Last time: We studied Diophantine eqns, e.g., \( x^2 + y^3 = z^2 \).

Another technique: Proving diophantine eqns has only many solns w/o determining all solns.

**Proof:** The eqn \( x^2 + y^2 + z^2 = w^2 \) has only many pos int solns.

**Proof:** Consider the special case \( w = x+y \).

\[
\Rightarrow \quad x^2 + y^2 + z^2 = (x+y)^2 = x^2 + 2xy + y^2.
\]

\[
\Rightarrow \quad z^2 = 2xy.
\]

Let \( x = 2a^2, \ y = b^2 \), \( a, b \in \mathbb{Z} \).

Then \( 2xy = 4a^2b^2 = (2ab)^2 \).

Let \( z = 2ab \).

**Summary:** \( \begin{cases} x = 2a^2, \ y = b^2, \ z = 2ab, \ w = x+y = 2a^2 + b^2 \\ \text{w/} \ a, b \in \mathbb{Z} \end{cases} \) then \( w^2 = x^2 + y^2 + z^2 \).

Double check: \( x^2 + y^2 + z^2 = (2a^2)^2 + (b^2)^2 + (2ab)^2 \)

\[
= 4a^4 + 4a^2b^2 + b^4
\]

\[ \left( \sqrt{n}, \ n \right)^2 \]
\[
= 4a^4 + b^2
\]
\[
= (2a^2 + b^2)^2
\]
\[
= w^2 \quad \text{.}
\]

More on Pythagorean Triples

If \( A^2 + B^2 = C^2 \) then \( \left( \frac{A}{C} \right)^2 + \left( \frac{B}{C} \right)^2 = 1 \)

\( w/ \frac{A}{C}, \frac{B}{C} \) are \#s if \( A, B, C \) are nonzero integers

Conversely, if \( a^2 + b^2 = 1 \) \( w/ \ a, b \) rational \#s, then

\[ a = \frac{A}{C}, \quad b = \frac{B}{C} \quad \text{for some } A, B, C \in \mathbb{Z} \]

\( \Rightarrow \) \( A^2 + B^2 = C^2 \).

\[ \text{Upshot: Finding rational solns to the eqn } x^2 + y^2 = 1 \text{ is closely related to finding integer solns to } A^2 + B^2 = C^2. \]

Thm: All rational solns to \( x^2 + y^2 = 1 \) are given by

\[ (x, y) = \begin{cases} (-1, 0) \quad \text{or} \\ x = 1 - t^2 \quad , \quad y = \frac{2t}{1 + t^2} \quad w/ \ t \text{ a rational } \# \\ \end{cases} \]

Pf: First observe these are solns:
It's observed these are solutions:

\[ (-1)^2 + 0^2 = 1 \]

\[ \left( \frac{1-t^2}{1+t^2} \right)^2 + \left( \frac{2t}{1+t^2} \right)^2 = \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2} = \frac{t^4 + 2t^2 + 1}{(t^2 + 1)^2} = \frac{(t^2 + 1)^2}{(t^2 + 1)^2} = 1. \]

Moreover, if \( t \) is rational then \( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \) are rational.

Looking for: Rational points on the unit circle.

Conversely, suppose \((x,y)\) is a rational solution of \( x^2 + y^2 = 1. \)

The slope of the line from \((x,y)\) to \((-1,0)\) is rational, call it \( t. \)

\[ \frac{y}{x+1} = t. \]

\[ x^2 + y^2 = 1 \]

\[ y = t(x+1) \] for some rational \( t. \)

\[ \downarrow \]

\[ y^2 = t^2(x^2 + 2x + 1) \]

\[ 1 = x^2 + t^2(x^2 + 2x + 1) = (1 + t^2)x^2 + 2t^2x + t^2 \]

\[ \Rightarrow (1 + t^2)x^2 + 2t^2x + t^2 - 1 = 0. \]

\[ \Rightarrow \left[ \frac{x^2 + 2t^2}{1 + t^2} x + \frac{t^2 - 1}{1 + t^2} \right] = 0. \]
KEY STEP: We knew \( x = -1 \) is a solution to this equation.

Thus, can factor out an \( x + 1 \) to obtain

\[
(x + 1) \left( x + \frac{t^2 - 1}{t^2 + 1} \right) = 0.
\]

Double check: \( x^2 + \left( 1 + \frac{t^2 - 1}{t^2 + 1} \right) x + \frac{t^2 - 1}{t^2 + 1} = \frac{2t}{t^2 + 1} \)

\[
x = \frac{1 - t^2}{1 + t^2}, \quad y = t(x + 1) = t \left( 1 + \frac{1 - t^2}{1 + t^2} \right) = \frac{2t}{1 + t^2}.
\]

Ex: Show that the eqn \( x^2 + y^4 = z^2 \) has only many positive integer solutions.

Idea: \( x^2 + (y^2)^2 = z^2 \)

So: we’re looking for Pythagorean triples with one term a 0.

\[
x = m^2 - n^2
\]

Let \( y^2 = 2mn \) \( \text{w/ } m, n \in \mathbb{Z} \).

\[
z = m^2 + n^2
\]

Then \( x^2 + y^4 = z^2 \).
We found all Pythagorean triples in terms of int. $m,n$:

If $A^2 + B^2 = C^2$ and $\gcd(A,B,C) = 1$

then $A = 2mn$
$B = \sqrt{m^2 - n^2}$
$C = \sqrt{m^2 + n^2}$

(up to switching $A$ and $B$)

Now, choose $m,n$ s.t. $2mn$ is a square.

$m = 2u^2$, $n = v^2$ then $2mn = 4u^2v^2 = (2uv)^2$.

$x = m^2 - n^2 = 4u^4 - v^4$

$y = 2mn = 4u^2v^2 \Rightarrow y = 2uv$, these will be solns.

$z = m^2 + n^2 = 4u^4 + v^4$

Double check:

$x^2 + y^4 = (4u^4 - v^4)^2 + (4u^2v^2)^2$

$= 16u^8 - 8u^4v^4 + v^8 + 16u^4v^4$

$= 16u^8 + 8u^4v^4 + v^8$

$= (4u^4 + v^4)^2 = z^2$