cdot 17$

divisors of n: $1, 3, 3^2$

	1	3	3^2
1	$1 \cdot 1$	$1 \cdot 3$	$1 \cdot 3^2$
2	$2 \cdot 1$	$2 \cdot 3$	$2 \cdot 3^2$
17	$17 \cdot 1$	$17 \cdot 3$	$17 \cdot 3^2$
$2 \cdot 17$	$2 \cdot 17 \cdot 1$	$2 \cdot 17 \cdot 3$	$2 \cdot 17 \cdot 3^2$

exactly the divisors of $m \cdot n$

$= 2 \cdot 17 \cdot 3^2$

Last time: We defined a notion of multiplicative fcns

$f: \mathbb{Z}_{>0} \to \mathbb{C}$ is <u>multiplicative</u> if $f(mn) = f(m) f(n)$ whenever m, n are <u>relatively prime</u>.

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$\sum_{d|n}' f(d)$: denotes the sum of $f(d)$ over all positive divisors of n.

Ex $\sum_{d|10}' f(d) = f(1) + f(2) + f(5) + f(10)$

Thm: If f is a multiplicative function, and $g: \mathbb{Z}_{>0} \to \mathbb{C}$ defined by

$$g(n) = \sum_{d|n}' f(d),$$ then g is multiplicative.

Corollary: a) $d(n) :=$ # of pos int. divisors of n is multiplicative

b) $\sigma(n) := \sum$ of pos. int. divisors of n is multiplicative.

Pf of Cor: a) $d(n) = \sum_{d|n}' 1$ and $f(n) = 1$ is multiplicative

b) $\sigma(n) = \sum_{d|n}' d$ and $f(d) = d$ is multiplicative. ∎

Pf of Thm: Suppose $n, m > 0$, $(n,m) = 1$. Want to prove: $g(m)\,g(n) = g(mn)$.

Have: $$g(m)\,g(n) = \left(\sum_{d_1|m}' f(d_1)\right)\left(\sum_{d_2|n}' f(d_2)\right)$$

$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} \underbrace{f(d_1) f(d_2)}$$

- $d_1 \mid m$, $d_2 \mid n$ and $(n,m) = 1 \implies (d_1, d_2) = 1$

$\implies f(d_1) f(d_2) = f(d_1 d_2)$ b/c f is multiplicative by assumption.

$$\implies g(m)g(n) = \boxed{\sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(\underline{d_1 d_2})}$$

By previous theorem, $= \sum_{\boxed{d \mid mn}} f(d) = g(mn)$

So $g(m)\, g(n) = g(mn)$ when $(m,n) = 1$. ∎

Aside $m = 3$ $n = 3$

	1	3
1	1	3
3	3	9

- <u>Thm</u> Suppose $n = p_1^{a_1} \cdots p_k^{a_k}$. Then

$$d(n) = (a_1+1)(a_2+1) \cdots (a_k+1)$$

$$\sigma(n) = \frac{p_1^{a_1+1}-1}{} \cdot \cdots \cdot \frac{p_k^{a_k+1}-1}{}$$

$$\sigma(n) = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \cdots \cdot \frac{p_k^{a_k+1}-1}{p_k-1}$$

<u>Pf:</u> Because d is multiplicative,

- $d(n) = d(p_1^{a_1}) \cdots d(p_k^{a_k})$
- $d(p^a)$: The divisors of p^a are $\underbrace{\boxed{1, p, p^2, \ldots, p^a}}_{a+1 \text{ terms}}$

$$d(p^a) = a+1$$

$$= (a_1+1)(a_2+1) \cdots (a_k+1).$$

$$\sigma(n) = \sigma(p_1^{a_1}) \cdots \sigma(p_k^{a_k})$$

$$\sigma(p^a) = 1 + p + p^2 + \cdots + p^a = \frac{p^{a+1}-1}{p-1}$$

$$\sigma(n) = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \cdots \cdot \frac{p_k^{a_k+1}-1}{p_k-1}$$

<u>Ex:</u> $d(36) = d(2^2 \cdot 3^2) = (1+2)(1+2) = 9$

$$\sigma(36) = \sigma(2^2)\,\sigma(3^2) = (1+2+4)(1+3+9)$$

$$= 7 \cdot 13 = 91$$

<u>Def$^{\underline{n}}$</u> A positive integer n is <u>perfect</u> if

$\sum d = n$. In other words, n is perfect

$$\sum_{\substack{d \mid n \\ d \neq n}} d = n.$$

In other words, n is perfect if $\sigma(n) = 2n$.

(Aside: $\sigma(n) \geq n$, $\underbrace{\sigma(n) - n = n}_{\iff \sigma(n) = 2n}$ if n is perfect)

Ex: 6 is perfect $6 = 1+2+3$

28 is perfect $\sigma(28) = \sigma(4)\sigma(7) = (1+2+4)(1+7)$

$= 7 \cdot 8 = 56 = 2 \cdot 28.$

Conjecture: Every perfect number is even.

Know: There are no odd perfect numbers $< 10^{1500}$

In fact, If p is prime and $2^p - 1$ is also prime, then $n = 2^{p-1}(2^p - 1)$ is perfect.

ex: $p = 2$, $2^2 - 1 = 3$ is also prime

$n = 2^{2-1} \cdot 3 = 6$ is perfect

$p = 3$, $2^3 - 1$ is also prime

$\Rightarrow n = 2^2 \cdot (2^3 - 1) = 4 \cdot 7 = 28$ is perfect

A Mersenne prime is a prime of the form $2^p - 1$ w/ p a prime

Euler: If n is even and perfect then $n = 2^{p-1}(2^p - 1)$ w/ p prime and $2^p - 1$ prime.

Linear Diophantine Equations

Aim: Find all x, y integers so that

$$ax + by = c \quad \text{where } a, b, c \text{ are fixed integers.}$$

Ex: Find all integer sol$^{\underline{n}}$s to $14x + 22y = 10$

Thm: Suppose $a, b \neq 0$ integers and $d = (a, b)$. If $d \nmid c$, then the eq$^{\underline{n}}$ $ax + by = c$ has no integral solutions. If $d \mid c$, then the eq$^{\underline{n}}$ has infinitely many solutions.

If x_0, y_0 is one integral sol$^{\underline{n}}$, then all sol$^{\underline{n}}$s are given by

$$x = x_0 + t\frac{b}{d}, \quad t \in \mathbb{Z}.$$
$$y = y_0 - t\frac{a}{d}$$