Last time: Transcendental #’s: The complex #’s that are not algebraic

**Algebraic numbers:** The complex #’s \( \gamma \) s.t. \( \gamma \) is a root of a polynomial \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \) \( \forall \)

\( a_0 \neq 0, \ a_i \in \mathbb{Z} \).

**THM:** Suppose \( \gamma \) is algebraic, a root of \( f \). Then, \( \exists \ \delta > 0 \)

s.t. \( \forall \ \frac{p}{q} \neq \gamma \), \( |\gamma - \frac{p}{q}| > \frac{\delta}{q^n} \).

**THM:** Let \( \gamma = \sum_{k=1}^{\infty} 2^{-k!} = 2^{-1} + 2^{-2} + 2^{-6} + 2^{-24} + 2^{-120} + \ldots \) \( \approx \gamma \)

Then \( \gamma \) is transcendental.

**Idea of proof:**

\[ \text{Let } \frac{p_k}{q_k} \text{ be the rational approximations.} \]

\[ \sum_{k=1}^{K} 2^{-k!} \]

Then \( \frac{p_k}{q_k} < \gamma \)

We'll show: This approximation is so good that it contradicts the conclusion of the previous theorem.

**Pf:** By contradiction. Suppose \( \gamma \) is algebraic. Then

\[ \exists \ \delta > 0, \ n \in \mathbb{Z}, \ n > 0 \text{ s.t. if } \gamma \neq \frac{p}{q} \text{ then} \]
\[ \exists \delta > 0, n \in \mathbb{Z}, n > 0 \text{ s.t. if } q \neq \frac{p}{q} \text{ then } \]

\[ |y - \frac{p}{q}| > \frac{\delta}{q^n} . \]

\( [\text{note: } n \text{ is fixed, it depends upon } \alpha, \text{ but is indep of the ratio } \neq \frac{p}{q}]. \)

\[ \frac{p_k}{q_k} = \sum_{k=1}^{\infty} \frac{2^{-k!}}{k!} = \sum_{k=1}^{\infty} \frac{2^{k!-k!}}{k!}. \]

\( [\text{let } p_k = \sum_{k=1}^{\infty} 2^{k!-k!}, q_k = 2^{k!}.] \)

\( \text{Observe!} \]

\[ \frac{p_k}{q_k} < \gamma \]

\[ 0 < y - \frac{p_k}{q_k} = \sum_{k=k+1}^{\infty} \frac{2^{-k!}}{k!} < \sum_{m=(k+1)!}^{\infty} \frac{2^{-m}}{m} \]

\( \text{geometric series} \)

\[ = \frac{2^{-(k+1)!}}{1 - \frac{1}{2}} = 2 \cdot 2^{-(k+1)!} = 2 \left(2^{k!}\right)^{-(k+1)} \]

\[ = 2 \cdot 2^{-k_{(k+1)}} \]

\[ < 2^{k_{(k+1)}}. \]

\( \text{Thus:} \)

\[ |y - \frac{p_k}{q_k}| < 2 \cdot 2^{-k_{(k+1)}} = \frac{2}{9_k} = \frac{2}{9_k} \cdot \frac{2^{k}}{9_k} = \frac{2}{9_k} \cdot \frac{2^{k}}{9_k} \]

\( \delta \text{ is fixed: } \Rightarrow \text{ we can take } K > 0 \text{ s.t. } 2 \cdot 2^{k} < \delta. \)
$\delta$ is fixed: \[2^{2^{-k!}} < \delta\]

$n$ is fixed: \[\Rightarrow \text{we can take } K > n \text{ s.t. } K > n\]

Then \[|q - \frac{p_k}{q_k}| < \frac{2^{2^{-k!}}}{q_k} < \frac{\delta}{q_k}, \quad \Rightarrow \]

---

Magic $\square$'s

<table>
<thead>
<tr>
<th>12</th>
<th>19</th>
<th>21</th>
<th>3</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>25</td>
<td>2</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>8</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>14</td>
<td>16</td>
<td>23</td>
</tr>
<tr>
<td>6</td>
<td>13</td>
<td>20</td>
<td>22</td>
<td>4</td>
</tr>
</tbody>
</table>

Properties:
- It contains the integers 1, 2, 3, \ldots, 25
- The sum of any row and any column is "fixed" and equal (to 65)

Def: An $n \times n$ $\square$ w/ $n^2$ entries is said to be a magic $\square$ if the sum of the entries of any row or any column is always the same. The common sum is called the magic sum.

How constructed?
- Put 1 anywhere
- Move up and right
• Move up and right, placing #'s sequentially.

• When you get to an edge, "wrap around" vertically and horizontally.

• When you can't move because you're blocked, put the next # 1 down.

• Continue.

**THM!** If n is odd, the above strategy always works to construct a magic \( \square \).

**In fact:** there's a much more general strategy that constructs magic \( \square \)s for even and odd \( n \).

**Convenience:** We'll show how to put the #:s \( 0, 1, 2, \ldots, n^2 - 1 \) into a magic \( \square \). (Instead of \( 1, 2, \ldots, n^2 \).)

**Notation:** Each of the \( n^2 \) possible locations for a number in a \( \square \) will be called a cell.

Set \( x_j, y_j \) to be the X and Y coordinates of the cell containing \( j \).
the cell containing $j$.

$(x_0, y_0) = (4, 3)$

$(x_1, y_1) = (5, 5)$

$(x_2, y_2) = (1, 2)$

$y_{19} = y_7 = y_{20} = y_7 = y_1 = 5$

Example of Uniform Step Method

- Put 0 anywhere
- Put #s in sequentially going 1 right and up 2.

- Wrap around horizontally and vertically
- When blocked, go over 1 and up 2 but then move over 1 and up 3
- Continue, going over 1 and up 2

Question: How to describe the $(x_j, y_j)$ succinctly in terms of $j$?