EXCEPTIONAL SIEGEL WEIL THEOREMS FOR COMPACT Spin_8

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Abstract. Let E be a cubic étale extension of the rational numbers which is totally real, i.e., $E \otimes \mathbf{R} \simeq \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. There is an algebraic $\mathbf{Q}$-group $S_E = \mathrm{Spin}_{8,E}^c$ defined in terms of E, which is semisimple simply-connected of type D_4 and for which $S_E(\mathbf{R})$ is compact. We let G_E denote a certain semisimple simply-connected algebraic $\mathbf{Q}$-group of type D_4, defined in terms of E, which is split over $\mathbf{R}$. Then $G_E \times S_E$ maps to quaternionic E_8. This latter group has an automorphic minimal representation, which can be used to lift automorhpic forms on S_E to automorphic forms on G_E. We prove a Siegel-Weil theorem for this dual pair: I.e., we compute the lift of the trivial representation of S_E to G_E, identifying the automorphic form on G_E with a certain degenerate Eisenstein series.

Along the way, we prove a few more "smaller" Siegel-Weil theorems, for dual pairs $M \times S_E$ with $M \subseteq G_E$. The main result of this paper is used in the companion paper "Exceptional theta functions and arithmeticity of modular forms on G_2" to prove that the cuspidal quaternionic modular forms on G_2 have an algebraic structure, defined in terms of Fourier coefficients.

Contents

1. Introduction

Suppose G' is a reductive $\mathbf{Q}$-group with an automorphic minimal representation V_{min}, and $G \times S \subseteq G'$ is a dual pair. Let $\phi \in V_{min}$ and $\Theta_\phi \in \mathcal{A}(G')$ the associated automorphic form. Assuming the integral $\Theta_\phi(\mathbf{1})(g) := \int_{S(\mathbf{Q})\backslash S(\mathbf{A})} \Theta(g,h)\,dh$ converges (or can be regularized), one would like to identify explicitly the automorphic form $\Theta_\phi(\mathbf{1})(g) \in \mathcal{A}(G)$. By a Siegel-Weil formula we mean an identity $\Theta_\phi(\mathbf{1})(g) = E_\phi(g)$, where $E_\phi(g)$ is some degenerate Eisenstein series on G defined in terms of ϕ.

Siegel-Weil theorems have a long history. The classical setting is when $S = O(V)$ is the orthogonal group of a quadratic space of even dimension, $G = \mathrm{Sp}_{2m}$, $G' = \mathrm{Sp}_{2m\dim(V)}$ (or rather the metaplectic double cover), and Θ_ϕ comes from the Weil representation. This setting has been

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studied by Siegel [Sie51], Weil [Wei65], Rallis [Ral87], and Kudla-Rallis [KR88a, KR88b, KR94] among others. We refer to [KR88a, KR94] for a more extensive history.

The Weil representation on symplectic groups are not the only automorphic minimal representations. The group G' could, for instance, be of type D_n or exceptional of type E. Beautiful examples of this sort of "exceptional" Siegel Weil theorems appear in work of Gan [Gan00b, Gan08, Gan11] and Gan-Savin [GS20]. In this paper, we prove a family of Siegel-Weil theorems for G' of type E_8. We also prove some Siegel-Weil theorems when G' is of type D_6, D_7 and E_7. In all of these cases, S is simply connected of type D_4 with $S(\mathbf{R})$ compact.

Siegel-Weil theorems are frequently used in conjunction with Rankin-Selberg integrals to prove results connecting special values of L-functions, theta lifts, and periods of automorphic forms. In such a framework, the Siegel-Weil Eisenstein series $E_\phi(g)$ is realized as a special value $E_\phi(g, s = s_0)$ of an Eisenstein series $E_\phi(g, s)$, and this latter Eisenstein series participates in a Rankin-Selberg integral. Relating $E_\phi(g, s = s_0)$ to theta functions then gives a non-trivial relationship between L-values of the cusp forms appearing in the Rankin-Selberg integral and their theta lifts.

The situation in this paper is similar. The main result of this paper will be used in the companion paper [Pol23] to prove that every cuspidal quaternionic modular form on G_2 of even weight at least 6 lifts to an anisotropic group of type F_4. Combined with the other main results of [Pol23], this proves that the cuspidal quaternionic modular forms on G_2 of even weight at least 6 have an algebraic structure, defined in terms of Fourier coefficients. We refer to [Pol23] for more details.

We remark that our main result, Theorem 9.4.1, can be considered as part of the theory of D_4 modular forms, in the sense of [Wei06]. In fact, when $E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$, Weissman hypothesized the existence of a Siegel-Weil formula as in Theorem 9.4.1.

1.1. **Statement of results.** We give rough version of the main result of this paper, deferring the precise statement until after all the group theoretic notation has been defined.

Let E be a cubic étale extension of the rational numbers which is totally real, i.e., $E \otimes \mathbf{R} \simeq \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. There is an algebraic $\mathbf{Q}$-group $S_E = \mathrm{Spin}^c_{8,E}$ defined in terms of E, which is semisimple simply-connected of type D_4 and for which $S_E(\mathbf{R})$ is compact.

We let G_E denote a certain semisimple simply-connected algebraic $\mathbf{Q}$-group of type D_4, defined in terms of E, which is split over $\mathbf{R}$. Then $G_E \times S_E$ maps to a group denoted G_J below, which is quaternionic E_8. This latter group is split at all finite places and $G_J(\mathbf{R}) = E_{8,4}$ has real rank four.

The group G_J has an automorphic minimal representation, V_{min}, defined in [Gan00a] and studied further in [GS05]. We review the construction and properties of this and other automorphic minimal representations in sections 4 and 5 below.

Let P_J be the maximal Heisenberg parabolic subgroup of G_J, so that P_J has Levi subgroup of type $GE_{7,3}$. Using the map $G_E \to G_J$, one can intersect $G_E \cap P_J$ to obtain a maximal Heisenberg parabolic subgroup of G_E, call it P_E.

Let $\phi \in V_{min}$ and Θ_ϕ the associated automorphic form on G_J. We can realize V_{min} as a submodule[1] of $Ind_{P_J(\mathbf{A})}^{G_J(\mathbf{A})}(\delta_{P_J}^{5/29})$. Using this realization, let $Res(\phi)$ be the restriction of ϕ to G_E. Then $Res(\phi)$ lands in $Ind_{P_E(\mathbf{A})}^{G_E(\mathbf{A})}(\delta_{P_E})$. Extending $Res(\phi)$ to a section $f_\phi(g, s)$ in $Ind_{P_E(\mathbf{A})}^{G_E(\mathbf{A})}(\delta_{P_E}^s)$, one can define[2] an Eisenstein series $E_\phi(g, s) = \sum_{\gamma \in P_E(\mathbf{Q}) \backslash G_E(\mathbf{Q})} f_\phi(\gamma g, s)$. Using the fact that $f_\phi(g, s)$ came from the minimal representation on G_J, the Eisenstein series $E_\phi(g, s)$ turns out to be regular at $s = 1$. The associated automorphic form $E_\phi(g) \in \mathcal{A}(G_E)/\mathbf{1}$ (modding out by the trivial representation) is independent of the extension of ϕ to an inducing section $f_\phi(g, s)$. Here is our main theorem.

Let $\Theta_\phi(\mathbf{1})(g) := \int_{S_E(\mathbf{Q}) \backslash S_E(\mathbf{A})} \Theta_\phi(g, h)\, dh$ be the theta lift of $\mathbf{1}$ to $\mathcal{A}(G_E)$.

[1] At the archimedean place, this realization is somewhat convoluted; see section 9.5

[2] In the body of the paper, we normalize the parameter s in these Eisenstein series differently.

Theorem 1.1.1 (See section 9)**.** *Let the notation be as above. Normalize the Haar measure on S_E so that the automorphic quotient $S_E(\mathbf{Q})\backslash S_E(\mathbf{A})$ has measure 1; this is the Tamagawa measure. Then one has an identity of automorphic forms $\Theta_\phi(\mathbf{1})(g) = E_\phi(g)$ in $\mathcal{A}(G_E)/\mathbf{1}$.*

We also prove similar theorems for dual pairs of type $D_2 \times S_E \subseteq D_6$ when $E = \mathbf{Q} \times F$ (F quadratic étale), $D_3 \times S_E \subseteq D_7$ when $E = \mathbf{Q} \times F$, and $\mathrm{SL}_{2,E} \times S_E \subseteq E_7$. These "smaller" theorems are, in fact, used in the proof of Theorem 1.1.1.

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2. Group theory

In this section, we define various groups and embeddings of groups that we use throughout the paper.

2.1. **Generalities.** We begin by recalling the following well-known result in the theory of linear algebraic groups. See [Mil17, Proposition 18.8, Theorem 22.53, and Theorem 23.70].

Proposition 2.1.1. *Let k be a field of characteristic 0.*

(1) Suppose $\mathfrak{g}$ is a semisimple Lie algebra over k. There exists a connected, semsisimple, simply-connected algebraic k-group $G(\mathfrak{g})$ with $Lie(G(\mathfrak{g})) = \mathfrak{g}$. The group $G(\mathfrak{g})$ with these properties is unique up to isomorphism.

(2) Suppose H is an algebraic k-group and $L : \mathfrak{g} \to Lie(H)$ a morphism of Lie algebras over k. Then there exits a unique map of algebraic groups $G(\mathfrak{g}) \to H$ whose differential is L.

We also recall:

Proposition 2.1.2. *Let k be a field of characteristic 0. Suppose G, S are connected semisimple algebraic k-groups, and each maps to an algebraic k-group H via maps ι_G, ι_S. Suppose moreover that the differential $d\iota_G : Lie(G) \to Lie(H)$ lands in $Lie(H)^S$. Then $\iota_G(G)$ lands in the centralizer of $\iota_S(S)$, so that one obtains a map $\iota_G \times \iota_S : G \times S \to H$.*

Proof. Let $\overline{k}$ denote the algebraic closure of k. It suffices to check that if $x \in S(\overline{k})$ then x centralizes $\iota_G(G)$. For this, consider the map ι' given by composing ι_G with conjugation by x. This gives a potentially new map $\iota' : G \to H$ over $\overline{k}$. But $d\iota' = d\iota_G$ because $d\iota_G : Lie(G) \to Lie(H)$ lands in $Lie(H)^S$. But the differential gives a fully faithful functor $Rep(G) \to Rep(Lie(G))$ [Mil17, Theorem 22.53], so $\iota' = \iota_G$. □

2.2. **The group S_E.** Let Θ denote the octonion $\mathbf{Q}$ algebra with positive-definite norm form N_Θ. We write tr_Θ for the octonionic trace on Θ.

Set $J = H_3(\Theta)$ the exceptional cubic norm structure. We write a typical element X of J as

$$X = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix}. \tag{1}$$

Let E be an étale cubic extension of $\mathbf{Q}$ that is totally real. We assume given an embedding $E \hookrightarrow J$ with the following properties:

(1) E lands in $H_3(\mathbf{Q}) \subseteq J$;
(2) if N_J denotes the cubic norm on J and N_E the cubic norm on E, then $N_J(x) = N_E(x)$ for all $x \in E$;
(3) $1 \in E$ maps to $1 \in J$;
(4) if $E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$, then the map $E \to J$ is $(c_1, c_2, c_3) \mapsto \mathrm{diag}(c_1, c_2, c_3)$;
(5) if $E = \mathbf{Q} \times F$ with F a real quadratic field, then $(1, 0) \mapsto \mathrm{diag}(1, 0, 0)$, $(0, 1) \mapsto \mathrm{diag}(0, 1, 1)$, and in general the image of $X \in E$ has x_2 and x_3 coordinates equal to 0.

Let V_E denote the orthogonal complement of $E \subseteq J$ under the trace pairing on J. Let M_J^1 denote group of linear automorphisms of J that fix the cubic norm N_J. Then M_J^1 is simply connected of type E_6. We let S_E denote the subgroup of M_J^1 that is the identity on E. The group S_E is connected, simply connected of type D_4 and has $S_E(\mathbf{R})$ compact.

We set $E_{sp} = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$. The group $S_{E_{sp}}$ can be recognized as the group of triples $(g_1, g_2, g_3) \in \mathrm{SO}(\Theta)$ with $(g_1x_1, g_2x_2, g_3x_3)_{\mathrm{tr}_\Theta} = (x_1, x_2, x_3)_{\mathrm{tr}_\Theta}$ for all $x_1, x_2, x_3 \in \Theta$, where $(x_1, x_2, x_3)_{\mathrm{tr}_\Theta} = \mathrm{tr}_\Theta(x_1(x_2x_3))$. We identify $H_2(\Theta) \subseteq J$ as those elements with c_1, x_2, and x_3 coordinates equal to 0. When $E = \mathbf{Q} \times F$, then $V_E = \Theta^2 \oplus \Theta_F$, where Θ_F is the orthogonal complement in $H_2(\Theta)$ to the image of F.

2.3. **Exceptional groups.** For a cubic norm structure A over a field k of characteristic 0, we have the Freudenthal space $W_A = k \oplus A \oplus A^\vee \oplus k$. This space comes equipped with a symplectic form and a quartic form, see [Pol20a, Section 2.2]. We let H_J denote the identity component of the group of linear automorphisms of that preserve these forms up to similitude $\nu : H_J \rightarrow \mathrm{GL}_1$ (see [Pol20a, Section 2.2]) and H_J^1 the kernel of ν. When $A = J$ is the exceptional cubic norm structure, H_J^1 is simply connected of type E_7; it is split at every finite place and $H_J^1(\mathbf{R}) = E_{7,3}$. When $A = E$ is cubic étale over k, then $H_E^1 = \mathrm{SL}_{2,E}$. (For an explicit map $\mathrm{SL}_{2,E} \rightarrow H_E^1$, one can see [Pol18, Section 4.4].) We have a map $H_E^1 \times S_E \rightarrow H_J^1$.

We recall from [Pol20a, Section 4] the Lie algebra $\mathfrak{g}(A)$. One has $\mathfrak{g}(A) = \mathfrak{sl}_2 \oplus \mathfrak{h}(A)^0 \oplus (V_2 \otimes W_A)$. Here $\mathfrak{h}(A)^0 = Lie(H_A^1)$ and V_2 denotes the standard representation of $\mathfrak{sl}_2$. We denote by[3] G_A the connected, simply connected group with $Lie(G_A) = \mathfrak{g}(A)$. When $A = J$, G_J is of type E_8, split at every finite place and $G_J(\mathbf{R}) = E_{8,4}$. When $A = E$, G_E is of type D_4 with $G_E(\mathbf{R})$ split.

The group G_J has rational root system of type F_4. When $E = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$, D_E is split and has rational root system of type D_4; when $E = \mathbf{Q} \times F$ with F field, D_E has rational root system of type B_3, and when E is a field, D_E has rational root system of type G_2. The group M_J^1 has rational root system of type A_2, and H_J^1 has rational root system of type C_3.

2.4. **Classical root types.** We now define some groups that we will use that have classical root types.

Let $H = \mathbf{Q} \oplus \mathbf{Q}$ be a hyperbolic plane. Set $V_{10} = H \oplus \Theta$ with bilinear form $((h_1, \theta_1), (h_2, \theta_2)) = (h_1, h_2) - (\theta_1, \theta_2)_\Theta$. Let $V_{12} = H \oplus V_{10} = H^2 \oplus \Theta$ and $V_{14} = H \oplus V_{12} = H^3 \oplus \Theta$. We let $G_{5,\Theta}$ be the simply-connected cover of $\mathrm{SO}(V_{10})$, and similarly define $G_{6,\Theta}$, $G_{7,\Theta}$ as the simply connected cover of $\mathrm{SO}(V_{12})$, respectively, $\mathrm{SO}(V_{14})$.

For a quadratic étale extension F of $\mathbf{Q}$, we let $V_{4,F} = H \oplus F$ with quadratic form $q(h, f) = q_H(h) + N_F(f)$. Observe that one has $V_{10} = F \oplus \Theta_F$, so that $V_{12} = V_{4,F} \oplus \Theta_F$. Similarly, we set $V_{6,F} = H^2 \oplus F = H \oplus V_{4,F}$, and $V_{14,\Theta} = V_{6,F} \oplus \Theta_F$. Let $G_{2,F}$ be the simply connected cover of $\mathrm{SO}(V_{4,F})$ and let $G_{3,F}$ be the simply connected cover of $\mathrm{SO}(V_{6,F})$. From the inclusions $\mathrm{SO}(V_{4,F}) \times \mathrm{SO}(\Theta_F) \subseteq \mathrm{SO}(V_{12})$ and $\mathrm{SO}(V_{6,F}) \times \mathrm{SO}(\Theta_F) \subseteq \mathrm{SO}(V_{14,\Theta})$, we obtain maps $G_{2,F} \times S_{\mathbf{Q}\times F} \rightarrow G_{6,\Theta}$ and $G_{3,F} \times S_{\mathbf{Q}\times F} \rightarrow G_{7,\Theta}$.

2.5. **Parabolic subgroups.** We will need to keep track of numerous parabolic subgroups of various reductive groups. We use the following naming convention: Suppose G is a reductive group, with a fixed split torus T and simple roots $\alpha_1, \ldots, \alpha_r$ for a rational root system $\Phi(G, T)$. Then for each subset I of the set of simple roots $\{\alpha_1, \ldots, \alpha_r\}$, there is an associated standard parabolic $P_{G,I} = M_{G,I}N_{G,I}$ with those simple roots appearing in its unipotent radical $N_{G,I}$. Thus if I is a singleton then $P_{G,I}$ is maximal, whereas if $I = \{\alpha_1, \ldots, \alpha_r\}$ then $P_{G,I} =: P_{G,0}$ is minimal. We will also write $P_{G,j}$ for $P_{G,\{\alpha_j\}}$, $P_{G,jk}$ for $P_{G,\{\alpha_j,\alpha_k\}}$ and so on.

[3]This notation slightly conflicts with the notation of [Pol20a], where G_A denoted the adjoint group associated to this Lie algebra.

3. General facts about automorphic forms and representation theory

In this section, we collect various general facts about automorphic forms and representation theory that we will use later in the paper.

3.1. **Weyl groups.** Suppose G is a reductive group with a torus $T \subseteq G$ that is maximal among split tori. Fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ for the roots $\Phi(G,T)$; $\Phi(G,T)^+$ denotes the set of positive roots. Let W denote the Weyl group of $\Phi(G,T)$. If $P = MN$ is a standard parabolic, we write W_M for the subgroup of W generated by the simple reflections corresponding to elements $\Delta \cap \Phi(M,T)$.

Following [Cas, Section 1], set

$$[W/W_M] = \{w \in W : w(\alpha) \in \Phi(G,T)^+ \forall \alpha \in \Delta \cap \Phi(M,T)\},$$

$$[W_M \backslash W] = \{w \in W : w^{-1}(\alpha) \in \Phi(G,T)^+ \forall \alpha \in \Delta \cap \Phi(M,T)\}.$$

If $P = MN$, $Q = LV$ are two standard parabolic subgroups, then we set

$$[W_L \backslash W / W_M] = [W_L \backslash W] \cap [W/W_M].$$

3.2. **Intertwining operators.** We now review standard facts about intertwining operators. Suppose P_0 is the minimal standard parabolic for the root system $\Phi(G,T)$. Let k be a local field, either p-adic or archimedean. Let $\chi : P_0(k) \rightarrow \mathbf{C}^\times$ be a character. Consider the induced representation $Ind_{P_0(k)}^{G(k)}(\delta_{P_0}^{1/2}\chi)$. If $w \in W$, we will define the intertwining operator

$$M(w) : Ind_{P_0(k)}^{G(k)}(\delta_{P_0}^{1/2}\chi) \rightarrow Ind_{P_0(k)}^{G(k)}(\delta_{P_0}^{1/2}w(\chi))$$

associated to w.

Let U_α be the root subgroup corresponding to the root $\alpha \in \Phi(G,T)$ and set $N_w = \prod_\alpha U_\alpha$ with the product over positive roots α for which $w^{-1}(\alpha)$ is negative. If $w \in W$ and $f \in Ind_{P_0(k)}^{G(k)}(\delta_{P_0}^{1/2}\chi)$, we set

$$M(w)f(g) = \int_{N_w(k)} f(w^{-1}ng)\, dn$$

if the integral is absolutely convergent. In this case, $M(w) : Ind_{P_0(k)}^{G(k)}(\delta_{P_0}^{1/2}\chi) \rightarrow Ind_{P_0(k)}^{G(k)}(\delta_{P_0}^{1/2}w(\chi))$ is $G(k)$-intertwining. If $w = w_1w_2$ with $\ell(w) = \ell(w_1)+\ell(w_2)$, and each of $M(w_1)$, $M(w_2)$ are defined (i.e., defined by absolutely convergent integrals), then so is $M(w)$, and $M(w) = M(w_1) \circ M(w_2)$.

Let $h_\alpha : \mathrm{GL}_1 \rightarrow T$ be the cocharacter associated to the root α. If $w = s_\alpha$ is a simple reflection, and $N_w = U_\alpha$ is one-dimensional, then the integral defining $M(s_\alpha)$ is absolutely convergent if $\chi(h_\alpha(t)) = |t|^s$ with $Re(s) > 0$.

We will frequently be in the following situation: $P \subseteq G$ is a maximal parabolic subgroup, $\nu : P \rightarrow \mathrm{GL}_1$ is a character, and $I(s) := Ind_{P(k)}^{G(k)}(|\nu|^s)$. We assume $\delta_P = |\nu|^{s_P}$ with $s_P > 0$ as opposed to $s_P < 0$. We set $\lambda_s = \delta_{P_0}^{-1/2}|\nu|^s$, so that $I(s) \subseteq Ind_{P_0(k)}^{G_0(k)}(\delta_{P_0}^{1/2}\lambda_s)$. Now suppose $w \in [W/W_M]$ where $P = MN$. Then if $\alpha > 0$ and $w^{-1}(\alpha) < 0$, we must have that $w^{-1}(U_\alpha) \subseteq \overline{N}$, the unipotent radical opposite to P. In this case, $M(w)$ is absolutely convergent for $Re(s) >> 0$, and has a meromorphic continuation in s, in the following sense. Given s_0, there is an integer k so that if $f(g,s) \in I(s)$ is a flat section, then $(s-s_0)^kM(w)f(g,s)$ is meromorphic in s and regular at s_0. In this case,

$$(s-s_0)^kM(w) : I(s_0) \rightarrow Ind_{P_0(k)}^{G_0(k)}(\delta_{P_0}^{1/2}w(\lambda_s))$$

is $G(k)$-intertwining.

In the situation of the above paragraph, there is one intertwiner to which we give a fixed name and notation: the long intertwining operator. Namely, the set $[W/W_M]$ has a unique longest element (see [Cas, Proposion 1.1.4]), which we denote by w_0. While this notation is independent of the group G, the underlying group G will always be clear from context. When we say the long intertwining operator, we mean $M(w_0)$.

Throughout, we normalize the Haar measure on a reductive group G over $\mathbf{Q}$ so that the unipotent quotient groups $U_\alpha(\mathbf{Q})\backslash U_\alpha(\mathbf{A})$ have measure 1.

3.3. **Eisenstein series.** Suppose G is a reductive group over $\mathbf{Q}$, and $P \subseteq G$ is a maximal parabolic subgroup. Let $\nu : P \to \mathrm{GL}_1$ be a character, and $I(s) = Ind_P^G(|\nu|^s)$ (sometimes meaning a local induction, and sometimes meaning a global induction). We will always write $I_v(s)$ for the local induction at a place v and $I_f(s)$ for the induction at the finite places, We let s_P be the real number for which $\delta_P = |\nu|^{s_P}$, and we assume s_P is positive (as oppposed to negative). Let $f(g,s) \in I(s)$ be a global flat section. The Eisenstein series associated to f is

$$E(g,f,s) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g, s).$$

The sum converges absolutely for $Re(s) > s_P$ and defines an analytic function of s in that range. The Eisenstein series has meromorphic continuation in s.

Suppose $f_\infty(g,s)$ is a K_∞-finite flat section, where $K_\infty \subseteq G(\mathbf{R})$ is a flat section. Fixing f_∞, we can let $f_{fte} \in I_f(s)$ vary, and consider the Eisenstein map $Eis : I_f(s) \to \mathcal{A}(G)$ for $Re(s) > s_P$. Now, given s_0, there is an integer k so that $(s-s_0)^k E(g,f,s)$ is regular at $s = s_0$ for all $f_{fte}(g,s) \in I_f(s)$, and in that case, the Eisenstein map is well-defined and intertwining from $I_f(s_0)$ to $\mathcal{A}(G)$.

Recall that to test if the Eisenstein series $(s-s_0)^k E(g,f,s)$ is regular at s_0, it suffices to see that its constant term along any parabolic Q of G is regular. If $Q = LV$ is standard, the constant term $E_V(g,s)$ is a finite sum of "Eisenstein series" for the Levi L, one for each element of $[W_L\backslash W/W_M]$, where $P = MN$. Precisely, one has the following relation.

Suppose $w \in [W_L\backslash W/W_M]$ and $f(g,s) \in I(s)$, global induction. Then one has the intertwined inducing section $f^w(g,s) := M(w)f(g,s)$, and the new functions

$$E^w(g,f,s) = \sum_{\gamma (wPw^{-1}\cap L)(\mathbf{Q})\backslash L(\mathbf{Q})} f^w(\gamma g, s).$$

Then

$$E_V(g,s) = \sum_{w \in [W_L\backslash W/W_M]} E^w(g,f,s),$$

and everything in sight converges absolutely for $Re(s) >> 0$. Here, $g \in G(\mathbf{A})$, but we abuse notation and call the $E^w(g,f,s)$ Eisenstein series; if g were restricted to the Levi subgroup $L(\mathbf{A})$, then they would be Eisenstein series on L.

The parabolic subgroup $wPw^{-1} \cap L$ of L can be described in terms of simple roots. Namely, suppose $\{\alpha_1, \ldots, \alpha_r\} = I \sqcup J$ are the simple roots for $\Phi(G,T)$ with those elements of I appearing in the unipotent radical V of Q and those elements of J appearing in $\Phi(L,T)$. To describe $wPw^{-1} \cap L$, we would like to know those simple roots β of $\Phi(L,T)$ which appear in the unipotent radical of $wPw^{-1} \cap L$. Equivalently, we would like to find those roots $\beta \in J$ for which $U_{-\beta}$ is not contained in $wPw^{-1} \cap L$. We therefore set

$$\Delta^w(L) = \{\beta \in J = \Phi(L,T) : w^{-1}(\beta) \in \Phi(N,T)\}.$$

The parabolic subgroup $wPw^{-1} \cap L$ of L is thus $P_{L,\Delta^w(L)}$, in the notation of subsection 2.5. We call the set $\Delta^w(L)$ *the associated simple roots of* w. Note that $\Delta^w(L)$ is sometimes empty, in which case $E^w(g,f,s) = f^w(g,s)$ is just a single term.

3.4. **Cuspidal and Eisenstein projection.** Suppose G is a reductive group, either actually semisimple, or we fix some unitary central character. Suppose $\varphi \in \mathcal{A}(G)$ is a (moderate growth) automorphic form. We say that φ has an *Eisenstein projection*, if there is a (moderate growth) automorphic form φ_E satisfying

(1) φ_E is orthogonal to all cusp forms;
(2) $\varphi_C := \varphi - \varphi_E$ is cuspidal.

If φ_E exists, it is unique: Indeed, suppose $\varphi_{E,1}$ and $\varphi_{E,2}$ are Eisenstein projections of φ. Then on the one hand, $\varphi_{E,1} - \varphi_{E,2}$ is orthogonal to all cusp forms, because each $\varphi_{E,j}$ is. On the other hand, $\varphi_{E,1} - \varphi_{E,2} = (\varphi - \varphi_{E,2}) - (\varphi - \varphi_{E,1})$ is cuspidal. Thus, it is 0, so $\varphi_{E,1} = \varphi_{E,2}$. When φ has an Eisenstein projection φ_E, we set $\mathcal{P}_E(\varphi) = \varphi_E$ and $\mathcal{P}_C(\varphi) = \varphi - \varphi_E$ and call them the Eisenstein and cuspidal projections of φ, respectively. One checks immediately that if φ has an Eisenstein projection, and $g \in G(\mathbf{A}_f)$, then $g \cdot \varphi$ has an Eisenstein projection, and $\mathcal{P}_E(g \cdot \varphi) = g \cdot \mathcal{P}_E(\varphi)$, $\mathcal{P}_C(g \cdot \varphi) = g \cdot \mathcal{P}_C(\varphi)$.

3.5. **Irreducibility and unitarizability of principal series.** We will require the following well-known theorem.

Theorem 3.5.1. *Let G be a split semisimple group over a p-adic field k. Let $P \subseteq G$ be a maximal parabolic subgroup. If $s \in \mathbf{R}$ and $s > 1$, then the representation $Ind_P^G(\delta_P^s)$ is irreducible and not unitarizable.*

Proof. The irreducibility can be proved using the criterion in [Jan93, Theorem 3.1.2].

Now, the set of $s \in \mathbf{R}$ for which $Ind_P^G(\delta_P^s)$ has a unitarizable subquotient is compact. This is a theorem of Tadic, see [Mui97, Lemma 5.1]. If for any $s_0 \in \mathbf{R}$ with $s_0 > 1$ one had that $Ind_P^G(\delta_P^{s_0})$ were unitarizable, then by continuity all $Ind_P^G(\delta_P^s)$ with $s > 1$ would be unitarizable, violating the compactness. This proves the theorem. □

3.6. **Moving between isogenous groups.** Throughout the paper, we will prove Siegel-Weil identities and theorems about Eisenstein series on groups that are semisimple and simply connected. We will then utilize these results on isogenous groups. We explain now the principle that lets us go between isogenous groups in the context of these sorts of results.

Suppose G is reductive, and let G_{sc} the simply-connected cover of the derived group of G. There is a canonical map $\pi_{sc} : G_{sc} \to G$. Let P be a parabolic subgroup of G. We begin with the following well-known lemma.

Lemma 3.6.1. *For any field k of characteristic 0, $P(k)\backslash G(k)/G_{sc}(k)$ is a singleton.*

Proof. The unipotent elements of $G(k)$ are in the image of $G_{sc}(k)$. Consequently, representatives of the Weyl group of G are in the image of π_{sc} as well. Combining the last two statements, the lemma follows from the Bruhat decomposition. □

Suppose now G is defined over $\mathbf{Q}$, $K_\infty \subseteq G(\mathbf{R})$ a maximal compact subgroup, and let U be a finite-dimensional representation of K_∞. Suppose we have a family of characters $\chi_s : P(\mathbf{A}) \to \mathbf{C}^\times$ depending on a complex parameter s. Suppose also that $f_\infty(g,s) \in I_{P(\mathbf{R})}^{G(\mathbf{R})}(\chi_s) \otimes U$ is K_∞-equivariant, i.e., $f_\infty(gk,s) = k^{-1} f_\infty(g,s)$ for all $g \in G(\mathbf{R})$ and $k \in K_\infty$. Let $f_{fte}(g,s) \in Ind_{P(\mathbf{A}_f)}^{G(\mathbf{A}_f)}(\chi_s)$ be some inducing section, $f(g,s) = f_{fte}(g,s) f_\infty(g,s)$, and $E(g,f,s)$ the Eisenstein series, defined by an absolutely convergent summation for $Re(s) >> 0$. We assume $f(g,s)$ is flat.

Lemma 3.6.2. *Let Z denote the center of G. Assume that $G(\mathbf{R}) = Z(\mathbf{R}) G_{sc}(\mathbf{R}) K_\infty$. Let the other notation and assumptions be as above. Then the restriction of $E(g,f,s)$ to $G_{sc}(\mathbf{A})$ is regular (or has a pole of order at most k) at some special piont $s = s_0$ for all $f_{fte}(g,s)$ if and only if $E(g,f,s)$ is regular (or has a pole of order at most k) on $G(\mathbf{A})$ for all $f_{fte}(g,s)$.*

Proof. Let k be the highest order of pole for the Eisenstein series on G and k_{sc} the highest order of pole for the Eisenstein series on G_{sc}. By Lemma 3.6.1,$P \cap G_{sc} =: P_{sc}$ is a parabolic subgroup of G_{sc}, and the restriction map from the induced representation on G to the induced representation on G_{sc} is surjective. Consequently, $k_{sc} \leq k$. For the reverse inequality, we have the $(s-s_0)^k E(g,f,s)$

gives an intertwining map from $I_{fte}(\chi_s)$ to $\mathcal{A}(G)$. There exists $f_{fte}(g,s)$ so that the associated Eisenstein series value is non-zero. By the fact that the map is intertwining, we can assume that $(s-s_0)^k E(g_\infty, f, s)$ is nonzero at $s = s_0$. But now because $G(\mathbf{R}) = Z(\mathbf{R})G_{sc}(\mathbf{R})K_\infty$, this Eisenstein series is nonzero on $G_{sc}(\mathbf{R})$. Hence $k_{sc} \geq k$. □

We now explain how we go between Siegel-Weil formulas proved for simply-connected groups, and Siegel-Weil formulas for more general groups. So, suppose we have a commuting pair $G \times S \to G'$. Let $G'_1 = G'_{sc}$ and $G_1 = G_{sc}$. Then we have maps of Lie algebras

$$Lie(G_1) \to Lie(G) \to Lie(G') \to Lie(G'_1)$$

so there is a map $G_1 \to G'_1$. Suppose H is simply connected, so the map $H \to G'$ factors through G'_1. Now $Lie(G)$ maps into $Lie(G')^S$, and thus $Lie(G'_1)^S$, so $Lie(G_1)$ maps into $Lie(G'_1)^S \subseteq Lie(G'_1)$. Consequently we have a map $G_1 \times H \to G'_1$.

Now suppose we have proved a Siegel-Weil theorem for the pair $G_1 \times S \to G'_1$, and we have a hypothetical Siegel-Weil theorem for the pair $G \times S \subseteq G'$. That is:

1. we have an induced representation on $G'(\mathbf{A}_f)$, $I_{G',fte}(s = s_0)$ and similarly an induced representation on $G'_1(\mathbf{A}_f)$;
2. we have two linear maps m_{theta}, m_{Eis}, $m_* : I_{G',fte}(s = s_0) \to (\mathcal{A}(G) \otimes U)^{K_{G,\infty}}$;
3. we have two linear maps m^1_{theta}, m^1_{Eis}, $m_* : I_{G'_1,fte}(s = s_0) \to (\mathcal{A}(G_1) \otimes U)^{K_{G_1,\infty}}$;
4. the maps are compatible with restrictions, i.e. $m^1_* Res^{G'_1}_{G'}(f_{fte}) = Res^{G_1}_{G} m_*(f_{fte})$;
5. the maps $m^1_{Theta} = m^1_{Eis}$ agree on $I_{G'_1,fte}(s = s_0)$;
6. the maps m_* are $G(\mathbf{A}_f)$ intertwining, and the maps m^1_* are $G_1(\mathbf{A}_f)$ intertwining.

Proposition 3.6.3. *Suppose m_{theta} and m_{Eis} both land in $\mathcal{A}(G)_\xi \otimes U$ for the same central character ξ. Suppose also that $G(\mathbf{R}) = Z_G(\mathbf{R})G_1(\mathbf{R})K_{\infty,G}$. Then in the above situation, we have a Siegel-Weil theorem for the pair $G \times S \subseteq G'$. I.e., $m_{Theta} = m_{Eis}$ on $I_{G',fte}(s = s_0)$.*

Proof. Set $d = m_{Theta} - m_{Eis}$. Then for all $f_{fte} \in I_{G',fte}(s = s_0)$, the hypotheses imply $d(f_{fte})|_{G_1} = 0$. Because the maps m_* are $G(\mathbf{A}_f)$ intertwining, we need only verify that $d(f_{fte})|_{G(\mathbf{R})} \equiv 0$ for all f_{fte}. Now the proposition follows from the assumption that $G(\mathbf{R}) = Z_G(\mathbf{R})G_1(\mathbf{R})K_{\infty,G}$ and the fact that $d(f_{fte})|_{G_1} = 0$. □

4. Automorphic minimal representations: type D

In this section, we consider degenerate Eisenstein series on groups of type D_5, D_6, D_7 that can be used to define automorphic minimal representations. The minimal representations appear as residues of the Eisenstein series at certain special points. Beyond reviewing these constructions, we also calculate the constant terms along maximal parabolic subgroups of the functions in the automorphic minimal representation.

4.1. **Degenerate principal series.** We review a few facts concerning degenerate principal series on p-adic groups of type D. Thus let $k = \mathbf{Q}_p$ and let G_n be split, simply-connected of type D_n. We set $V_{2n} = H^n$ over k. Thus $G_n \to \mathrm{SO}(V_{2n})$. Let b_j, b_{-j} be the standard basis of the j^{th} copy of the hyperbolic plane H in V_{2n}. Let $P \subseteq G$ be the parabolic stabilizing the line kb_1, and $\nu : P \to \mathrm{GL}_1$ as $pb_1 = \nu(p)b_1$. One has $\delta_P = |\nu|^{2n-2}$. We set $I^G_P(s) = Ind^G_P(|\nu|^s)$; the long intertwiner $M(w_0)$ relates $I(s)$ with $I(2n-2-s)$.

Proposition 4.1.1 (Weissman [Wei03]). *Let the notation be as above.*

1. *The representations $I^G_P(n)$ and $I^G_P(n-2)$ have a non-split composition series of length two.*
2. *The spherical representation is a subrepresentation of $I^G_P(n-2)$ and a quotient of $I^G_P(n)$.*
3. *The spherical vector generates $I^G_P(n)$.*

(4) *The meromorphically continued intertwining operator* $M(w_0) : I_P^G(s) \to I_P^G(2n-2-s)$ *defines a regular intertwining map at* $s = n$. *The image of the intertwining operator is the unique irreducible subrepresentation of* $I_P^G(s = n-2)$*; it is spherical.*

Proof. This is essentially all contained in [Wei03]. The structure of a composition series of $I_P^G(s)$ is computed using the Fourier-Jacobi functor, which reduces the calculation to the case of principal series on SL_2.

For the second part, one computes that the c-function for $M(w_0)$ is

$$c(s) = \frac{\zeta(s+1-n)\zeta(s+3-2n)}{\zeta(s)\zeta(s+2-n)}.$$

This vanishes at $s = n-2$, so the spherical representation is a subrepresentation at $s = n-2$ and a quotient at $s = n$. The third part follows from the second part. Because the spherical vector generates $I(n)$, that $M(w_0)$ is regular follows from the fact that $c(n)$ is regular. □

4.2. **Type D_5.** Recall the group $G_5 = G_{5,\Theta}$, which is simply-connected of type D_5. This group acts on $V_{10} = H \oplus \Theta$. Let b_1, b_{-1} be the standard basis of H, and P_{G_5} the parabolic subgroup stabilizing the line $\mathbf{Q}b_1$. We define ν and $I(s)$ as in the previous subsection.

Suppose $f = \otimes f_v \in Ind(|\nu|^s)$ is a flat section for every v, and f_∞ is spherical. We have the Eisenstein series $E(g,f,s) = \sum_{\gamma \in P_{G_5}(\mathbf{Q})\backslash G_5(\mathbf{Q})} f(\gamma g, s)$.

Proposition 4.2.1. *The Eisenstein series* $E(g,f,s)$ *is regular at* $s = 5$.

Proof. Observe that G_5 is split at every finite place. We first verify the proposition for the spherical Eisenstein series, then deduce the general case from the spherical case.

Let $N_{P_{G_5}}$ denote the unipotent radical of P_{G_5}. The constant term of $E(g,f,s)$ along $N_{P_{G_5}}$ is (see subsection 3.3)

$$E_{N_{P_{G_5}}}(g,f,s) = f(g,s) + M(w_0)f(g,s).$$

The c-function is $\frac{\zeta(s-4)\zeta(s-7)}{\zeta(s)\zeta(s-3)}$. The archimedean interwiner is $\frac{\Gamma(s-4)}{\Gamma(s)}$ [Pol22, Lemma 4.1.1]. The global zeta function $\zeta(s-4)$ has a simple pole at $s = 5$, but $\zeta(s-7)$ has a simple zero at $s = 5$. The archimedean intertwiner is regular at $s = 5$, so we conclude that the lemma holds in case f is spherical everywhere.

The general case follows because the spherical vector generates the module $I_p(s)$ for $s = 5$ and every $p < \infty$; see Proposition 4.1.1. □

4.3. **Type D_6.** Recall that we set $V_{12} = H^2 \oplus \Theta$, and $G_6 = G_{6,\Theta} \to \mathrm{SO}(V_{12})$ the simply-connected cover. In this subsection, we discuss the automorphic minimal representation on G_6, and compute constant terms of the automorphic forms in this representation for the standard maximal parabolic subgroups of G_6.

To begin, let b_j, b_{-j} be the standard basis of the j^{th} copy of the hyperbolic plane H. We set $P_{G_6,1}$ the maximal parabolic that stabilizes the line $\mathbf{Q}b_1$, with associated character ν. We set $P_{G_6,2}$ the maximal parabolic that stabilizes the plane $\mathbf{Q}b_1 \oplus \mathbf{Q}b_2$. As usual, we set $I(s) = Ind_{P_{G_6,1}}^{G_6}(|\nu|^s)$.

Set $v_j = \frac{1}{\sqrt{2}}(b_j + b_{-j})$, so that $(v_i, v_j) = \delta_{ij}$. We let $\iota \in O(V_{12})$ be the map that exchanges b_j with b_{-j}, and is minus the identity on Θ. We define a maximal compact subgroup of $\mathrm{SO}(V_{12})(\mathbf{R})$ as those group elements that commute with ι; similarly one has a maximal compact subgroup $K_{6,\infty}$ of $G_6(\mathbf{R})$. Observe that the action of $K_{6,\infty}$ on $v_1 + iv_2 \in V_{12} \otimes \mathbf{C}$ defines a representation $j(\bullet, i) : K_{6,\infty} \to \mathbf{C}^\times$.

For an even integer ℓ, let $f_{\infty,\ell}(g,s) \in I_\infty(s)$ be the flat section with $f_{\infty,\ell}(k_\infty, s) = j(k_\infty, i)^\ell$. Let $f_{fte}(g,s) \in I_f(s)$ be a flat section, let $f_\ell(g,s) = f_{fte}(g,s)f_{\infty,\ell}(g,s)$ and $E(g,f_\ell,s)$ the associated Eisenstein series.

Proposition 4.3.1. *Let the notation be as above.*

(1) [HS20, Theorem 6.4] *The Eisenstein series $E(g, f_\ell, s)$ has at most a simple pole at $s = 6$.*
(2) *This simple pole is achieved if $\ell = \pm 4$ and $f_{fte}(g, s)$ is spherical.*
(3) *Fixing $\ell = 4$ or $\ell = -4$, the residual representation is irreducible.*
(4) *If $\ell \in \{-2, 0, 2\}$, then $E(g, f_\ell, s)$ is regular at $s = 6$.*

Proof. As written, part 1 is from [HS20]. Part 2 follos form the computation of the constant term of the Eisenstein series in the proof of Theorem 7.0.1 of [Pol22]. Part 3 again follows from [HS20, Theorem 6.4].

It remains to prove the fourth part. Because the spherical vector generates $I_f(s = 6)$, it suffices to check the regularity on the spherical vector. Its constant term down to the minimal parabolic P_0 is a sum of four terms (see Proposition 6.1.1 of [Pol22] and the proof of Theorem 7.0.1 of that paper.). These four terms give c-functions (see [Pol22] for notation)

(1) $c(1) = 1$
(2) $c(w_{12}) = \frac{\zeta(s-1)}{\zeta(s)} \frac{\Gamma_{\mathbf{C}}(s-1)}{\Gamma_{\mathbf{R}}(s-j)\Gamma_{\mathbf{R}}(s+j)}$
(3) $c(w_2 w_{12}) = c(w_{12}) \frac{\zeta(s-5)\zeta(s-8)}{\zeta(s-1)\zeta(s-4)} \frac{\Gamma(s-5)}{\Gamma(s-1)}$
(4) $c(w_{12} w_2 w_{12}) = c(w_2 w_{12}) \frac{\zeta(s-9)}{\zeta(s-8)} \frac{\Gamma_{\mathbf{C}}(s-9)}{\Gamma_{\mathbf{R}}(s-8+v)\Gamma_{\mathbf{R}}(s-8-v)}$

All these terms are regular at $s = 6$. Thus the spherical Eisenstein series is regular at $s = 6$. This proves the proposition. □

Given $f_{fte} \in I_f(s = 6)$, we let $\Theta_f^+ = Res_{s=6} E(g, f_4, s)$ and $\Theta_f^- = Res_{s=6} E(g, f_{-4}, s)$, where we have extended f_{fte} to a flat section to define the Eisenstein series map. We set $V_{min,p}$ the unique irreducible quotient of $I_p(s = 6)$, or equivalently the unique irreducible subrepresentation of $I_p(s = 4)$. We set $V_{min} = \otimes_p' V_{min,p}$. Then the residue of the Eisenstein map gives an intertwining operator $V_{min} \to \mathcal{A}(G_6)$ for either the $+$ or $-$ cases.

Recall the standard maximal parabolic subgroups $P_{G_6,j} = M_{G_6,j} N_{G_6,j}$ of G_6, for $j = 1, 2$. We will now compute the constant terms $\Theta^+ f(g)_{N_{G_6,j}}$; the computation for Θ_f^- is identical.

The structure of the computation is explained in section 3.3. What one has to do is to compute which intertwining operators $M(w)$ and which Eisenstein series $E^w(g, f_4, s)$ are absolutely convergent and thus do not contribute to the residue at $s = 6$. And for those intertwining operators and Eisenstein series which can potentially contribute to the residue at $s = 6$, one must make a finer computation to determine if they actually do contribute.

Let $f^1(g, s) = M(w_0) f_4(g, s)$, the result of applying the global long intertwining operator $M(w_0)$ to $f_4(g, s)$. Then $f^1(g, s) \in I(10 - s)$, and has a simple pole at $s = 6$. Indeed, one can check that there is a simple pole for the finite spherical vector, and then there is at most a simple pole for the entire induced representation $I_f(s)$ because $I_f(s = 6)$ is generated by the spherical vector. We let $\overline{f}(g) = Res_{s=6} f^1(g, s)$.

Proposition 4.3.2. *Let $f(g, s) = f_{fte}(g, s) f_{\infty,4}(g, s) \in I(s)$ be a flat section and let the notation be as above.*

(1) *One has $\Theta_f(g)_{N_{G_6,2}} = E_{\mathrm{GL}_2}(g, \overline{f})$, an absolutely convergent Eisenstein series of* GL_2*-type, defined as a sum over $P_{G_6,0} \backslash P_{G_6,2}$.*
(2) *One has $\Theta_f(g)_{N_{G_6,1}} = \overline{f}(g)$.*

Proof. To give notation for the computation, suppose $t \in T$ the split standard torus, and $t b_j = t_j b_j$ for $j = 1, 2$. We set $r_j(t) = |t_j|$. Writing unramified characters of T in additive notation, we have $|\nu|^s$ is $s r_1$ and $\lambda_s = \delta_{P_0}^{-1/2} |\nu|^s$ is $(s - 5) r_1 + (-4) r_2$.

For the first part of the proposition, observe that the set $[W_{M_{G_6,2}} \backslash W_{G_6} / W_{M_{G_6,1}}]$ has size 2, with elements $1, w$ where $w(r_1) = -r_2$ and $w(r_2) = r_1$. The Eisenstein series corresponding to $w = 1$ is absolutely convergent at the special point $s = 6$, so this term does not contribute to the residue.

The other term can be understood in terms of $f^1(g,s) = M(w_0)f_4(g,s)$ using Langlands functional equation for Eisenstein series.

For the second part of the proposition, one has that the set $[W_{M_{G_6,1}}\backslash W_{G_6}/W_{M_{G_6,1}}]$ has size three. The elements are $\{1, w_0, w_1\}$ where $w_0(r_1) = -r_1$, $w_0(r_2) = r_2$ and $w_1(r_1) = r_2$, $w_1(r_2) = r_1$. The intertwining operator $M(w_1)$ is checked to be absolutely convergent at $s = 6$. One finds that the Eisenstein series $E^{w_1}(g, f^{w_1}, s)$ is exactly the one (up to isogenous groups, see Lemma 3.6.2) studied in Proposition 4.2.1. The result follows. □

4.4. **Type D_7.** In this subsection, we prove results about the automorphic minimal representation on the group $G_7 = G_{7,\Theta}$ of type D_7. Recall that $V_{14} = H^3 \oplus \Theta$. We write b_j, b_{-j} for the standard basis of the j^{th} copy of the hyperbolic plane H.

Define $\iota \in O(V_{14})$ exactly as in the case of $O(V_{12})$, so that ι exchanges b_j with b_{-j} and is minus the identity of Θ. We write $K_{G_7,\infty} \subseteq G_7(\mathbf{R})$ for the associated maximal compact subgroup. We let $v_j = \frac{1}{\sqrt{2}}(b_j + b_{-j})$ so that $(v_i, v_j) = \delta_{ij}$. The action on $V_3 := \mathrm{Span}(v_1, v_2, v_3)$ defines a map $K_{G_7,\infty} \to O(3)$.

Suppose $\ell \geq 1$ is an integer. The ℓ^{th} symmetric power $S^\ell(V_3)$ has an irreducible highest weight quotient of dimension $2\ell + 1$; this is the usual theory of spherical harmonics. Let $\mathbf{V}_\ell$ be this $2\ell+1$-dimensional space, which is an $O(3)$ and thus $K_{G_7,\infty}$ representation.

Let $f_{\infty,\ell}(g,s) \in I(s) \otimes \mathbf{V}_\ell$ be the flat section, for which $f_{\infty,\ell}(gk,s) = k^{-1}f_{\infty,\ell}(g,s)$ and $f_{\infty,\ell}(1,s)$ is the image of v_1^ℓ in $\mathbf{V}_\ell$. Suppose $f_{fte}(g,s) \in I_f(g,s)$ is a flat section for this induced representation at the finite places. Let $f_\ell(g,s) = f_{fte}(g,s)f_{\infty,\ell}(g,s)$ and let $E(g,f,s)$ be the associated Eisenstein series. **We now set $\ell = 4$.**

Proposition 4.4.1. *Let the notation be as above.*

1. *The Eisenstein series $E(g,f,s)$ has at most a simple pole at $s = 7$, which is achieved by the finite-place spherical vector.*
2. *The residue defines a $G_7(\mathbf{A}_f)$ intertwining map $I_f(s = 7) \to \mathcal{A}(G_7)$. The image is nonzero and irreducible.*
3. *The global long intertwining operator $M(w_0)$ has at worst a simple pole at $s = 7$.*

Proof. That the poles of the Eisenstein series are at most simple follows from [HS20, Proposition 6.3]. As usual, by the structure of the degenerate principal series $I(s = 7)$ reviewed in Proposition 4.1.1, to prove that $M(w_0)$ has at worst a simple pole, it suffices to analyze the case when the finite part of our inducing section is spherical.

In this case, the intertwining operator $M(w_0)$ is computed in [Pol22, Proposition 4.1.2] (archimedean part) and [Pol22, Proposition 6.2.1] (finite part). One obtains

$$c(s) = \frac{\zeta(s-6)\zeta(s-11)}{\zeta(s)\zeta(s-5)} \frac{((s-5)/2)_2\Gamma(s-6)((s-14)/2)_2}{((s-2)/2)_3\Gamma(s-2)((s-11)/2)_3}.$$

Here $(z)_n = z(z+1)\cdots(z+n-1)$ is the Pochammer symbol. One sees that $c(s)$ has a simple pole at $s = 7$. This proves the third part of the proposition.

Going back to the first part, because $E(g, f_s) = E(g, M(w_0)f_s)$, and the spherical Eisenstein series is regular at $s = 5$ [Pol22, Theorem 7.0.1], the spherical Eisenstein series $E(g,f,s)$ has an honest pole at $s = 7$.

Part 2 of the proposition follows as in previous cases from [HS20]. □

We write $\Theta_f(g) = Res_{s=7}E(g,f,s)$. This is an element of $\mathcal{A}(G_{7,\Theta}) \otimes \mathbf{V}_4$ that is $K_{G_7,\infty}$-equivariant.

For $j = 1,2,3$, let $P_{G_7,j} = M_{G_7,j}N_{G_7,j}$ be the parabolic subgroup of G_7 that stabilizes the isotropic subspace spanned by b_1 through b_j. We compute the constant terms of Θ_f along the $N_{G_7,j}$ for $j = 1,2,3$.

Similar to our analysis of the minimal representation on G_6, we set $f^1(g,s) = M(w_0)f_4(g,s)$ and $\overline{f}(g) = Res_{s=7} f^1(g,s)$. Set

$$E_{M_{G_7,2},1,2}(g,\overline{f}) = \sum_{\gamma \in (P_{G_7,1} \cap P_{G_7,2})(\mathbf{Q}) \backslash P_{G_7,2}(\mathbf{Q})} \overline{f}(\gamma g)$$

and

$$E_{M_{G_7,3},1,3}(g,\overline{f}) = \sum_{\gamma \in (P_{G_7,1} \cap P_{G_7,3})(\mathbf{Q}) \backslash P_{G_7,3}(\mathbf{Q})} \overline{f}(\gamma g).$$

These sums are absolutely convergent.

Proposition 4.4.2. *Let the notation be as above.*

(1) $\Theta_f(g)_{N_{P_{G_7,2}}} = E_{M_{G_7,2},1,2}(g,\overline{f})$.
(2) $\Theta_f(g)_{N_{P_{G_7,3}}} = E_{M_{G_7,3},1,3}(g,\overline{f})$.

Proof. One first computes the sets $[W_{M_{G_7,j}} \backslash W_{G_7} / W_{M_{G_7,1}}]$ for $j = 2,3$. For $j = 3$, the set has size two, with elements $1, w_3$ where $w_3(r_1) = -r_3$, $w_3(r_2) = r_1$, and $w_3(r_3) = r_2$. For $j = 2$, the set has size three, with elements $1, w_1, w_2$ where

(1) $w_1(r_1) = r_3$, $w_1(r_2) = r_1$, $w_1(r_3) = r_2$;
(2) $w_2(r_1) = -r_2$, $w_2(r_2) = r_1$ and $w_2(r_3) = r_3$.

Now, for $j = 2$ or 3 and $w = 1$, one checks that the Eisenstein series $E^{w=1}(g,f,s)$ are regular at $s = 7$ by absolute convergence. Thus, these terms do not contribute to the residue at $s = 7$. One deduces the second part of the proposition from Langlands functional equation for Eisenstein series.

For the first part of the proposition, we analyze the Eisenstein series coming from $w = w_1$. In this case, we are reduced to the Eisenstein series studied in Proposition 4.2.1. Thus this term does not contribute to the residue. The first part of the proposition now follows from Langlands functional equation. □

We now consider the constant term of Θ_f along $N_{P_{G_7,1}}$. Let w_{12} be the Weyl group element that exchanges r_1 with r_2 (leaving r_3 fixed). Set $f^{w_{12}}(g,s) = M(w_{12})f(g,s)$ and $E^{w_{12}}(g,f,s)$ the associated Eisenstein series (see subsection 3.3).

Proposition 4.4.3. *One has*

$$\Theta_f(g)_{N_{G_7,1}} = \overline{f}(g) + res_{s=7} E^{w_{12}}(g,f,s).$$

This latter Eisenstein series residue can be identified with theta functions on $G_{6,\Theta}$.

Proof. The set $[W_{M_{G_7,1}} \backslash W_{G_7} / W_{M_{G_7,1}}]$ has size three, with elements $1, w_{12}$ and w_0. The part of the constant term fcorresponding to $w = 1$ is just $f(g,s)$, which of course is regular at $s = 7$. The $w = w_{12}$ and $w = w_0$ terms do contribute nontrivially to the residue at $s = 7$. That $res_{s=7}E^{w_{12}}(g,f,s)$ can be identified with theta functions on $G_{6,\Theta}$ follows from Proposition 4.3.1. Specifically, a priori, the residue at $s = 7$ might involve terms that do not arise as theta functions, but these terms disappear by Proposition 4.3.1 part 3. □

5. Automorphic minimal representations: type E

In this section, we discuss automorphic minimal representations on group of type E.

5.1. **Type E_6.** Recall the group M_J^1 defined in section 2; it is semisimple, simply connected of type E_6. Moreover, it is split at every finite place. It has an A_2 rational root system. Specifically, there is an action of SL_3 on J, defined by the formula $g \cdot X = gXg^t$. Taking the diagonal torus of SL_3 gives rise to this root system of M_J^1. We write $\{r_i - r_j\}_{i \neq j}$ for the roots, where $i, j \in \{1, 2, 3\}$.

Let e_{11} be the element of J with c_1 coordinate equal to 1 and all other coordinates equal to 0; see equation (1). Let $Q_{M_J^1}$ be the parabolic subgroup of M_J^1 that fixes the line $\mathbf{Q}e_{11}$. Define $\mu : Q \to \mathrm{GL}_1$ as $qe_{11} = \mu(q)e_{11}$. Let $I(s) = Ind_{Q_{M_J^1}}^{M_J^1}(|\mu|^{s/2})$. We will consider Eisenstein series associated to this induced representation.

If T is the diagonal torus of SL_3, as mentioned, we have a map $T \to M_J^1$. If $t = (t_1, t_2, t_3)$, this map satisfies $\mu(t) = t_1^2$. One has $\delta_P(t) = 8(2r_1 - r_2 - r_3)$, where $r_j(t) = |t_j|$, and $\rho_{P_0} = 8(r_1 - r_3)$. We have $\lambda_s = |\mu|^{s/2}\delta_{P_0}^{-1/2} = (s-8)r_1 + 8r_3$. (Remember that $r_1 + r_2 + r_3 = 0$ on T.)

We assume $f(g, s)$ is a flat section in $I(s)$, spherical at the archimedean place. We let $E(g, f, s)$ be the associated Eisenstein series. We are interested in this Eisenstein series at $s = 18$. According to [Wei03], the spherical vector generates the $I(s = 18)$ at every finite place.

We now have the following proposition.

Proposition 5.1.1. *The Eisenstein series $E(g, f, s)$ is regular at $s = 18$.*

Proof. Write $Q_{M_J^1} = L_{M_J^1} V_{M_J^1}$. We compute the constant term down to Levi $L_{M_J^1}$ of $Q_{M_J^1}$. The set $[W_{L_{M_J^1}} \backslash W / W_{L_{M_J^1}}]$ consist of $w = 1$ and $w = w_{r_1 - r_2}$, the simple reflection corresponding to this root. We have $w_{r_1-r_2}(\lambda_s) + \rho_{P_0} = 8r_1 + (s-8)r_2$. The global intertwining operator $M(w_{r_1-r_2})$ is absolutely convergent at $s = 18$, using that $\langle \lambda_s, r_1 - r_2 \rangle = s - 8$. But now, the Eisenstein series on $L_{M_J^1}$ associated to $M(w_{r_1-r_2})f(g, s)$ at $s = 18$ was proved to be regular in Proposition 4.2.1. This completes the proof. □

5.2. **Type E_7.** Recall from section 2 the group H_J^1; it is simply connected of type E_7, and split at every finite place. In this subsection, we define and compute with the automorphic minimal representation on H_J^1.

There is a map $\mathrm{Sp}_6 \hookrightarrow H_J^1$, defined by realizing $W_J \subseteq \wedge^3 W_6 \otimes \Theta$, where W_6 is the standard representation of Sp_6. Let T be the diagonal torus of Sp_6. Then T gives H_J^1 a rational root system of type C_3. We write $r_i \pm r_j$, $i, j \in \{1, 2, 3\}$ for these roots.

We let $P_{H_J^1}$ denote the Siegel parabolic subgroup of H_J^1, defined as the as the stabilizer of the line $\mathbf{Q}(0, 0, 0, 1)$. In terms of the C_3 root system, the Siegel parabolic subgroup corresponds to the simple root $2r_3$. We define $\lambda : P \to \mathrm{GL}_1$ via $p(0, 0, 0, 1) = \lambda(p)(0, 0, 0, 1)$. Define $j(g, Z) \in \mathbf{C}^\times$ via the action on $r_0(Z) := (1, -Z, Z^\#, -N(Z))$, i.e., $gr_0(Z) = j(g, Z)r_0(gZ)$. (See [Pol20a, Proposition 2.3.1].) We define sections for $I(s) := Ind_{P_{H_J^1}}^{H_J^1}(|\lambda|^s)$. Specifically, for an even integer ℓ, let $f_{\infty,\ell}(g, s)$ be the associated flat section in $I_\infty(s)$ with $f(k, s) = j(k, i)^\ell$ for all $k \in K_{H_J^1,\infty}$. Here the maximal compact subgroup $K_{H_J^1,\infty}$ is defined as $k \in H_J^1(\mathbf{R})$ with $r_0(k \cdot i) = r_0(i)$.

Consider a flat section $f_{fte}(g, s) \in I_f(s)$. Let $f_\ell(g, s) = f_{fte}(g, s)f_{\infty,\ell}(g, s)$. We define an Eisenstein series $E(g, f, s) = \sum_{\gamma \in P_{H_J^1}(\mathbf{Q}) \backslash H_J^1(\mathbf{Q})} f(\gamma g, s)$. The modulus character of $P_{H_J^1}$ is $\delta_{P_{H_J^1}}(p) = |\lambda(p)|^{18}$, so the Eisenstein series converges absolutely for $Re(s) > 18$.

We will be interested in the residue at $s = 14$ when $\ell = \pm 4$. We fix now $\ell = 4$; the results for $\ell = -4$ are identical and proved identically.

We recall from [HS20, Theorem 3.3], see also [Wei03], that the p-adic representations $I_p(s = 14), I_p(s = 4)$ have a nonsplit composition series of length two. The spherical representation is the unique irreducible subrepresentation of $I_p(s = 4)$, while it is the unique irreducible quotient of $I_p(s = 14)$. Finally, as usual, the intertwining operator locally gives a defined surjection from $I_p(s = 14)$ to the proper spherical subrepresentation in $I_p(s = 4)$.

Proposition 5.2.1. *Let the notation be as above.*

1. *The Eisenstein series has at most a simple pole at $s = 14$. This pole is achieved for the inducing section that is spherical at every finite place.*
2. *The residue representation is nonzero and irreducible, and thus defines an intertwining map $I_f(s = 14) \to \mathcal{A}(H_J^1)$.*

Proof. That the Eisenstein series has at most a simple pole at $s = 14$ is proved in [HS20, Proposition 6.3]. That this pole is achieved by the vector that is spherical at every finite place is due to Kim [Kim93]. □

We define $\Theta_f^+(g) = Res_{s=14}E(g, f_4, s)$, and $\Theta_f^- = Res_{s=14}E(g, f_{-4}, s)$. We compute the constant terms of Θ_f^+ along the unipotent radicals of the standard maximal parabolic subgroups. Our simple roots are $\alpha_1 = r_1 - r_2, \alpha_2 = r_2 - r_3$ and $\alpha_3 = 2r_3$. Thus, following the naming convention of subsection 2.5, the standard maximal parabolic subgroups of H_J^1 are $P_{H_J^1,j} = M_{H_J^1,j}N_{H_J^1,j}$ for $j = 1, 2, 3$. The Siegel parabolic occurs for $j = 3$.

We begin by computing the constant term down to the Siegel Levi. As usual, let $M(w_0)$ be the long intertwining operator. Set $f^1(g, s) = M(w_0)f_4(g, s)$. We will see below that $f^1(g, s)$ has at most a simple pole at $s = 14$. Let $\overline{f}(g) = Res_{s=14}f^1(g, s)$.

Proposition 5.2.2. *One has $\Theta_f(g)_{N_{H_J^1,3}} = \overline{f}(g)$.*

Proof. The set $[W_{M_{H_J^1,3}}\backslash W_{H_J^1}/W_{M_{H_J^1,3}}]$ has size four. The four elements are 1, w_{2r_3}, $w_{r_2+r_3} = w_{2r_3}w_{r_2-r_3}w_{2r_3}$, $w_0 = w_{2r_3}w_{r_2-r_3}w_{r_1-r_2}w_{r_2+r_3}$, of lengths 1, 2, 3, and 6.

We now compute how these Weyl elements move around our inducing character. We begin by observing $|\lambda| = r_1 + r_2 + r_3$ and $\delta_{P_0}^{1/2} = 17r_1 + 9r_2 + r_3$.

1. We set $\lambda_s = |\lambda|^s\delta_{P_0}^{-1/2} = (s - 17)r_1 + (s - 9)r_2 + (s - 1)r_3$.
2. applying w_{2r_3}, get $w_{2r_3}(\lambda_s) = (s - 17)r_1 + (s - 9)r_2 + (1 - s)r_3$, with $\langle\lambda_s, r_3\rangle = s - 1$ and $w_{2r_3}(\lambda_s) + \rho_{P_0} = sr_1 + sr_2 + (2 - s)r_3$.
3. applying $w_{r_2-r_3}$, get $w_{r_2-r_3}w_{2r_3}(\lambda_s) = (s-17)r_1 + (1-s)r_2 + (s-9)r_3$, with $\langle w_{2r_3}(\lambda_s), r_2 - r_3\rangle = 2s - 10$.
4. applying w_{2r_3}, get $w_{2r_3}w_{r_2-r_3}w_{2r_3}(\lambda_s) = (s-17)r_1+(1-s)r_2+(9-s)r_3$, with $\langle w_{r_2-r_3}w_{2r_3}(\lambda_s), r_3\rangle = s - 9$ and $w_{r_2+r_3}(\lambda_s) + \rho_{P_0} = sr_1 + (10 - s)r_2 + (10 - s)r_3$. At $s = 14$, this is $6(2r_1 - r_2 - r_3) + 2(r_1 + r_2 + r_3)$.
5. applying $w_{r_1-r_2}$, get $w_{r_1-r_2}w_{r_2+r_3}(\lambda_s) = (1-s)r_1+(s-17)r_2+(9-s)r_3$, with $\langle w_{r_2+r_3}(\lambda_s), r_1 - r_2\rangle = 2s - 18$.
6. applying $w_{r_2-r_3}$, get $w_{r_2-r_3}w_{r_1-r_2}w_{r_2+r_3}(\lambda_s) = (1 - s)r_1 + (9 - s)r_2 + (s - 17)r_3$, with $\langle w_{r_1-r_2}w_{r_2+r_3}(\lambda_s), r_2 - r_3\rangle = 2s - 26$.
7. applying w_{2r_3}, get $w_0(\lambda_s) = (1-s)r_1+(9-s)r_2+(17-s)r_3$, with $\langle w_{r_2-r_3}w_{r_1-r_2}w_{r_2+r_3}(\lambda_s), r_3\rangle = s - 17$ and $w_0(\lambda_s) + \rho_{P_0} = (18 - s)(r_1 + r_2 + r_3)$.

At the finite places, we obtain the following c-functions. Set $\zeta_\Theta(s) = \zeta(s)\zeta(s - 3)$.

1. $c(1) = 1$
2. $c(w_{2r_3}) = \frac{\zeta(s-1)}{\zeta(s)}$
3. $c(w_{r_2+r_3}) = c(w_{2r_3})\frac{\zeta_\Theta(s-5)}{\zeta_\Theta(s-1)}\frac{\zeta(s-9)}{\zeta(s-8)} = \frac{\zeta(s-5)\zeta(s-9)}{\zeta(s)\zeta(s-4)}$
4. $c(w_0) = c(w_{r_2+r_3})\frac{\zeta_\Theta(s-9)}{\zeta_\Theta(s-5)}\frac{\zeta_\Theta(s-13)}{\zeta_\Theta(s-9)}\frac{\zeta(s-17)}{\zeta(s-16)} = \frac{\zeta(s-9)\zeta(s-13)\zeta(s-17)}{\zeta(s)\zeta(s-4)\zeta(s-8)}$

Observe that the global intertwining operators $M(w_{2r_3})$ and $M(w_{r_2+r_3})$ are absolutely convergent at $s = 14$. Moreover, $w_{2r_3}(\lambda_s) + \rho_{P_0} = sr_1 + sr_2 + (2 - s)r_3$ at $s = 14$ becomes $(8\frac{2}{3})(r_1 + r_2 - 2r_3) + \frac{16}{3}(r_1 + r_2 + r_3)$. Because $8\frac{2}{3} > 8$, the associated Eisenstein series is absolutely convergent at $s = 14$. Thus the $E^w(g, f, s)$ for $w = 1$ and $w = w_{2r_3}$ do not contribute to the residue at $s = 14$. Additionally, the $E^w(g, f, s)$ for $w = w_{r_2+r_3}$ is regular at $s = 14$, by Proposition 5.1.1.

The only term that can contribute is thus $M(w_0)f(g,s)$. By explicitly computing the intertwining operator at the archimedean place, we see that $M(w_0)f(g,s)$ has at most a simple pole at $s=14$. Indeed, because $I_p(s=14)$ is generated by the spherical vector for every $p<\infty$, it suffices to check the simplicity of the pole when the inducing section is spherical at every finite place. □

Remark 5.2.3. One can use identical computations to those in the proof of Proposition 5.2.2 to prove that if $\ell \in \{-2,0,2\}$ then the Eisenstein series is regular at $s=14$.

We now compute the constant term $\Theta_f(g)$ down to parabolic with Levi of type $D_{5,1}\times \mathrm{SL}_2$. This is the parabolic $P_{H_J^1,2}=M_{H_J^1,2}N_{H_J^1,2}$. The simple roots in its Levi are $r_1-r_2, 2r_3$. Set

$$E_{M_{H_J^1,2}}(g,\overline{f}) = \sum_{\gamma\in(P_{H_J^1,3}\cap P_{H_J^1,2})(\mathbf{Q})\backslash P_{H_J^1,2}(\mathbf{Q})} \overline{f}(\gamma g).$$

The sum defining this Eisenstein series is absolutely convergent.

Proposition 5.2.4. *One has* $\Theta_f(g)_{N_{H_J^1,2}} = E_{M_{H_J^1,2}}(g,\overline{f})$.

Proof. The set $[W_{M_{H_J^1,2}}\backslash W_{H_J^1}/W_{M_{H_J^1,3}}]$ has size three. Its elements are $1, w_{r_2-r_3}w_{2r_3}$ and $w = w_{r_2-r_3}w_{r_1-r_2}w_{2r_3}w_{r_2-r_2}w_{2r_3}$

Both the global intertwining operator and the sum defining the Eisenstein series on $M_{H_J^1,2}$ are absolutely convergent for $w=1$ and $w=w_{r_2-r_3}w_{2r_3}$ at $s=14$. Thus, these terms do not contribute to the residue at $s=14$. The proposition follows from Langlands functional equation of Eisenstein series. □

The parabolic $P_{H_J^1,1}=M_{H_J^1,1}N_{H_J^1,1}$ has Levi of type $D_{6,2}$. We now compute the constant term of $\Theta_f(g)$ along $N_{H_J^1,1}$. Let $w_1=w_{r_1-r_2}w_{r_2-r_3}w_{2r_3}$. Set $f^{w_1}(g,s)=M(w_1)f(g,s)$ and $E^{w_1}(g,f,s)$ the associated Eisenstein series.

Proposition 5.2.5. *One has* $\Theta_f(g)_{N_{H_J^1,1}} = Res_{s=14}E^{w_1}(g,f,s)$.

Proof. The set $[W_{M_{H_J^1,1}}\backslash W_{H_J^1}/W_{M_{H_J^1,3}}]$ has size two. Its elements are 1 and w_1. The Eisenstein series $E^w(g,f,s)$ for $w=1$ is absolutely convergent at $s=14$, so does not contribute to the residue. The proposition follows. □

Note that $Res_{s=14}E^{w_1}(g,f,s)$ can be considered a theta function on $G_{6,\Theta}$ associated to f^{w_1}.

5.3. **Type E_8.** The automorphic minimal representation on quaternionic E_8, i.e., the group G_J, was constructed in [Gan00a]. In this subsection, we review results from [Gan00a] and compute the constant terms of the functions in this automorphic minimal representation along for the standard maximal parabolic subgroups of G_J.

We fix a maximal compact subgroup $K_{G_J,\infty}\subseteq G_J(\mathbf{R})$ as in [Pol20a, Paragraph 4.1.3]. The Lie algebra of this compact subgroup maps to $\mathfrak{su}_2 = V_3$, affording a three-dimensional representation of $K_{G_J,\infty}$ via the adjoint action. For a positive integer ℓ, we have the $(2\ell+1)$-dimensional vector space $\mathbf{V}_\ell$ defined as the highest weight quotient of $S^\ell(V_3)$; this is again a representation of $K_{G_J,\infty}$. Fixing an $\mathfrak{sl}_2$-triple of $\mathfrak{su}_2\otimes\mathbf{C}$ gives us an associated basis $\{x^{\ell+v}y^{\ell-v}\}_{-\ell\le v\le\ell}$ of $\mathbf{V}_\ell$.

The relative root system is of type F_4. The simple roots are $\alpha_1=(0,1,-1,0)$, $\alpha_2=(0,0,1,-1)$, $\alpha_3=(0,0,0,1)$ and $\alpha_4=(1/2,-1/2,-1/2,-1/2)$ in a Euclidean coordinate system. The highest root in these coordinates is $(1,1,0,0)$.

The Heisenberg parabolic subgroup of G_J is defined to be the stabilizer of the highest root space. In terms of the decomposition $\mathfrak{g}(J)=\mathfrak{sl}_2\oplus\mathfrak{h}(J)^0\oplus V_2\otimes W_J$, the highest root space is spanned by $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\in\mathfrak{sl}_2$. In the notation of subsection 2.5, it is the parabolic $P_{G_J,1}=M_{G_J,1}N_{G_J,1}$. The derived group of the Levi $M_{G_J,1}$ is the group H_J^1 with a C_3 root system. We define $\nu: P_{G_J,1}\to \mathrm{GL}_1$ as $p\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)=\nu(p)\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$.

We will consider the induced representation $I(s) = Ind_{P_{G_J,1}}^{G_J}(|\nu|^s)$. The modulus character $\delta_{P_{G_J}} = |\nu|^{29}$.

See [Gan00a, Proposition 3.2] for the following proposition. One can also see [HS22].

Proposition 5.3.1. *The representation $I_p(s = 24)$ has a unique irreducible quotient, and the representation $I_p(s = 5)$ has a unique irreducible subrepresentation. These irreducible representations are spherical.*

We now have the following result from [Gan00a].

Proposition 5.3.2. *For any flat inducing section $f(g, s) \in I(s)$, the Eisenstein series $E(g, f, s)$ has at most a simple pole at $s = 24$.*

In the course of computing constant terms of $E(g, f, s)$ to the various maximal parabolic subgroups of G_J, we will reprove this result (in a different way).

We write elements of the Weyl group W_{F_4} in notation that indicates how they are a product of simple reflections. Specifically, if w_j denotes the reflection corresponding to the simple root α_j, and $w = w_{i_1} \cdots w_{i_N}$, we denote w by $[i_1, i_2, \ldots, i_N]$.

We begin by computing the constant term of $E(g, f, s)$ to the parabolic with Levi of type $D_{7,3}$. This is the parabolic $P_{G_J,4} = M_{G_J,4}N_{G_J,4}$. It has simple roots $\alpha_1, \alpha_2, \alpha_3$ in its Levi. Let $w_3 = [4, 3, 2, 3, 4, 1, 2, 3, 2, 1]$ and $f^{w_3}(g, s) = M(w_3)f(g, s)$ and $E^{w_3}(g, f, s)$ the associated Eisenstein series on $M_{G_J,4}$.

Proposition 5.3.3. *For a general flat section $f(g, s) \in I(s)$, the constant term $E_{N_{G_J,4}}(g, f, s)$ has at most a simple pole at $s = 24$. The residue is $Res_{s=24}E^{w_3}(g, f, s)$.*

Proof. The set $[W_{M_{G_J,4}} \backslash W/W_{M_{G_J,1}}]$ has size three, with elements

(1) []
(2) $[4, 3, 2, 1]$
(3) $w_3 = [4, 3, 2, 3, 4, 1, 2, 3, 2, 1]$

We analyze the terms from $[W_{M_{G_J,4}} \backslash W/W_{M_{G_J,1}}]$ one-by-one:

(1) []: The associated simple roots are [1]. (See subsection 3.3 for the meaning of this terminology.) This yields an Eisenstein series associated to the $D_{6,2}$ Levi on $D_{7,3}$. The intertwining operator is trivial, and because $s = 24 > 12$, the Eisenstein series is absolutely convergent. Thus this term is regular at $s = 24$.
(2) $[4, 3, 2, 1]$: The associated simple roots are [2]. The 2 parabolic of $D_{7,3}$ will have Levi $\mathrm{SL}_2 \times D_{5,1}$. The intertwining operator is absolutely convergent. Setting $\lambda' = [4, 3, 2, 1](\lambda_s) + \rho_{P_0}$, we obtain $\langle \lambda', \alpha_2^\vee \rangle = s - 9$. Because $24 - 9 = 15 > 10$, this Eisenstein series is absolutely convergent, so is regular at $s = 24$.
(3) $w_3 = [4, 3, 2, 3, 4, 1, 2, 3, 2, 1]$: The associated simple roots are [1]. The intertwining operator is absolutely convergent, as $s - 19 - 3 > 1$ at $s = 24$. Setting $\lambda' = [4, 3, 2, 3, 4, 1, 2, 3, 2, 1](\lambda_s) + \rho_{P_0}$, we obtain $\langle \lambda', \alpha_1^\vee \rangle = s - 17$. At $s = 24$, the associated Eisenstein series on $D_{7,3}$ has at most a simple pole by [HS20].

This completes the proof of the proposition. □

We will now make a special choice of flat inducing section at the infinite place. Namely, set $f_{\infty,\ell}(g, s) \in I(s) \otimes \mathbf{V}_\ell$ the flat section satisfying $f(gk, s) = k^{-1}f(g, s)$ for all $k \in K_{G_J,\infty}$ and $f(1, s) = x^\ell y^\ell \in \mathbf{V}_\ell$. Up to scalar multiple, the vector $x^\ell y^\ell$ is the image in $\mathbf{V}_\ell$ of h^ℓ, where $e, h, f \in \mathfrak{su}_2 \otimes \mathbf{C}$ is our fixed $\mathfrak{sl}_2$ triple.

For a flat section $f_{fte}(g, s) \in I_f(s)$, we set $f_\ell(g, s) = f_{fte}(g, s)f_{\infty,\ell}(g, s)$. **We fix** $\ell = 4$ and consider the Heisenberg Eisenstein series $E(g, f, s)$.

We now have the following proposition.

Proposition 5.3.4 (Gan, see [Gan00a, Gan00b]). *The Eisenstein series $E(g,f,s)$ with f spherical at every finite place attains the pole at $s=24$. The residue of the Eisenstein map $I_f(s=24) \to \mathcal{A}(G_J)$ is defined and intertwining, and the residual representation, is irreducible.*

We write $\Theta_f(g) = Res_{s=24}E(g,f,s)$.

We now compute the constant terms of Θ_f along the parabolic subgroups $P_{G_J,j}$ with $j=1,2,3$. We begin with the constant term down to the Heisenberg parabolic. As usual, we write $f^1(g,s) = M(w_0)f(g,s)$ and $\overline{f}(g) = Res_{s=24}f^1(g,s)$. (We will see momentarily that $f^1(g,s)$ has at most a simple pole at $s=24$.)

Proposition 5.3.5. *Let $w_2 = [1,2,3,4,2,3,2,1]$, $f^{w_2}(g,s) = M(w_2)f(g,s)$ and let the other notation be as above. Then $f^1(g,s)$ has a simple pole at $s=24$ while the integral defining $M(w_2)$ is absolutely convergent. One has*

$$\Theta_f(g)_{N_{G_J,1}} = \overline{f}(g) + Res_{s=24}E_{w_2}(g,f,s).$$

The Eisenstein series on $M_{G_J,1}$ is for its Siegel parabolic, and yields a vector in the minimal representation on H_J^1.

Proof. The set $[W_{M_{G_J,1}}\backslash W/W_{M_{G_J,1}}]$ has five elements:

(1) []
(2) [1]
(3) $w_2 = [1,2,3,4,2,3,2,1]$
(4) $1,2,3,2,1]$
(5) $w_0 = [1,2,3,4,2,3,1,2,3,4,1,2,3,2,1]$

We analyze the them in turn:

(1) The term [] yields the inducing section, which is of course defined at $s=24$.
(2) The term [1] yields a Siegel Eisenstein series on the Levi, evaluated at $s = 23 > 18$, (observe $|\lambda(h_{\alpha_2}(t))| = |t|$ as $\langle r_1+r_2+r_3, r_3\rangle = 1$) and from an absolutely convergent intertwining operator. Thus this term does not contribute to the residue.
(3) The term $[1,2,3,2,1]$ gives an Eisenstein series for $M_{G_J,1}$ with the simple root 4 excluded. The associated Levi in $M_{G_J,1}$ is of type $D_{5,1}\times \mathrm{SL}_2$. The intertwining operator $M([1,2,3,2,1])$ is seen to be absolutely convergent. Setting $\lambda' = [1,2,3,2,1](\lambda_s)+\rho_{P_0}$, one has $\langle \lambda', \frac{1}{2}\alpha_4^\vee\rangle = s-6$. As $24-6 = 18 > 8$, the associated Eisenstein series is absolutely convergent. Thus this term does not contribute to the residue.
(4) The term $w_2 = [1,2,3,4,2,3,2,1]$ yields an absolutely convergent intertwining operator. The associated Eisenstein series on $M_{G_J,1}$ is for the Siegel parabolic. We have analyzed this Eisenstein series in subsection 5.2, see Remark 5.2.3.
(5) The long intertwining operator $M(w_0)$ has a simple pole at $s=24$; see Proposition 4.1.3 of [Pol20b], which handles the spherical case. The general case follows from Proposition 5.3.1.

□

We next compute the constant term down to the parabolic $P_{G_J,2} = M_{G_J,2}N_{G_J,2}$. This is the one with Levi of form $\mathrm{SL}_2 \times E_{6,2}$. The roots in the SL_2 are α_1, and the roots in the E_6 are α_3,α_4.

Proposition 5.3.6. *Let the notation be as above. Then $\Theta_f(g)_{N_{G_J,2}} = E_{\mathrm{SL}_2}(g,\overline{f})$, an absolutely convergent SL_2 Eisenstein series on $M_{G_J,2}$.*

Proof. We have $[W_{M_{G_J,2}}\backslash W/W_{M_{G_J,1}}]$:

(1) []
(2) $[2,3,4,2,3,2,1]$
(3) $[2,3,2,1]$
(4) $[2,3,4,2,3,1,2,3,4,1,2,3,2,1]$

(5) $[2,1]$
(6) $[2,3,4,1,2,3,2,1]$
(7) $[2,3,1,2,3,4,1,2,3,2,1]$

We handle the corresponding terms one-by-one.

(1) $[]$: This yields an Eisenstein series on the SL_2 part of $M_{G_J,2}$. It is absolutely convergent, so does not contribute to the residue at $s=24$.
(2) $[2,3,4,2,3,2,1]$: This again yields an Eisenstein series on the SL_2 part of $M_{G_J,2}$. The intertwining operator is absolutely convergent, and so is the Eisenstein series. Thus this term does not contribute to the residue.
(3) $[2,3,2,1]$: The associated simple roots for this Eisenstein series are $[1,4]$. Thus this term yields an Eisenstein series on SL_2 part of the Levi, and an Eisenstein on E_6 part. The intertwining operator is absolutely convergent, and so is the SL_2 Eisenstein series. Setting $\lambda' = [2,3,2,1](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda_s, \frac{1}{2}\alpha_4^\vee \rangle = s-6$. As $24-6=18>12$, this Eisenstein series on E_6 is also absolutely convergent. Thus this term is regular at $s=24$.
(4) $[2,3,4,2,3,1,2,3,4,1,2,3,2,1]$: This yields an Eisenstein series on the SL_2 bit. Neither the intertwining operator, nor the Eisenstein series, is absolutely convergent. However, applying the Langlands functional equation, one obtains the Eisenstein series in the statement of the proposition.
(5) $[2,1]$: This yields an Eisenstein series on the E_6 part, with simple root $[3]$ not in the new Levi. The intertwining operator is absolutely convergent, and so is the Eisenstein series. Thus this term does not contribute to the residue.
(6) $[2,3,4,1,2,3,2,1]$: The associated roots for this term are $[1,3]$, so there is an SL_2 Eisenstein series and an E_6 Eisenstein series. The intertwining operator is absolutely convergent. Setting $\lambda' = [2,3,4,1,2,3,2,1](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \frac{1}{2}\alpha_3^\vee \rangle = s-11$. As $24-11=13>12$, the Eisenstein series on E_6 is absolutely convergent. One also sees that the SL_2 Eisenstein series is absolutely convergent. Thus this term does not contribute to the residue.
(7) $[2,3,1,2,3,4,1,2,3,2,1]$: The associated roots for this term is $[4]$, so this term yields an Eisenstein series on E_6. One sees that the intertwining operator is absolutely convergent. Setting $\lambda' = [2,3,1,2,3,4,1,2,3,2,1](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \frac{1}{2}\alpha_4^\vee \rangle = s-15$. One sees, because the K-equivariance is preserved by the intertwining operator, that the above inducing section on E_6 will be spherical at the archimedean place. Thus we know from 5.1.1 that this Eisenstein series is regular.

The proposition is proved. □

The parabolic $P_{G_J,3} = M_{G_J,3}N_{G_J,3}$ has Levi of type $\mathrm{SL}_3 \times D_{5,1}$. The simple roots in its Levi are α_1, α_2 (in the SL_3) and α_4 in the $D_{5,1}$.

Proposition 5.3.7. *Let the notation be as above. Let $E_{\mathrm{SL}_3}(g,\overline{f})$ be absolutely convergent Eisenstein series on SL_3 for the simple root $[1]$ for the inducing section $\overline{f}$. Then $\Theta_f(g)_{N_{G_J,3}} = E_{\mathrm{SL}_3}(g,\overline{f})$.*

Proof. We have $[W_{M_{G_J,3}} \backslash W / W_{M_{G_J,1}}]$:

(1) $[]$
(2) $[3,4,2,3,2,1]$
(3) $[3,2,1]$
(4) $[3,4,2,3,1,2,3,4,1,2,3,2,1]$
(5) $[3,2,3,4,1,2,3,2,1]$

We analyze the terms one-by-one:

(1) $[]$: The associated simple roots is $[1]$. This yields a maximal parabolic Eisenstein series on SL_3, for the $(1,2)$ parabolic. The associated Eisenstein series is absolutely convergent.

(2) $[3,4,2,3,2,1]$: The associated simple roots are $[1,2]$. This yields a Borel Eisenstein series on SL_3. The intertwining operator is absolutely convergent. Setting $\lambda' = [3,4,2,3,2,1](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \alpha_1^\vee \rangle = s - 10$ and $\langle \lambda', \alpha_2^\vee \rangle = s - 17$. Because $24 - 10 > 24 - 17 = 7 > 3$, this Borel Eisenstein series is absolutely convergent. Thus, this term does not contribute to the residue.
(3) $[3,2,1]$: The associated simple roots are $[2,4]$. This yeilds a maximal parabolic Eisenstein series on SL_3 times a maximal parabolic Eisenstein series on $D_{5,1}$. The intertwining operator is absolutely convergent. Setting $\lambda' = [3,2,1](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \alpha_2^\vee \rangle = s - 9$ and $\langle \lambda', \frac{1}{2}\alpha_4^\vee \rangle = s - 6$. The SL_3 Eisenstein series is absolutely convergent because $24 - 9 = 15 > 3$. The $D_{5,1}$ Eisenstein series is absolutely convergent because $24 - 6 = 18 > 8$. Thus this term does not contribute to the residue.
(4) $[3,4,2,3,1,2,3,4,1,2,3,2,1]$: The associated simple root is $[2]$. This yields a maximal parabolic Eisenstein series on SL_3. Neither the intertwining operator nor the Eisenstein series will be in the range of absolute convergence. Thus we analyze it using Langlands functional equation, and obtain the Eisenstein series in the statment of the proposition.
(5) $[3,2,3,4,1,2,3,2,1]$: The associated simple roots are $[1,4]$. This yields a maximal parabolic Eisenstein series on SL_3 times a maximal parabolic Eisenstein series on $D_{5,1}$. The intertwining operator is absolutely convergent. Setting $\lambda' = [3,2,3,4,1,2,3,2,1](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \alpha_1^\vee \rangle = s - 17$ and $\langle \lambda', \frac{1}{2}\alpha_4^\vee \rangle = s - 15$. Because $24 - 17 = 7 > 3$ and $24 - 15 = 9 > 8$, the two Eisenstein series are absolutely convergent. Thus this term does not contribute to the residue.

□

6. Twisted Jacquet functors

In this section, we compute various twisted Jacquet functors of p-adic minimal representations. We will use these computations as part of the eventual proof of the Siegel-Weil theorems in section 9.

More specifically, in this section, we prove results of the following sort. Suppose $G \times S \subseteq G'$ is a commuting pair, and $V_{min,p}$ is a minimal representation of $G'(\mathbf{Q}_p)$. Let U be the unipotent radical of a parabolic subgroup of G, and $\chi : U(\mathbf{Q}_p) \to \mathbf{C}^\times$ a non-degenerate character. Let $(V_{min,p})_{(U,\chi)}$ be the twisted Jacquet functor. Then in this section, we prove that the $S(\mathbf{Q}_p)$-coinvariants $(V_{min,p})_{(U,\chi),S(\mathbf{Q}_p)}$ of $(V_{min,p})_{(U,\chi)}$ are one-dimensional in various cases. Additionally, we explicitly write down a nonzero element–thus, a basis–of the dual space of $(V_{min,p})_{(U,\chi),S(\mathbf{Q}_p)}$.

6.1. **Orbits.**

To prove the one-dimensionality of the space of coinvariants as mentioned above, we will need to show that $S_E(\mathbf{Q}_p)$ acts transitively on the $\mathbf{Q}_p$ points of a certain algebraic set Ω_x, in various cases. To prove this transitivity of action, we consistently use the following method.

(1) We prove that $S_E(\overline{\mathbf{Q}}_p)$ acts transitively on $\Omega_x(\overline{\mathbf{Q}}_p)$;
(2) We verify that the stabilizer of a point $\lambda \in \Omega_x(\mathbf{Q}_p)$ is an algebraic group that is semisimple and simply-connected.

In the above setting, it then follows that $S_E(\mathbf{Q}_p)$ acts transitively on $\Omega_x(\mathbf{Q}_p)$ using [BG14, Proposition 1] and the triviality of the Galois cohomology of a simply-connected group over a p-adic field.

Throughout this section, we write $C = \Theta \otimes k$ for a p-adic local field k and C^0 for the subspace of trace 0 elements. The group $\mathrm{Spin}(C)$ acts on three copies of C. We write a typical element g of $\mathrm{Spin}(C)$ as $g = (g_1, g_2, g_3)$, with $g_j \in \mathrm{SO}(C)$. We begin by recalling the following well-known lemma.

Lemma 6.1.1. *For the action of* $\mathrm{Spin}(C)$ *on* C^3*, one has the following stabilizers:*

(1) The set of $g \in \mathrm{Spin}(C)$ *with* $g_1(1) = 1$ *is a copy of* Spin_7.

(2) The set of $g \in \mathrm{Spin}(C)$ *with* $g_1(1) = 1$ *and* $g_2(1) = 1$ *is* G_2.
(3) Suppose $v \in C^0$ *has nonzero norm. The set of* $g \in \mathrm{Spin}(C)$ *with* $g_1(1) = 1$, $g_2(1) = 1$ *and* $g_3(v) = v$ *is a copy of* SU_3.

We now recall the construction of some specific elements in $\mathrm{Spin}(C)$, from [SV00, Section 3.6].

Lemma 6.1.2 ([SV00])**.** *For* $c \in C$ *an octonion with nonzero norm, let* s_c *denote the reflection in* c, ℓ_c *left multiplication by* c *and* r_c *right multiplication by* c. *Suppose* $a_1, \ldots, a_r, b_1, \ldots, b_r \in C$ *with* $\prod_i N(a_i)N(b_i) = 1$. *Set* $t_1 = s_{a_1}s_{b_1}\cdots s_{a_r}s_{b_r}$, $t_2 = \ell_{a_1}\ell_{b_1^*}\cdots\ell_{a_r}\ell_{b_r^*}$, *and* $t_3 = r_{a_1}r_{b_1^*}\cdots r_{a_r}r_{b_r^*}$. *Finally, let* $\widehat{t}(x) = (t(x^*))^*$. *Then* $(\widehat{t_1}, t_2, t_3) \in \mathrm{Spin}(C)$.

Proof. It is proved in [SV00, section 3.6] that under the conditions above, $t_1(xy) = t_2(x)t_3(y)$ for all $x, y \in C$, and that the t_j are in $\mathrm{SO}(C)$. But now one checks immediately that this means $(\widehat{t_1}, t_2, t_3) \in \mathrm{Spin}(C)$. □

Lemma 6.1.3. *Suppose* (v_1, v_2) *and* (v_1', v_2') *in* C^2 *satisfies* $N(v_j) = N(v_j') \neq 0$ *for* $j = 1, 2$. *Then there exists* $g = (g_1, g_2, g_3) \in \mathrm{Spin}(C)$ *so that* $g_1(v_1) = v_1'$ *and* $g_2(v_2) = v_2'$.

Proof. We first work over the algebraic closure of k. By Lemma 6.1.2, we can move v_1 to v_1', so we can assume $v_1 = v_1' \in \overline{k}1$. Now, there exists $u \in C^0$ with $N(u) \neq 0$ so that $(u, v_2) = 0$. Hence $uv_2 \in C^0$. Now we take $u' = N(v_2)^{-1/2}(uv_2)$. Then $N(u') = N(u)$ and $(u')^{-1}(uv_2) \in \overline{k}1$. Because $u, u' \in C^0$, the reflections by u, u' do not move $v_1 \in \overline{k}1$. By choosing the squareroot of $N(v_2)$ appropriately, we see that the lemma is proved over $\overline{k}$.

To descend from $\overline{k}$ to k, we use Galois cohomology, applying Lemma 6.1.1. □

Lemma 6.1.4. *Suppose* $E_j \simeq k \times k \times k$ *for* $j = 1, 2$ *are embedded in* J *as cubic norm structures, both inside* $H_3(k) \subseteq J$. *Then there exists* $m \in M_J^1$ *so that* $m(E_1(a, b, c)) = E_2(a, b, c)$.

Proof. We can consider both E_j in $M_3(k)$, and then they can be moved to one another by $\mathrm{SL}_3(k)$. □

Lemma 6.1.5. *We work over a p-adic field* k. *Let* F *be an étale quadratic extension of* k. *Assume we have an embedding* $E = k \times F \hookrightarrow J$ *satisfying the assumptions in subsection 2.2. Let* C_F *be* $(F)^\perp \subseteq H_2(C)$. *Then* Spin_E *acts transitively on elements of* C_F *with the same nonzero norm.*

Proof. We first work over the algebraic closure of k. In that case, by Lemma 6.1.4, we can move $E = k \times F$ to $E_1 = k \times k \times k$ embedded diagonally, via some element $g \in M_J^1$. Then the claim follows from the same claim for E_1, which we have already proved.

To descend to k, apply Galois cohomology and Lemma 6.1.1. □

Lemma 6.1.6. *Let* $E \hookrightarrow J$ *be an embedding of a cubic étale* k*-algebra. Let* $x \in E$ *have* $N(x) \neq 0$, *and let* $\Omega_x = \{(x, v) \in E \oplus V_E : \text{ rank one}\}$. *Then* $S_E(k)$ *acts transitively on* $\Omega_x(k)$.

Proof. Over an algebraic closure, we may assume E is embedded diagonally in J. Then an (x, v) in Ω_x satisfies $x = (c_1, c_2, c_3)$ with all $c_j \neq 0$, and $v = (v_1, v_2, v_3)$ with $N(v_j) = c_{j-1}c_{j+1}$ and $v_1(v_2v_3) = c_1c_2c_3$.

In this case, by Lemma 6.1.3, we may move v_1 and v_2 to nonzero elements of $\overline{k}1$. Then v_3 is uniquely determined by the final equation in terms of v_1, v_2. Thus over $\overline{k}$, there is one orbit.

Because v_3 is determined by v_1, v_2 under the conditions of the lemma, the stabilizer of a $v = (v_1, v_2, v_3)$ is of type G_2. Thus the stabilizer is simply connected, so there is one k-orbit. □

Lemma 6.1.7. *Suppose* $y = (a, b, c, d) \in W_E$ *is non-degenerate. Let* $\Omega_y = \{(y, w) \in W_J = W_E \oplus V_E^2 : \text{ rank one}\}$. *Then* $S_E(k)$ *acts transitively on* $\Omega_w(k)$.

Proof. Using the action of $\mathrm{SL}_{2,E}$ on W_J, we may assume $y = (1, 0, c, d)$, with $d^2 + 4N(c) \neq 0$. In fact, working over $\overline{k}$ for now, we may assume $d = 0$, so that $N(c) \neq 0$.

Now, in this case, $w = (u, v)$, with $u = (u_1, u_2, u_3) \in C^3$, $N(u_j) = -c_j$, v determined by u, and $(u_1, u_2, u_3)_{\mathrm{tr}_C} = 0$. By Lemma 6.1.3, we can and do move u_1, u_2 to nonzero elements of $\overline{k}1$. Then $\mathrm{tr}(u_3) = 0$ and $N(u_3) = -c_3 \neq 0$. But such elements are in one orbit under the action of G_2. Moreover, the stabilizer is an SU_3 by Lemma 6.1.1, which is simply connected. Thus there is one orbit over $\overline{k}$, and in fact one orbit over k. This completes the proof. □

6.2. **Spaces of coinvariants.** For the split, simply-connected group G_n over k of type D_n, with standard representation $V_{2n} = H \oplus V_{2n-2}$, let Ω denote the nonzero isotropic vectors in V_{2n-2}. Let V_{min} be the minimal representation of G_n, which recall is the unique irreducible subrepresentation of $I(s = n-2)$, in the notation of subsection 4.1.

We recall the following theorem. Let $P_{G_n} = M_{G_n} N_{G_n}$ be the maximal parabolic of G_n stabilizing the line kb_1 in V_{2n}. One can define an action of P_{G_n} on $C_c^\infty(\omega)$ as in [MS97].

Theorem 6.2.1 (Savin, Maagard-Savin). *One has an exact sequence of P_{G_n}-modules,*

$$0 \to C_c^\infty(\omega) \to V_{min} \to V_{min,N_{G_n}} \to 0.$$

Remark 6.2.2. We remark that one does not need to use the exact argument of [Sav94] to prove this result. One can use the Fourier-Jacobi functor of [Wei03], [HS20] to obtain the theorem, if one wants.

We will use the following proposition in section 9.

Proposition 6.2.3. *Let F be a quadratic étale extension of $\mathbf{Q}_p$, and $E = \mathbf{Q}_p \times F$. Recall that we have maps $G_{2,F} \times S_E \to G_6$ and $G_{3,F} \times S_E \to G_7$; see subsection 2.4. Let $P_{2,F} \subseteq G_{2,F}$ and $P_{3,F} \subseteq G_{3,F}$ be the parabolic subgroups that stabilize the line $\mathbf{Q}_p b_1$ in the standard representation of these groups.*

1. *Let $N_{2,F}$ be the unipotent radical of $P_{2,F}$, which we identify with F via the exponential map. Suppose $x \in F$ has nonzero norm to $\mathbf{Q}_p$, and let $\chi_x : N_{2,F} \simeq F \to \mathbf{C}^\times$ be the character given by $\chi_x(y) = \psi((x, y))$. Then the space of coinvariants $(V_{min,G_6})_{(N_{2,F},\chi_x),S_E}$ is dimension one.*
2. *Let $N_{3,F}$ be the unipotent radical of $P_{3,F}$, which we identify with $H \oplus F$ via the exponential map. Suppose $x \in H \oplus F$ is non-degenerate, and let $\chi_x : N_{3,F} \simeq H \oplus F \to \mathbf{C}^\times$ be the character given by $\chi_x(y) = \psi((x, y))$. Then the space of coinvariants $(V_{min,G_7})_{(N_{3,F},\chi_x),S_E}$ is dimension one.*

Proof. We prove the first item. The proof of the second item is identical.

Let

$$\Omega_x = \{(x, v) \in F \oplus C_F : (x, v) \in \Omega \text{ is isotropic}\}.$$

By Theorem 6.2.1, the coinvariants $(V_{min,G_6})_{(N_{2,F},\chi_x)} \simeq C_c^\infty(\Omega_x)$ via the restriction map. See [MS97, Lemma 2.2] for a very similar argument. Now the claim follows from the transitivity of the action of S_E on Ω_x, which is proved in Lemma 6.1.5. □

We now consider similar spaces of coinvariants for the minimal representations on groups of type E_7 and E_8. We refer the reader to [GS05] and the references contained therein, especially section 12 of [GS05], for the fact that the minimal representation is the unique irreducible subrepresentation of the degenerate principal series we studied in section 5.

For E_7, we have the following. Let $P_{H_J^1} = M_{H_J^1} N_{H_J^1}$ be the Siegel parabolic subgroup of H_J^1. Let $\Omega \subseteq J$ be the set of rank one elements. One defines an action of $P_{H_J^1}$ on $C_c^\infty(\Omega)$ as in [MS97].

Theorem 6.2.4 (Savin, Magaard-Savin). *There is a short exact sequence of $P_{H_J^1}$ modules*

$$0 \to C_c^\infty(\Omega) \to V_{min,H_J^1} \to (V_{min,H_J^1})_{N_{H_J^1}} \to 0.$$

Again, this theorem can be proved using the Fourier-Jacobi functor.

For E_8, let $P_{G_J} = M_{G_J} N_{G_J}$ be the Heisenberg parabolic subgroup. Let $Z \subseteq N_{G_J}$ be the center of N_{G_J}, which is also highest root space of G_J. Denote by Ω the rank one elements of W_J. There is a representation of P_{G_J} on $C_c^\infty(\Omega)$; see [Gan11, section 2.3]. The following theorem (see [Gan11, Section 2.3] again) can be proved using the work in [GS05, Sections 11,12].

Theorem 6.2.5 (Gan, Savin)**.** *There is a short exact sequence of P_{G_J} modules*

$$0 \to C_c^\infty(\Omega) \to (V_{min,G_J})_Z \to (V_{min,G_J})_{N_{G_J}} \to 0.$$

We can now state and prove the analogues of Proposition 6.2.3 that we will need in the cases of minimal representation on E_7 and E_8.

Proposition 6.2.6. *Let E be a cubic étale algebra over $\mathbf{Q}_p$, and $x \in E$ an element with nonzero norm to $\mathbf{Q}_p$. Recall that we have a map $\mathrm{SL}_{2,E} \times S_E \to H_J^1$. Let U_E be the unipotent radical of the standard Borel of $\mathrm{SL}_{2,E}$, which we identify with E via the exponential map. Let $\chi_x : U_E \to \mathbf{C}^\times$ be the character given by $\chi_x(y) = \psi((x,y))$. Then the space of coinvariants $(V_{min,H_J^1})_{(U_E,\chi_x),S_E}$ is one-dimensional.*

Proof. This follows from Theorem 6.2.4 and Lemma 6.1.6, completely similar to the proof of Proposition 6.2.3. □

We now consider the case of minimal representation on E_8.

Proposition 6.2.7. *Let E be a cubic étale algebra over $\mathbf{Q}_p$, and $x \in W_E$ a non-degenerate element. Recall that we have a map $G_E \times S_E \to G_J$. Let N_E be the unipotent radical of the standard Heisenberg parabolic subgroup of G_E. We identify N_E/Z with W_E via the exponential map. Let $\chi_x : N_E \to \mathbf{C}^\times$ be the character given by $\chi_x(y) = \psi((x,y))$. Then the space of coinvariants $(V_{min,G_J})_{(N_E,\chi_x),S_E}$ is one-dimensional.*

Proof. This follows from Theorem 6.2.5 and Lemma 6.1.7, completely similar to the proof of Proposition 6.2.3. □

6.3. **Equivariant linear functionals.** In subsection 6.2, we showed that various spaces of coinvariants of the form $(V_{min,q})_{(U,\chi),S(\mathbf{Q}_q)}$ are one-dimensional, where $V_{min,q}$ is the minimal representation of the group $G'(\mathbf{Q}_q)$, where G' is one of the groups G_6, G_7, H_J, G_J and U depends upon G'. In this subsection, we identify an explicit nonzero element of the dual of this one-dimensional space. Here q is an arbitrary finite prime.

Recall that the local minimal representation $V_{min,q}$ of $G'(\mathbf{Q}_q)$ can be considered as a submodule of following induced representations:

(1) $G' = G_6$, $V_{min,q} \subseteq I_q(s=4) := Ind_{P'(\mathbf{Q}_q)}^{G'(\mathbf{Q}_q)}(s=4)$, $P' = P_{G_6}$;
(2) $G' = G_7$, $V_{min,q} \subseteq I_q(s=5) := Ind_{P'(\mathbf{Q}_q)}^{G'(\mathbf{Q}_q)}(s=5)$, $P' = P_{G_7}$;
(3) $G' = H_J^1$, $V_{min,q} \subseteq I_q(s=4) := Ind_{P'(\mathbf{Q}_q)}^{G'(\mathbf{Q}_q)}(s=4)$, $P' = P_{H_J^1}$;
(4) $G' = G_J$, $V_{min,q} \subseteq I_q(s=5) := Ind_{P'(\mathbf{Q}_q)}^{G'(\mathbf{Q}_q)}(s=5)$, $P' = P_J$.

We will use this realization to state and prove the following proposition.

Proposition 6.3.1. *Suppose (G,U) is one of the following four pairs:*

(1) $G = G_{2,F}$, $U = N_{2,F}$;
(2) $G = G_{3,F}$, $U = N_{3,F}$;
(3) $G = \mathrm{SL}_{2,E}$, $U = U_E$;
(4) $G = G_E$, $U = N_E$.

Let $\chi : U \to \mathbf{C}^\times$ be a non-degenerate unitary character of $U(\mathbf{Q}_q)$. Abusing notation, let w_0 be the long Weyl element in $G(\mathbf{Q}_q)$, see subsection 3.2. Define $L_\chi : V_{min,q} \to \mathbf{C}$ as

$$L_\chi(\overline{f}) = \int_{U(\mathbf{Q}_q)} \chi^{-1}(n)\overline{f}(w_0 n)\, dn.$$

Here $\overline{f} \in V_{min,q} \subseteq I_q(s = s_0)$ with $s_0 \in \{4, 5\}$ as above. Then the integral defining L_χ converges absolutely, and L_χ is not identically 0 on $V_{min,q}$.

One sees that the integral defining L_χ converges absolutely by a comparison with intertwining operators. Thus the proposition follows without much difficulty from the following lemma.

Lemma 6.3.2. *Let G be one of the groups in Proposition 6.3.1, and let $P = P' \cap G \subseteq G$ be the standard parabolic with unipotent radical U. Consider $V_{min,q} \subseteq I_q(s = s_0)$. Then the restriction map*

$$Res : V_{min,q} \to Ind_{P(\mathbf{Q}_q)}^{G(\mathbf{Q}_q)}(s = s_0) =: I_{G,q}(s = s_0)$$

is surjective.

Proof. There are four cases:

1. $G = G_{2,F}$, which is the simply connected cover of $\mathrm{SO}_{4,F}$. So, $G = \mathrm{SL}_{2,F}$.
2. $G = G_{3,F}$, which is the simply connected cover of $\mathrm{SO}_{6,F}$, so $G = \mathrm{SU}_{2,2,F}$.
3. $G = \mathrm{SL}_{2,E}$
4. $G = G_E$

It is easy to see that the image of the restriction map is a nonzero $G(\mathbf{Q}_q)$-submodule of $I_{G,q}(s = s_0)$. In cases 1 and 3, this principal series is known to be irreducible, because it is on a group of SL_2-type. In case 4, for G_E, the representation is known to be generated by any vector which is not annihilated by the long intertwiner. The restriction of the spherical section of $V_{min,q}$ to G_E satisfies this property. Thus, in cases 1,2, and 4, the restriction is surjective.

So, we only must argue for case 3. We have the canonical covering map $G_{3,F} \to \mathrm{SO}_{6,F}$. Let $J_{G,q}(s = 5)$ be the associated principal series on $\mathrm{SO}_{6,F}$. Applying Lemma 3.6.1, one sees that the restriction of functions from $\mathrm{SO}_{6,F}$ to $G_{3,F}$ defines an isomorphism $J_{G,q}(s = 5) \to I_{G,q}(s = 5)$. Likewise, we have a covering map $G_7 \to \mathrm{SO}_{14}$; let $J_{G',q}(s = 5)$ be the associated induced representation on SO_{14}. Restriction of functions yields an isomorphism $J_{G',q}(s = 5) \to I_q(s = 5)$. Moreover, because the minimal representation on $G'(\mathbf{Q}_q)$ was defined in terms of intertwining operators, $V_{min,q}$ extends to a $\mathrm{SO}_{14}(\mathbf{Q}_q)$ subrepresentation of $J_{G',q}(s = 5)$.

Thus, it suffices to verify that the induced representation $J_{G,q}(s = 5)$ on $\mathrm{SO}_{6,F}(\mathbf{Q}_q)$ is irreducible. The proof is to follow the argument in [HS20, Theorems 3.1 and 3.2]. In these theorems, the authors restrict to the simply connected case then exclude type D_3 with the condition $\dim D > 2$. However, [HS20, Theorem 3.1] goes over without assuming simply connected. Then, because we are working on $\mathrm{SO}_{6,F}$ instead of $\mathrm{Spin}_{6,F} = \mathrm{SU}_{2,2,F}$, the GL_2 Levi in the Heisenberg parabolic acts transitively on nontrivial characters of the highest root space. Thus, the argument of [HS20, Theorem 3.2] goes through to prove that this representation is irreducible. □

7. Siegel Weil Eisenstein series I

Our Siegel-Weil theorems are identities relating a theta lift to a special value of a degenerate Eisenstein series. We call the latter "Siegel-Weil Eisenstein series". In this section, we define some of these Siegel-Weil Eisenstein series and compute their constant terms along various maximal parabolic subgroups.

7.1. **The group $G_{2,F}$.** Let F be a real quadratic field. The group $G_{2,F}$ acts on the $V_{2,F} = H \oplus F$. Let $P_{2,F} = M_{2,F}N_{2,F}$ be the parabolic subgroup of $G_{2,F}$ that stabilizes the line $\mathbf{Q}b_1$. The action of $P_{2,F}$ on b_1 defines a character $\nu : P_{2,F} \to \mathrm{GL}_1$, and we consider the associated induced representation $I(s) = Ind_{P_{2,F}}^{G_{2,F}}(|\nu|^s)$.

Let $v_1 = \frac{1}{\sqrt{2}}(b_1 + b_{-1})$ and let $v_2 = \frac{1}{\sqrt{2}}(1_F)$. Observe that $(v_i, v_j) = \delta_{ij}$. Then v_1, v_2 can be used to define a maximal compact subgroup $K_{G_{2,F},\infty} \subseteq G_{2,F}(\mathbf{R})$, and the action of $K_{G_{2,F},\infty}$ on $v_1 + iv_2 \in V_{2,F} \otimes \mathbf{C}$ gives a character $j(\bullet, i) : K_{G_{2,F},\infty} \to \mathbf{C}^\times$. For an even integer ℓ, let $f_{\infty,\ell}(g,s) \in I_\infty(s)$ be the flat section with $f_{\infty,\ell}(k,s) = j(k,i)^\ell$.

Let $f_\ell(g,s) \in I(s)$ be a global flat section, with infinite component equal to $f_{\infty,\ell}$. Let $E(g, f_\ell, s) = \sum_{\gamma \in P_{2,F}(\mathbf{Q})\backslash G_{2,F}(\mathbf{Q})} f_\ell(\gamma g, s)$ be the associated Eisenstein series. The following proposition is well-known.

Proposition 7.1.1. *Suppose $s_0 := |\ell| > 2$. The Eisenstein series $E(g, f_\ell, s)$ converges absolutely at s_0, and the constant term $E(g, f_\ell, s = s_0)_{N_{2,F}} = f_\ell(g, s = s_0)$.*

Proof. The proof of the proposition boils down to verifying that the archimedean intertwining operator

$$M(w_0)f_{\infty,\ell}(g, s = |\ell|) = \int_{N_{2,F}(\mathbf{R})} f_{\infty,\ell}(w_0 n g, s = |\ell|)\, dn = 0. \tag{2}$$

As mentioned, this is well-known, and in any event, can be verified by the reader. □

7.2. **The group $G_{3,F}$.** Let F be a real quadratic étale extension of $\mathbf{Q}$, i.e., either $F = \mathbf{Q} \times \mathbf{Q}$ or F is a real quadratic field. The group $G_{3,F}$ acts on the vector space $V_{6,F} = H^2 \oplus F$. Let $P_{3,F}$ be the parabolic subgroup stabilizing the line $\mathbf{Q}b_1$. The action of $P_{3,F}$ on b_1 defines a character $\nu : P_{3,F} \to \mathrm{GL}_1$, and we consider the induced representation $I(s) = Ind_{P_{3,F}}^{G_{3,F}}(|\nu|^s)$.

Let $v_j = \frac{1}{\sqrt{2}}(b_j + b_{-j})$ for $j = 1, 2$ and $v_3 = \frac{1}{\sqrt{2}}(1_F)$. Then $(v_i, v_j) = \delta_{ij}$. From $V_3 = \mathrm{Span}(v_1, v_2, v_3)$ one obtains a maximal compact subgroup $K_{G_{3,F},\infty} \subseteq G_{3,F}(\mathbf{R})$. For an integer $\ell \geq 1$, let $f_{\infty,\ell}(g,s) \in I_\infty(s) \otimes \mathbf{V}_\ell$ be the flat section defined exactly as in subsection 4.4. Let $f_\ell(g,s) \in I(s)$ be a flat section with archimedean component equal to $f_{\infty,\ell}(g,s)$. We let $E(g, f_\ell, s) = \sum_{\gamma \in P_{3,F}(\mathbf{Q})\backslash G_{3,F}(\mathbf{Q})} f_\ell(\gamma g, s)$ be the associated Eisenstein series.

We now fix $\ell = 4$ and $s_0 = 5$. The Eisenstein series is absolutely convergent at $s = s_0 = 5$.

Proposition 7.2.1. *Suppose $F = \mathbf{Q} \times \mathbf{Q}$ so that $V_{6,F} = H^3$. Let $P_{3,F;3} = M_{3,F;3}N_{3,F;3}$ be the parabolic subgroup stabilizing* $\mathrm{Span}(b_1, b_2, b_3)$. *Then*

$$E(g, f_4, s = 5)_{N_{3,F;3}} = \sum_{(P_{3,F} \cap M_{3,F;3})(\mathbf{Q})\backslash M_{3,F;3}(\mathbf{Q})} f_4(\gamma g, s = 5)$$

an absolutely convergent SL_3 *Eisenstein series.*

Proof. The general form of a constant term is expressed in subsection 3.3. There are two relevant Weyl elements: 1 and w where $w(r_1) = -r_3$, $w(r_2) = r_1$ and $w(r_3) = -r_2$. Then $w = w_{r_2+r_3}w_{r_1-r_2}$ has length two. One finds that for this w, the global intertwining operator is absolutely convergent, and the associated SL_3 Eisenstein series is defined by an absolutely convergent sum. Thus, to prove that $E^w(g, f_4, s = 5)$ is 0, it suffices to prove that the archimedean intertwining operator $M(w)$ is 0 on $f_{4,\infty}(g, s = 5)$. One is reduced to showing the vanishing of

$$\int_{\mathbf{R}^\times \times \mathbf{R}^2} |t|^{s+\ell} pr(ub_1 + vb_2 + b_{-3})^\ell e^{-t^2(u^2+v^2+1)}\, dt\, du\, dv \tag{3}$$

at $s = \ell$. Here $pr()^\ell$ is the natural projection from $Sym^\ell(V_{6,F} \otimes \mathbf{R})$ to $\mathbf{V}_\ell$. One can verify the vanishing using [Pol22, proof of Proposition 4.1.4]. This completes the proof. □

Remark 7.2.2. There is a second standard A_2 maximal parabolic of $G_{3,F}$ when $F = \mathbf{Q} \times \mathbf{Q}$, defined as the stabilizer of $\mathrm{Span}(b_1, b_2, b_{-3})$. The computation of the constant term of $E(g, f_4, s = 5)$ along this parabolic is essentially identical to the computation just done.

Proposition 7.2.3. *Suppose F is a field. Let $P_{3,F;2} = M_{3,F;2}N_{3,F;2}$ be the parabolic subgroup of $G_{3,F}$ that stabilizes $\mathrm{Span}(b_1, b_2)$. Then*

$$E(g, f_4, s = 5)_{N_{3,F;2}} = \sum_{(P_{3,F} \cap M_{3,F;2})(\mathbf{Q}) \backslash M_{3,F;2}(\mathbf{Q})} f_4(\gamma g, s = 5),$$

an absolutely convergent SL_2-type Eisenstein series.

Proof. The constant term $E(g, f_4, s)_{N_{3,F;2}}$ is a sum of two Eisenstein series. The relevant Weyl elements are 1 and w, where $w = w_{r_2} w_{r_1 - r_2}$. The term for $w = 1$ gives the statement of the proposition, so we must verify that $E^w(g, f_4, s)$ vanishes at $s = 5$.

As before, the global intertwining operator $M(w)$ and the associated Eisenstein series $E^w(g, f_4, s)$ are absolutely convergent at $s = 5$. So it suffices to check that the archimedean component $M(w)f_{4,\infty}(g, s)$ vanishes at $s = 5$. To see this, one first applies $M(w_{r_1 - r_2})$ to $f_{4,\infty}(g, s)$. This intertwining operator is computed in [Pol22, Proposition 4.2.2] and [Pol20b, Proposition 3.3.2]. One then applies $M(w_{r_2})$ to the result, which is the integral of equation (2), which vanishes. □

We now compute the constant term long the unipotent radical $N_{3,F}$ of $P_{3,F}$.

Proposition 7.2.4. *Let F be either $\mathbf{Q} \times \mathbf{Q}$ or a real quadratic field. Let $w_{12} = w_{r_1 - r_2}$ be the simple reflection corresponding to the root $r_1 - r_2$. One has*

$$E(g, f_4, s = 5)_{N_{3,F}} = f_4(g) + E^{w_{12}}(g, f_4, s = 5),$$

the Eisenstein series $E^{w_{12}}(g, f_4, s)$ being absolutely convergent at $s = 5$.

Proof. The constant term along $N_{3,F}$ of $E(g, f_4, s)$ has three terms, $f_4(g, s)$, $E^{w_{12}}(g, f_4, s)$ and the long intertwining operator $M(w_0)f_4(g, s)$. Everything is absolutely convergent, so to prove the proposition it suffices to verify the $M(w_0)f_{4,\infty}(g, s)$ vanishes at $s = 5$. This quickly reduces to the vanishing of the integral in equation (3). This completes the proof. □

8. Siegel Weil Eisenstein series II

In this section, we define the Siegel-Weil Eisenstein series on G_E, and compute its constant terms along the various maximal parabolic subgroups.

Let $P_E = P_{G_E}$ be the Heisenberg parabolic subgroup of G_E, and ν the character $P_{G_E} \to \mathrm{GL}_1$ given by the action on the highest root space. We consider the induced representation $I(s) = Ind_{P_{G_E}}^{G_E}(|\nu|^s)$.

Define a flat archimedean inducing section $f_{\infty,\ell}(g, s)$ exactly as in subsection 5.3. We will take $\ell = 4$. We consider flat sections $f_{\ell=4}(g, s) \in I(s)$ with archimedean component equal to $f_{\infty,4}(g, s)$. Let $E(g, f_4, s)$ be the associated Eisenstein series. It converges for $Re(s) > 5$. We will show that the Eisenstein series is regular at $s = 5$, and we will be interested in the constant term of $E(g, f_4, s = 5)$ along the various maximal parabolic subgroups of G_E.

We break the computation into cases: E is a field; $E = E_{sp} = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$, and $E = \mathbf{Q} \times F$ with F a field.

To do many of the constant term computations below, we make precise calculations at the archimedean place. By the equivariance for the maximal compact subgroup, it always suffices to make the computation of the intertwined inducing section $M(w)f_4(g, s)$ at $g = 1$. Then, the way we do this is to factor intertwining operators into ones corresponding to simple reflections, and then to explicitly compute these latter intertwiners, using SL_2 theory. Then, what one must keep track of is how the various SL_2's sit inside the group G_E.

The papers [Pol20b] and [cDD+22] make very similar computations in slightly different contexts. We will use notation from these two papers, and refer the reader to [Pol20b] and [cDD+22] for a more thorough explanation.

We set $A = \begin{pmatrix} 2 & 2 & 1 \\ 56 & 8 & -4 \\ 140 & -20 & 6 \end{pmatrix}$. This matrix is the change-of-basis matrix between $x^8 + y^8, x^6y^2 + x^2y^4$, x^4y^4 and $f_1^8 + f_2^8$, $f_1^6 f_2^2 + f_1^2 f_2^6$, $f_1^4 f_2^4$; see [Pol20b]. Let $A_1 = A^t$. We let $d(s) = \mathrm{diag}(v_2(s), v_1(s), v_0(s))$, where $v_0(s) = 1$, $v_1(s) = \frac{((1-s)/2)_1}{((1+s)/2)_1}$ and $v_2(s) = \frac{((1-s)/2)_2}{((1+s)/2)_2}$.

8.1. **The case of E a field.** In this case, the group G_E has rational root system of type G_2. The simple roots are (in a Euclidean coordinate system) $\alpha_1 = (0, 1, -1)$ and $\alpha_2 = (1, -2, 1)$.

One has for $[W/W_{M_1}]$ the following elements:

(1) []
(2) [2]
(3) [1, 2, 1, 2]
(4) [1, 2]
(5) [2, 1, 2, 1, 2]
(6) [2, 1, 2].

For $[W_{M_1}\backslash W/W_{M_1}]$ one has

(1) []
(2) [2]
(3) [2, 1, 2]
(4) [2, 1, 2, 1, 2].

For $[W_{M_2}\backslash W/w_{M_1}]$, one has

(1) []
(2) [1, 2, 1, 2]
(3) [1, 2].

One finds that (in the above Euclidean coordinates) $\rho_{P_0} = (5, -1, -4)$ and the highest root is $(2, -1, -1)$. We thus set $\lambda_s = (2s - 5, 1 - s, 4 - s)$.

The long intertwining operator $w_0 = [2, 1, 2, 1, 2]$ includes all the ones of smaller length that we must study, so we write down what happens with c-functions for w_0. At each step, we compute $\langle \lambda', \frac{1}{3}\alpha_j\rangle$ if a simple reflection $[j]$ is being applied. Note that, if $j = 2$ so that the root is long, then $\alpha_j^\vee = \frac{1}{3}\alpha_j$ is the coroot. If $j = 1$ is short, then $\alpha_j = \alpha_j^\vee$, but then the associated c-function is $\zeta_E(\frac{1}{3}\langle\lambda', \alpha_1^\vee\rangle)$. One has

(1) $\lambda_s = (2s - 5, 1 - s, 4 - s)$
(2) apply [2], get $\langle \lambda', \frac{1}{3}\alpha_j\rangle = s - 1$, and the new $\lambda' = (s - 4, s - 1, 5 - 2s)$;
(3) apply [1], get $\langle \lambda', \frac{1}{3}\alpha_j\rangle = s - 2$, and the new $\lambda' = (s - 4, -2s + 5, s - 1)$;
(4) apply [2], get $\langle \lambda', \frac{1}{3}\alpha_j\rangle = 2s - 5$, and the new $\lambda' = (1 - s, 2s - 5, 4 - s)$;
(5) apply [1], get $\langle \lambda', \frac{1}{3}\alpha_j\rangle = s - 3$, and the new $\lambda' = (-s + 1, -s + 4, 2s - 5)$;
(6) apply [2], get $\langle \lambda', \frac{1}{3}\alpha_j\rangle = s - 4$, and the new $\lambda' = (-2s + 5, s - 4, s - 1)$.

Proposition 8.1.1. *For the constant term $E(g, f_4(g, s))_{N_1}$ at $s = 5$, set $f_4^{[2]}(g) = M([2])f_4(g, s)|_{s=5}$ (absolutely convergent) and $E_{\mathrm{GL}_{2,E}}(g, f_4^{[2]}(g))$, an absolutely convergent Eisenstein series on $\mathrm{GL}_{2,E}$. Then $E(g, f_4, s = 5)_{N_1} = f_4(g, s = 5) + E_{\mathrm{GL}_{2,E}}(g, f_4^{[2]}(g))$.*

Proof. We analyze the terms in $[W_{M_1}\backslash W/W_{M_1}]$ one-by-one:

(1) []: This yields the inducing section, which is regular at $s = 5$, as desired.

(2) [2]: The intertwining operator is absolutely convergent. Setting $\lambda'' = [2](\lambda_s) + \rho_{P_0}$, we obtain $\langle \lambda'', \frac{1}{3}\alpha_1^\vee\rangle = s-1$. As $5-1=4>2$, this gives an absolutely convergent Eisenstein series on $\mathrm{GL}_{2,E}$.
(3) [2, 1, 2]: The intertwining operator is absolutely convergent. Setting $\lambda'' = [2,1,2](\lambda_s)+\rho_{P_0}$, we obtain $\langle \lambda'', \frac{1}{3}\alpha_1^\vee\rangle = s-2$. As $5-2=3>2$, this will give us an absolutely convergent Eisenstein series on $\mathrm{GL}_{2,E}$. Looking at the archimedean component, we compute

$$v_{212}(s) = A_1^{-1}d(2s-5)A_1 d(s-2)^3 A_1^{-1} d(s-1)A_1(0,0,1)^t.$$

One has $v_{212}(s=5)=(0,0,0)$, so this term disappears from the constant term $E(g,f_4,s=5)_{N_1}$.
(4) [2, 1, 2, 1, 2]: The intertwining operator has a global simple pole at $s=5$, and locally the integrals are absolutely convergent. Now, we set

$$v_{21212}(s) = A_1^{-1}d(s-4)A_1 d(s-3)^3 v_{212}(s).$$

One finds $v_{21212}(s=5)=0$, and $v'_{21212}(s=5)=0$. Consequently, this term does not contribute to the constant term of $E(g,f_4,s=5)_{N_1}$.

The proposition is proved. □

We now compute the constant term of $E(g,f_4,s)$ down to M_2.

Proposition 8.1.2. *One has $E(g,f_4,s=5)_{N_2} = E_{\mathrm{GL}_2}(g, f_4|_{M_2})$, an absolutely convergent Eisenstein series obtained by restricting the inducing section $f_4(g)$ to M_2 and evaluating at $s=5$.*

Proof. We analyze the terms in $[W_{M_2}\backslash W/w_{M_1}]$ one-by-one:
(1) []: This term gives an absolutely convergent Eisenstein series on the long root GL_2.
(2) [1, 2, 1, 2]: The intertwining operator is globally absolutely convergent. Setting $\lambda'' = [1,2,1,2](\lambda_s)+\rho_{P_0}$, we have $\langle\lambda'', \alpha_2^\vee\rangle = s-3$. This is the point where the Eisenstein series has a simple pole, with residue a one-dimensional representation. We set

$$v_{1212}(s) = A_1 d(s-3)^3 v_{212}(s).$$

Then one computes that $v_{1212}(s=5)=(0,0,0)$ and $v'_{1212}(s=5)=(*,*,0)$. Because only the trivial representation could contribute to the residue, we see that this Eisenstein vanishes at $s=5$.
(3) [1, 2]: The intertwining operator is globally absolutely convergent. Setting $\lambda'' = [1,2](\lambda_s)+\rho_{P_0}$, one has $\langle\lambda'',\alpha_2^\vee\rangle = 2s-4$. At $s=5$, we thus obtain an absolutely convergent Eisenstein series. Setting

$$v_{12}(s) = A_1 d(s-2)^3 A_1^{-1} d(s-1) A_1 (0,0,1)^t$$

we have $v_{12}(s=5)=0$. Thus this term does not contribute to the constant term at $s=5$.

□

8.2. **The case of $E = \mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}$.** In this case G_E has a root system of type D_4. In Euclidean coordinates, the simple roots are $\alpha_1=(1,-1,0,0)$, $\alpha_2=(0,1,-1,0)$, $\alpha_3=(0,0,1,-1)$, $\alpha_4=(0,0,1,1)$. The simple root corresponding to the Heisenberg parabolic is α_2.

We again have the Eisenstein series $E(g,f_4,s)$. We will see that it is regular at $s=5$, and we will compute the constant terms along the maximal parabolic subgroups. The three maximal non-Heisenberg parabolic subgroups are related by triality, so we will only compute the constant term down to one of them.

The constant term down to the Heisenberg Levi involves the elements of the set $[W_{M_2}\backslash W/W_{M_2}]$, which are given as follows.

(1) [];
(2) [2, 4, 1, 2]
(3) [2, 3, 1, 2]

(4) $[2,4,3,2]$
(5) $w_0 = [2,3,1,2,4,2,3,1,2]$
(6) $[2]$
(7) $[2,4,3,1,2]$

The constant term down to the $D_{3,3}$ Levi involves the elements of the set $[W_{M_1}\backslash W/W_{M_2}]$, which are given as follows. We also list the simple roots corresponding to the associated new parabolic of M_1:

(1) $[]$; $[2]$
(2) $[1,2]$; $[3,4]$
(3) $[1,2,4,3,2]$; $[2]$.

One has $\rho = (3,2,1,0)$ and the highest root is $(1,1,0,0)$. We thus set $\lambda_s = (s-3, s-2, -1, 0)$.

Proposition 8.2.1. *The constant term of $E(g, f_4, s=5)$ along N_1 is $E_{M_1}(g, f_4|_{M_1})$, an absolutely convergent Eisenstein series on M_1 associated to the simple root $[2]$ of M_1.*

Proof. We evaluate one-by-one: The constant term down to the $D_{3,3}$ Levi involves the elements of the set $[W_{M_1}\backslash W/W_{M_2}]$, which are given as follows. We also list the simple roots corresponding to the associated new parabolic of M_1:

(1) $[]$: The associated simple root is $[2]$. This gives an absolutely convergent Eisenstein series on $D_{3,3}$ from its "Siegel" parabolic (stabilizing an isotropic line.)
(2) $[1,2]$: The associated simple roots are $[3,4]$. The intertwining operator gives:
   (a) apply $[2]$, get $\langle \lambda', \alpha_j \rangle = s-1$, and the new $\lambda' = (s-3, -1, s-2, 0)$;
   (b) apply $[1]$, get $\langle \lambda', \alpha_j \rangle = s-2$, and the new $\lambda' = (-1, s-3, s-2, 0)$.
   It is globally absolutely convergent. Setting $\lambda'' = [1,2](\lambda_s) + \rho$, one has $\langle \lambda'', \alpha_j^\vee \rangle = s-1$ for $j = 3,4$. Now, applying the modulus character of the $[3]$ parabolic of M_1 to $\alpha_3^\vee(t)$ gives $|t|^2$, and similarly applying the modulus character of the $[4]$ parabolic of M_1 to $\alpha_4^\vee(t)$ gives $|t|^2$. As $5-1 = 4 > 2$, this Eisenstein series is absolutely convergent. Setting $v_{12}(s) = d(s-2)A_1^{-1}d(s-1)A_1(0,0,1)^t$, we obtain $v_{12}(s=5) = (0,0,0)$. Thus this term does not contribute to the constant term $E(g, f_4, s=5)_{N_1}$.
(3) $[1,2,4,3,2]$: The associated simple root is $[2]$. The intertwining operator gives:
   (a) apply $[2]$, get $\langle \lambda', \alpha_j \rangle = s-1$, and the new $\lambda' = (s-3, -1, s-2, 0)$;
   (b) apply $[3]$, get $\langle \lambda', \alpha_j \rangle = s-2$, and the new $\lambda' = (s-3, -1, 0, s-2)$;
   (c) apply $[4]$, get $\langle \lambda', \alpha_j \rangle = s-2$, and the new $\lambda' = (s-3, -1, -s+2, 0)$;
   (d) apply $[2]$, get $\langle \lambda', \alpha_j \rangle = s-3$, and the new $\lambda' = (s-3, -s+2, -1, 0)$;
   (e) apply $[1]$, get $\langle \lambda', \alpha_j \rangle = 2s-5$, and the new $\lambda' = (-s+2, s-3, -1, 0)$.
   This intertwining operator is globally absolutely convergent at $s=5$. Setting $\lambda' = [1,2,4,3,2](\lambda_s) + \rho$, we have $\langle \lambda', \alpha_2^\vee \rangle = s-1$. Applying the modulus character of the "2" parabolic of this D_3 to $\alpha_2^\vee(t)$ gives $|t|^4$. Thus this Eisenstein series is at the edge of absolute convergence when $s=5$. We set

$$v_{12432}(s) = d(2s-5)A_1^{-1}d(s-3)A_1 d(s-2)^2 A_1^{-1} d(s-1)A_1(0,0,1)^t.$$

   Now $v_{12432}(s=5) = 0$ and $v'_{12432}(s=5) = (0,0,*)$. This is an Eisenstein series associated to a group with Jordan algebra $J_2(\mathbf{Q} \times \mathbf{Q})$, so it has a simple pole at this boundary point where $s=5$ by [HS20]. The residue is the trivial representation. However, since the K_∞ type of this Eisenstein series does not contain the trivial representation, we obtain vanishing. Thus this term does not contribute to the constant term $E(g, f_4, s=5)_{N_1}$.

□

We now compute the constant term to the Heisenberg parabolic.

Proposition 8.2.2. *The constant term $E(g, f_4, s)_{N_2}$ at $s = 5$ is $f_4(g, s = 5) + E_{\mathrm{GL}_{2,E}}(g, f_4^{[2]})$ where $f_4^{[2]}(g) = M([2])f_4(g, s = 5)$.*

Proof. We handle the elements of $[W_{M_2}\backslash W/W_{M_2}]$ one-by-one.

(1) []: The associated simple roots are [] (empty). This gives the inducing section $f_4(g, s = 5)$.
(2) $[2, 4, 1, 2]$: The associated simple roots are [3]. The intertwining operator is:
  (a) apply [2], get $\langle \lambda', \alpha_j \rangle = s - 1$, and the new $\lambda' = (s - 3, -1, s - 2, 0)$;
  (b) apply [1], get $\langle \lambda', \alpha_j \rangle = s - 2$, and the new $\lambda' = (-1, s - 3, s - 2, 0)$;
  (c) apply [4], get $\langle \lambda', \alpha_j \rangle = s - 2$, and the new $\lambda' = (-1, s - 3, 0, -s + 2)$;
  (d) ppply [2], get $\langle \lambda', \alpha_j \rangle = s - 3$, and the new $\lambda' = (-1, 0, s - 3, -s + 2)$.
  Setting $\lambda' = [2, 1, 4, 2](\lambda_s) + \rho$, we obtain $\langle \lambda_s, \alpha_3^\vee \rangle = 2s - 4$. Thus the intertwining operator and the Eisenstein series are absolutely convergent. One calculates the archimedean intertwiner and finds that it vanishes at $s = 5$. Thus this term does not contribute to the constant term $E(g, f_4, s = 5)_{N_2}$.
(3) $[2, 3, 1, 2]$: The associated simple roots are [4]. This case is nearly identical to the previous case; there is no contribution to the constant term.
(4) $[2, 4, 3, 2]$: The associated simple roots are [1]. This case is nearly identical to the previous two cases; there is no contribution to the constant term.
(5) $w_0 = [2, 3, 1, 2, 4, 2, 3, 1, 2]$: The associated simple roots are [] (empty). One finds that the intertwining operator is locally absolutely convergent but globally has a simple pole. One computes that the archimedean intertwining operator vanishes to order at least two at $s = 5$, so this term does not contribute to the constant term along N_2.
(6) [2]: The associated simple roots are $[1, 3, 4]$. This gives an intertwining operator and Eisenstein series that are both absolutely convergent, and do contribute to the constant term.
(7) $[2, 4, 3, 1, 2]$: The associated simple roots are $[1, 3, 4]$. This gives an intertwining operator that is absolutely convergent globally, and Eisenstein series that is also absolutely convergent. The archimedean intertwining operator is computed to vanish at $s = 5$, so this term does not contribute.

□

8.3. **The case $E = \mathbf{Q} \times F$.** In this case, G_E has a rational root system of type B_3. The simple roots (in a Euclidean coordinate system) are $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$, $\alpha_3 = (0, 0, 1)$. The parabolic subgroups M_j for $j = 1, 2, 3$ have the following Levi types: M_1 has Levi of type $D_{3,3}$ with rational root system of type B_2; M_2 is the Heisenberg Levi, isogenous to $\mathrm{GL}_2 \times \mathrm{GL}_{2,F}$; M_3 is isogoneous to $\mathrm{GL}_3 \times \mathrm{SO}_{2,F}$.

The constant term down to the Heisenberg Levi involves terms in $[W_{M_2}\backslash W/W_{M_2}]$, which has elements

(1) []
(2) [2]
(3) $[2, 3, 2]$
(4) $[2, 3, 1, 2]$
(5) $[2, 3, 1, 2, 3, 1, 2]$.

The constant term down to M_1 involves $[W_{M_1}\backslash W/W_{M_2}]$, which has elements

(1) []
(2) $[1, 2]$
(3) $[1, 2, 3, 2]$

The constant term down to M_3 involves $[W_{M_3}\backslash W/W_{M_2}]$, which has elements

(1) []
(2) $[3, 2]$

(3) $[3,2,3,1,2]$

One finds that $\rho_{P_0} = (3,2,1)$ and the highest root is $(1,1,0)$. We set $\lambda_s = (s-3, s-2, -1)$.

Proposition 8.3.1. *The constant term $E(g, f_4, s=5)_{N_1} = E_{M_1}(g, f_4(g, s=5)|_{M_1})$, an absolutely convergent Eisenstein series for the $[2]$ parabolic of M_1.*

Proof. We consider the elements of $[W_{M_1}\backslash W/W_{M_2}]$ one-by-one.

1. $[]$: The associated simple root is $[2]$. Applying the modulus character of the 2-parabolic of M_1 to $\alpha_2^\vee(t)$ gives $|t|^4$. As $\langle (s,s,0), \alpha_2^\vee\rangle = s$, at $s=5$ this Eisenstein series is absolutely convergent.
2. $[1,2]$: The associated simple root is $[3]$. The intertwining operator is absolutely convergent. Applying the modulus character $\delta_{1;3}$ of the 3-parabolic of M_1 to $\alpha_3^\vee(t)$ gives $|t|^4$, so $\langle \delta_{1;3}, \frac{1}{2}\alpha_3^\vee\rangle = 2$. Now, if $\lambda' = [1,2](\lambda_s) + \rho_{P_0}$, then $\langle \lambda', \frac{1}{2}\alpha_3^\vee\rangle = s-1$. As $5-1 = 4 > 2$, this Eisenstein series is absolutely convergent. One computes that the archimedean intertwiner vanishes at $s=5$, so this term does not contribute.
3. $[1,2,3,2]$: The associated simple root is $[2]$. The intertwining operator is absolutely convergent. Setting $\lambda' = [1,2,3,2](\lambda_s) + \rho_{P_0}$, one finds $\langle \lambda', \alpha_2^\vee\rangle = s-1$. So the Eisenstein series is at the reducibility point corresponding to the trivial representation. It is for a group associated to the Jordan algebra $J_2(F)$, so the pole is simple by [HS20]. Set $v_{1232}(s)$ the function of s from the archimedean intertwining operator. One finds $v_{1232}(s=5) = 0$ and $v_{1232}'(s=5) = (0,0,*)$. But the archimedean K-type is not trivial, so this term does not contribute.

□

We now consider the constant term to the 3-parabolic M_3 of $G_{\mathbf{Q}\times F}$.

Proposition 8.3.2. *The constant term $E(g, f_4, s=5)_{N_3} = E_{M_3}(g, f_4(g, s=5)|_{M_3})$, an absolutely convergent Eisenstein series on M_3 for the parabolic associated to the simple root $[2]$.*

Proof. The constant term down to M_3 involves $[W_{M_3}\backslash W/W_{M_2}]$. We consider the terms one-by-one.

1. $[]$: The associated simple root is $[2]$. One has $\langle (s,s,0), \alpha_2^\vee\rangle = s$ and $\langle \delta_{3;2}, \alpha_2^\vee\rangle = 3$. Thus this Eisenstein series is absolutely convergent.
2. $[3,2]$: The associated simple roots are $[1,2]$. The intertwining operator is globally absolutely convergent. Setting $\lambda' = [3,2](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \alpha_1^\vee\rangle = s-1$ and $\langle \lambda', \alpha_2^\vee\rangle = s-2$. The Eisenstein series converges absolutely [Art79, Lemma 4]. Let $v_{32}(s)$ be the archimedean multiplier. Then one has $v_{32}(s=5) = (0,0,0)$, so this term does not contribute.
3. $[3,2,3,1,2]$: The associated simple root is $[1]$. The intertwining operator is globally absolutely convergent. Setting $\lambda' = [3,2,2,1,2](\lambda_s) + \rho_{P_0}$, one has $\langle \lambda', \alpha_1^\vee\rangle = s-2$. Thus this Eisenstein series will be at the boundary of absolute convergence. Let $v_{32312}(s)$ be the archimedean multiplier,
$$v_{32312}(s) = A_1 d(s-3)^2 A_1^{-1} d(2s-5) A_1 d(s-2)^3 A_1^{-1} d(s-1) A_1 (0,0,1)^t.$$
One has $v_{32312}(s=5) = 0$ and $v_{32312}'(s) = (0,0,*)$. But now mirabolic Eisenstein series on GL_n have simple poles at the modulus character point, with one-dimensional residue. As the archimedean K-type does not contain the trivial representation of $\mathrm{SO}(3) \subseteq \mathrm{SL}_3(\mathbf{R})$, this Eisenstein series will vanish at $s=5$.

□

Finally, we consider the constant term down to the Heisenberg parabolic.

Proposition 8.3.3. *One has $E(g, f_4, s=5)_{N_2} = f_4(g, s=5) + E_{\mathrm{GL}_{2,E}}(g, f_4^{[2]})$, where $f_4^{[2]} = M([2])(f_4(g, s=5))$. Both the intertwining operator and the Eisenstein series are (globally) absolutely convergent.*

Proof. The constant term down to the Heisenberg Levi involves terms in $[W_{M_2}\backslash W/W_{M_2}]$. We consider the elements one-by-one.

(1) []: The associated simple roots are [] (empty). Here we have the inducing section, which does contribute to the constant term.
(2) [2]: The associated simple roots are $[1,3]$. The intertwining operator is globally absolutely convergent. So is the associated Eisenstein series.
(3) $[2,3,2]$: The associated simple roots are [1]. The intertwining operator is globally absolutely convergent, and the Eisenstein series is as well. One finds that the archimedean multiplier $v_{232}(s)$ vanishes at $s=5$.
(4) $[2,3,1,2]$: The associated simple roots are $[1,3]$. The intertwining operator is globally absolutely convergent. The Eisenstein series again is absolutely convergent. One finds that the archimedean multiplier $v_{2312}(s)$ vanishes at $s=5$, so this term does not contribute.
(5) $[2,3,1,2,3,1,2]$: The associated simple roots are [] (empty). The intertwining operator is locally absolutely convergent, and globally has a simple pole at $s=5$. One finds that the archimedean multiplier vanishes to order 2 at $s=5$, so this term also does not contribute.

□

9. Main theorems

We now come to the Siegel-Weil theorems. Throughout, we normalize Haar measure on the groups S_E so that $S_E(\mathbf{Q})\backslash S_E(\mathbf{A})$ has measure 1; this is the Tamagawa measure. The proofs of the results in this section follows the strategy of [Gan00b].

To help orient the reader, we now outline the proof of Theorem 1.1.1 in the special case where the archimedean data defining the theta lift $\Theta_f(\mathbf{1})(g)$ and the Eisenstein series $E(g,\overline{f})$ on G_E is as simple as possible, specifically, when this archimedean data is in a certain minimal K-type on $G_J(\mathbf{R})$. See Theorem 9.4.1 below. In subsection 9.5, we explain how using representation theoretic results of [GS05] and [HPS96] this minimal case implies the general case.

To setup the result, suppose $f(g,s)\in I_{G_J}(s)$ is a flat section, with vector-valued archimedean component fixed as in subsection 5.3. Let $\overline{f}(g) = Res_{s=24}M(w_0)f(g,s)$. Restricting $\overline{f}$ to G_E, one obtains an element in $I_{G_E}(s=5)$. Extending this to a flat section $\overline{f}(g,s)$, one can define the Eisenstein series $E_{G_E}(g,\overline{f},s) = \sum_{\gamma \in P_E(\mathbf{Q})\backslash G_E(\mathbf{Q})} \overline{f}(\gamma g,s)$, where P_E is the standard Heisenberg parabolic of G_E. The sum converges absolutely for $Re(s)>5$, and we proved in section 8 that the Eisenstein series is regular at $s=5$.

The Siegel-Weil Eisenstein series is defined as $E_{G_E}(g,\overline{f}) := E_{G_E}(g,\overline{f},s=5)$. The theta lift is

$$\Theta_f(\mathbf{1})(g) = \int_{S_E(\mathbf{Q})\backslash S_E(\mathbf{A})} \Theta_f(g,h)\,dh.$$

Theorem (See Theorem 9.4.1 below)**.** *With notation as above, one has an identity* $\Theta_f(\mathbf{1})(g) = E_{G_E}(g,\overline{f})$.

This theorem asserts an identity between two automorphic functions on G_E. Here is an outline of how this identity is proved.

(1) **Pure tensors**: It suffices to prove the identity of automorphic forms when the inducing section f is a pure tensor.
(2) **Use of representation theory**: Fix a prime p, which for reasons later we assume is such that G_E is split over $\mathbf{Q}_p$. Write the inducing section f_{fte} as a product $f_{fte}(g'g_p) = f_{fte}^p(g')f_p(g_p)$, where $g_p \in G_E(\mathbf{Q}_p)$ and $g' \in G_E(\mathbf{A}_f^{\{p\}})$, $\mathbf{A}_f^{\{p\}}$ the finite adeles away from p. To prove the Siegel-Weil identity for this single inducing section, we consider it as one element of a family of inducing sections, where f_{fte}^p is fixed but f_p is allowed to vary. Let σ_p be the $G_E(\mathbf{Q}_p)$ representation formed on the set of Siegel-Weil Eisenstein series $E(g,\overline{f})$

for these varying f_{fte}, and let τ_p be the $G_E(\mathbf{Q}_p)$ representation formed on the theta lifts $\Theta_f(\mathbf{1})(g)$ for these varying f_{fte}.

(3) **Base case**: Now, by computing constant terms of $\Theta_f(\mathbf{1})(g)$ and $E(g,\overline{f})$, one sees that the difference $E(g,\overline{f}) - \Theta_f(\mathbf{1})(g)$ is a cusp form. This step uses "smaller" Siegel-Weil theorems, i.e., Siegel-Weil theorems for dual pairs $M \times S_E$ with $M \subseteq G_E$. In other words, to prove the Siegel-Weil theorem, it suffices to prove that the cuspidal projection $\mathcal{P}_C(\Theta_f(\mathbf{1})(g))$ of $\Theta_f(\mathbf{1})(g)$ is equal to 0. Thus, it suffices to prove that the representation τ_p does not have a quotient appearing in the space of cusp forms.

(4) **Injectivity of Eisenstein projection**: The Eisenstein projection defines a map $\tau_p \to \sigma_p$. We prove that this map is injective. This is a key step, which we further outline momentarily.

(5) **The Siegel-Weil theorem**: Because of σ_p is a subquotient of $Ind_{G_E(\mathbf{Q}_p)}(s=5)$, and $\tau_p \subseteq \sigma_p$, τ_p is also a subquotient of this induced representation. When $G_E(\mathbf{Q}_p)$ is split, it is then easy to see that none of the subquotients of $Ind_{G_E(\mathbf{Q}_p)}(s=5)$ can appear in the space of cusp forms.

So, one key step in the above argument is proving the injectivity of the map $\tau_p \to \sigma_p$. We sketch this implication now.

(1) **Identification of the Eisenstein projection**: Because, as mentioned above, the difference $E(g,\overline{f}) - \Theta_f(\mathbf{1})(g)$ is a cusp form, the Eisenstein projection of $\Theta_f(\mathbf{1})(g)$ is $E(g,\overline{f})$.

(2) **Linearity**: We wish to show that the Eisenstein projection is injective. By linearity, this means that we must check that if $E(g,\overline{f}) = 0$, as an automorphic form, then $\Theta_f(\mathbf{1})(g) = 0$, as an automorphic form. Note that $E(g,\overline{f}) = 0$ implies that $\Theta_f(\mathbf{1})(g)$ is a cusp form. Assume from now on that $\Theta_f(\mathbf{1})(g)$ is a cusp form.

(3) **No singular cusp forms**: Because $\Theta_f(\mathbf{1})(g)$ is a quaternionic cuspidal modular form, all of its *degenerate* Fourier coefficients along the Heisenberg parabolic subgroup are 0. This is a consequence of the analysis done in [Pol20a] of the generalized Whittaker model of quaternionic modular forms: The generalized Whittaker model attached to degenerate characters of the Heisenberg parabolic are unbounded as functions on $G_E(\mathbf{R})$, and thus can't appear for cusp forms. So, to prove that $\Theta_f(\mathbf{1})(g) = 0$, it suffices to prove that all of its non-degenerate Fourier coefficients are equal to 0.

(4) **Twisted Jacquet modules**: Non-degenerate Fourier coefficients of $\Theta_f(\mathbf{1})(g)$ can be interpreted as linear functionals on the minimal representation $V_{min,q}$ that have certain equivariance properties. The work done in section 6, using many results of [Sav94, MS97, GS05] on models/properties of the minimal representation, proves that the space of such linear functionals is one-dimensional. *Consequently, if we can write down any explicit such nonzero linear functional, this explicit linear functional will control the Fourier coefficients of* $\Theta_f(\mathbf{1})(g)$. Proposition 6.3.1 exactly gives such a functional L_χ.

(5) **Finishing the argument**: We can now put the pieces together. Suppose $E(g,\overline{f}) = 0$. Then, it is not hard to show that the inducing section itself $\overline{f}(g)$, restricted to G_E, is 0. (One takes constant terms.). But now, if the inducing section $\overline{f}|_{G_E}$ itself is 0, one sees that $L_\chi(g_q \cdot f) = 0$ for $g_q \in G_E(\mathbf{Q}_q)$, by the formula for the explicit linear functional. But then, this means that the non-degenerate Fourier coefficients of $\Theta_f(\mathbf{1})(g)$ are equal to 0, by the "Twisted Jacquet module" step; here it is helpful that f_{fte} is assumed to be a pure tensor. Thus $\Theta_f(\mathbf{1})(g) \equiv 0$, by the "No singular cusp forms" step.

We now proceed with the rest of the section, first stating and proving the "smaller" Siegel-Weil theorems for the dual pairs $M \times S_E$. The proof of these smaller Siegel-Weil theorems proceeds similarly to the argument just sketched for $G_E \times S_E$.

9.1. **The case $G_{2,F}$.** Suppose F is a totally real quadratic étale extension of $\mathbf{Q}$. Recall we have a map $G_{2,F} \times S_{\mathbf{Q}\times F} \to G_6$. Let $f(g,s) \in I_{G_6}(s)$ be a flat section with archimedean part fixed as in subsection 4.3 and $\Theta_f \in \mathcal{A}(G_6)$ the associated element of the automorphic minimal representation. Let $\overline{f}(g) = Res_{s=6} M(w_0) f(g,s)$. Set

$$\Theta_f(\mathbf{1})(g) = \int_{S_E(\mathbf{Q})\backslash S_E(\mathbf{A})} \Theta_f(g,h)\,dh$$

the theta lift of the trivial representation. Finally, let $E_{G_{2,F}}(g,\overline{f})$ be the absolutely convergent Eisenstein series on $G_{2,F}$ for the parabolic $P_{2,F}$ (stabilizing an isotropic line in $V_{4,F} = H \oplus F$.)

Theorem 9.1.1. *The theta lift* $\Theta_f(\mathbf{1})(g) = E(g,\overline{f})$

Proof. First consider the case that F is a field.

Observe that $\Theta_f(g)_{N_{2,F}} = \Theta_f(g)_{N_{G_6,1}}$. This follows from the fact that Θ_f only has rank 0 and rank one Fourier coefficients along $N_{G_6,1}$. Now, applying Proposition 4.3.2 and Proposition 7.1.1, we observe that $\Theta_f(\mathbf{1})(g)$ and $E(g,\overline{f})$ have the same constant term to $P_{2,F} = M_{2,F}N_{2,F}$. Because $E(g,\overline{f})$ is orthogonal to cusp forms, we obtain that the Eisenstein part of $\Theta_f(\mathbf{1})(g)$ is $E(g,\overline{f})$.

Let S be an arbitrary finite set of finite primes. For $v \in V_{min,S}$, let $\overline{f}_v$ be the associated element of $I_{S,G_{2,F}}(s=4)$. Let χ be a non-degenerate unitary character of $N_{2,F}(\mathbf{A}_S)$. Set

$$L_\chi(v) = \int_{N_{2,F}(\mathbf{A}_S)} \overline{f}_v(w_0 n)\chi^{-1}(n)\,dn.$$

By Proposition 6.2.3 and Proposition 6.3.1, L_χ is the unique nonzero χ-linear functional on the $S_{\mathbf{Q}\times F}(\mathbf{A}_S)$ coinvariants of $V_{min,S}$, up to scalar multiple.

Claim 9.1.2. *Suppose $f \in I_{G_6,fte}(s=6)$, and $\varphi = \Theta_f(\mathbf{1})(g)$. If the Eisenstein projection of φ is 0, equivalently, if φ is cuspidal, then $\varphi = 0$.*

Proof. Let χ be a non-degenerate unitary character of $N_{2,F}(\mathbf{Q})\backslash N_{2,F}(\mathbf{A})$. Suppose $\varphi = \Theta_f(\mathbf{1})$ is nonzero, and $\varphi_\chi(g_S g_f^S g_\infty) \neq 0$, for some $g = g_S g_f^S g_\infty \in G_{2,F}(\mathbf{A})$. Here S is a finite set of finite places, and g_S, respectively g_f^S, denote the component of g_f at the places in S, respectively away from S. We choose S large enough so that

(1) $f = f_S \otimes f^S$ is a tensor
(2) f^S is the normalized spherical vector
(3) $g_f^S \in K^S$, the product of the hyperspecial maximal compact subgroups of G_6 away from S. (Recall that G_6 is split at every finite place.)

With S this large, we have $\varphi_\chi(g) = \varphi(g_S g_\infty) \neq 0$.

Fix the normalized spherical vectors away from S. This gives an embedding $V_{min,S} \to V_{min,f}$. Taking χ-Fourier coefficients of theta lifts and evaluating at g_∞ then gives a linear map $M_\chi : V_{min,S} \to \mathbf{C}$. We have $M_\chi(g_S\overline{f}) \neq 0$. The linear map factors through the $S_{\mathbf{Q}\times F}(\mathbf{A}_S)$-coinvariants of $V_{min,S}$. Thus there is a nonzero constant $c_\chi(g_\infty)$ so that $M_\chi(v) = c_\chi(g_\infty)L_\chi(v)$ for all $v \in V_{min,S}$.

Now suppose that φ is cuspidal, or equivalently, that its Eisenstein projection is 0. Taking the constant term of the Eisenstein series, we see that $\overline{f}|_{G_{2,F}(\mathbf{A})} \equiv 0$. But $\overline{f}$ is spherical outside S, and of our special form at infinity. Consequently, the away from S part of $\overline{f}$ is nonzero at the identity. Consequently, $\overline{f}_S(x_S) = 0$ for all $x_S \in G_{2,F}(\mathbf{A}_S)$. We obtain that $L_\chi(x_S\overline{f}_S)$ is identically 0. This contradicts the nonvanishing of $\varphi_\chi(g_S g_\infty)$.

We conclude that all of φ's non-degenerate Fourier coefficients are equal to 0, so $\varphi = 0$. □

Now fix p to be a split place of F. Fix inducing data in $I_{G_6}(s)$ away from p, and we let inducing data at p vary. The theta lift then gives a linear map $I_{G_6,p}(s=6) \to \mathcal{A}(G_{2,F})$. This map is $G_{2,F}(\mathbf{Q}_p)$-intertwining and factors through the coinvariants $(V_{min,p})_{S_{\mathbf{Q}\times F}(\mathbf{Q}_p)}$. Let τ_p be the image of the map.

Similarly, fixing the same data away from p, the absolutely convergent Eisenstein series gives a map $Eis: I_{G_{2,F},p}(s=4) \to \mathcal{A}(G_{2,F})$. Let σ_p denote the image of this map. Note that, by Theorem 3.5.1, $I_{G_{2,F},p}(s=4)$ is irreducible so $\sigma_p \simeq I_{G_{2,F},p}(s=4)$ or is 0.

Because the Eisenstein projection of the theta lift is the Siegel-Weil Eisenstein series, we obtain an equivariant map $\tau_p \to \sigma_p$. From Claim 9.1.2, this map is injective.

Because σ_p is irreducible or 0, we obtain $\tau_p \simeq \sigma_p \simeq I_{G_{2,F},p}(s=4)$ or $\tau_p = 0$. But $I_{G_{2,F},p}(s=4)$ is not unitarizable, by Theorem 3.5.1. Consequently, the cuspidal projection of τ_p is 0. Consequently the Eisenstein projection on τ_p is the identity, which proves the theorem in case F is a field.

The case of $F = \mathbf{Q} \times \mathbf{Q}$ goes through similarly to the case when F is a field, by applying the following lemma. □

Lemma 9.1.3. *Let Z denote the center of the Heisenberg unipotent radical $N_{G_2,2}$ on G_6. Then $\Theta_{f,Z} \equiv \Theta_{f,N_{G_6,2}}$.*

Proof. To prove this, one again only needs to use that Θ_f has only rank 0 and rank 1 Fourier coefficients along $N_{G_6,1}$. □

9.2. **The case of $G_{3,F}$.** Let F be a quadratic étale extension of $\mathbf{Q}$ that is totally real. Recall that we have the map $G_{3,F} \times S_{\mathbf{Q}\times F} \to G_7$.

Suppose $f(g,s) \in I_{G_7}(s)$ is a flat section, with our fixed vector-valued archimedean component. Let $\Theta_f(g) = Res_{s=7}E(g,f,s)$ be the associated theta function on G_7. As usual, we set $\overline{f}(g) = Res_{s=7}M(w_0)f(g,s)$. Let $E_{G_{3,F}}(g,\overline{f})$ be the absolutely convergent Siegel-Weil Eisenstein series (see subsection 7.2) associated to the parabolic $P_{G_3} \subseteq G_{3,F}$ that stabilizes an isotropic line in $V_{6,F}$.

Theorem 9.2.1. *With notation as above, $\Theta_f(\mathbf{1})(g) = E_{G_{3,F}}(g,\overline{f})$.*

Proof. Using the work in subsections 4.3, 7.2, 6.2, and 6.3, the proof is nearly identical to the proof of Theorem 9.1.1. We only explain the additional ingredients that are used:

To check that $\Theta_f(\mathbf{1})(g) - E_{G_{3,F}}(g,\overline{f})$ is cuspidal, besides the computations of 4.3 and 7.2, one also uses the Siegel-Weil theorem for $G_{2,F}$, i.e., Theorem 9.1.1. (One has to apply the work in subsection 3.6 to move between isogenous groups.)

One extra point that must be checked is that a cusp form φ in the image of the theta lift cannot be singular, i.e., if all of its non-degenerate Fourier coefficients of φ are 0, then $\varphi = 0$. In fact, it is true that in this case, φ *only* has non-degenerate Fourier coefficients along the unipotent radical of the parabolic $P_{G_3,1}$. To see this, observe that any theta function Θ_f on G_7 is a *modular form* in the sense of [Pol22]. This follows from Theorem 7.0.1 of [Pol22]. Then the theta lift $\Theta_f(\mathbf{1})(g)$ is again a modular form on $G_{3,F}$. (In [Pol22], we only worked with groups of the form $\mathrm{SO}(3,n)$ with $n \geq 4$, but everything carries over line-by-line for the case $n=3$.) But then from Theorem 3.2.4 of [Pol22], if φ is a cuspidal modular form, it can only have non-degenerate Fourier coefficients; this is because the generalized Whittaker functions of that theorem are unbounded for degenerate characters. □

9.3. **The case of $\mathrm{SL}_{2,E}$.** Let E be a totally real cubic étale $\mathbf{Q}$-algebra. We have the map of groups $\mathrm{SL}_{2,E} \times S_E \to H_J^1$. Let $P_3 = P_{H_J^1,3}$ be the Siegel parabolic subgroup of H_J^1, and $I_{H_J^1}(s) = Ind_{P_3}^{H_J^1}(|\lambda|^s)$ be the induced representation studied in subsection 5.2.

Suppose $f(g,s) \in I_{H_J^1}(s)$ is a flat section, with archimedean part fixed as in subsection 5.2. Let $\Theta_f \in \mathcal{A}(H_J^1)$ be the associated element of the automorphic minimal representation. For $g \in \mathrm{SL}_{2,E}(\mathbf{A})$, we have the theta lift

$$\Theta_f(\mathbf{1})(g) = \int_{S_E(\mathbf{Q})\backslash S_E(\mathbf{A})} \Theta_f(g,h)\,dh.$$

On the other hand, our Siegel-Weil Eisenstein series is $E_{\mathrm{SL}_{2,E}}(g, \overline{f})$. Here $\overline{f} = Res_{s=14} M(w_0) f(g,s)$ and the sum defining the Eisenstein series is over $B_{2,E}(\mathbf{Q})\backslash \mathrm{SL}_{2,E}(\mathbf{Q})$, where $B_{2,E}$ is the standard Borel subgroup. The sum is absolutely convergent.

The Siegel-Weil theorem is:

Theorem 9.3.1. *We have an identity* $\Theta_f(\mathbf{1})(g) = E_{\mathrm{SL}_{2,E}}(g, \overline{f})$.

Proof. The structure of the proof is the same as for Theorems 9.1.1 and 9.2.1. We only highlight the additional ingredients that are needed for Theorem 9.3.1.

We leave the computation of the constant terms of the Siegel-Weil Eisenstein series $E_{\mathrm{SL}_{2,E}}(g, \overline{f})$ to the reader, as this is an SL_2 calculation. The following claims are used to compute the constant terms of $\Theta_f(\mathbf{1})(g)$.

Recall $J = E \oplus V_E$.

Claim 9.3.2. *Over our global field* $\mathbf{Q}$*, if* $X \in J$ *has rank at most one, and* $X \in V_E$*, then* $X = 0$.

Proof. One has $\mathrm{tr}(V)^2 - \mathrm{tr}(V^2) = 2\,\mathrm{tr}(V^\#)$ for all $V \in J$. Thus if V is rank at most one, $\mathrm{tr}(V)^2 = \mathrm{tr}(V^2)$. If $V \in V_E$, then $\mathrm{tr}(V) = 0$, so $\mathrm{tr}(V^2) = 0$. But this form is positive definite on J, so $V = 0$. □

When E is a field, Claim 9.3.2 is enough to finish the proof of the Siegel-Weil theorem. We now consider the case when $E = \mathbf{Q} \times F$ with F quadratic étale.

Claim 9.3.3. *Suppose* $V \in J$ *is rank one, with* $c_1(V) = 0$*. Then* $x_2(V) = x_3(V) = 0$ *as well.*

Proof. This follows from the fact that $0 = c_j(V^\#)$ for $j = 2, 3$. □

Claim 9.3.4. *Let* Z *denote the root space of* H_J^1 *corresponding to* $e_{11} \in J$*. Let* $P_1 = M_1 N_1$ *denote the* $D_{6,2}$ *standard parabolic of* H_J^1*. If* Θ_f *is a theta function on* H_J^1*, then* $(\Theta_f)_Z = \Theta_{N_1}$.

Proof. The unipotent radical $N_1 = (XY)Z$ is a Heisenberg group. Here $YZ = N_1 \cap N_3$ and $X = N_1 \cap M_3$. Claim 9.3.3 implies $(\Theta_f)_Z = (\Theta_f)_{YZ}$. Now, there is an element $J_4 \in \mathrm{Sp}_4 \subseteq M_1$ that exchanges X with Y, so this proves that $(\Theta_f)_Z$ is also invariant by X. The claim follows. □

By Theorem 9.1.1 and our computation of $\Theta_f(g)_{N_1}$ in subsection 5.2, we can now compute the constant term $\Theta_f(\mathbf{1})(g)_Z$ in terms of Eisenstein series on $G_{2,F}$.

Let $N_{2,F}$ be the unipotent radical of the standard Borel of $\mathrm{SL}_{2,F}$, thought of as sitting inside $\mathrm{SL}_{2,E} = \mathrm{SL}_2 \times \mathrm{SL}_{2,F}$. We now consider the constant term of $\Theta_f(g)$ along N_F.

Claim 9.3.5. *Suppose* $V \in J$ *is rank one, and* V *is orthogonal to* $F \hookrightarrow H_2(\Theta)$*. Then* $V \in \mathbf{Q}e_{11}$.

Proof. Let U denote the image of V under the linear projection $J \to H_2(\Theta)$. Then if $U \neq 0$, the quadratic norm of U is negative, contradicting the fact that the c_1 coordinate of $V^\#$ is 0. Thus $U = 0$. Now one considers the c_2 and c_3 coordinate entry of $V^\# = 0$ to deduce $x_2(V) = x_3(V) = 0$. □

Let $P_2 = M_2 N_2$ be the standard $D_{5,1} \times \mathrm{SL}_2$ parabolic subgroup of H_J^1. Recall that we have $\mathrm{GL}_3 \subseteq \mathrm{Sp}_6 \subseteq H_J^1$. Let $\gamma \in \mathrm{GL}_3(\mathbf{Q})$ be the permutation matrix for which $Ad(\gamma)(\mathrm{SL}_{2,F})$ acts trivially on e_3, f_3 and $Ad(\gamma)(\mathrm{SL}_2)$ acts trivially on e_1, e_2, f_2, f_1. Here $e_1, e_2, e_3, f_3, f_2, f_1$ is the standard basis of Sp_6.

Conjugating the statement of Claim 9.3.5 by γ, it can be used to prove that $\Theta_f(g)_{\gamma \cdot N_F} = \Theta_f(g)_{N_2}$. For $g \in \mathrm{SL}_{2,E}(\mathbf{A})$ and $h \in S_E(\mathbf{A})$, we then have

$$\Theta_f(g,h)_{N_F} = \Theta_f(\gamma(g,h))_{\gamma \cdot N_F} = \Theta_f((\gamma h \gamma^{-1})\gamma g)_{N_2}.$$

The constant term $\Theta_f(x)_{N_2} = E_{\mathrm{SL}_2}(x, \overline{f})$, an SL_2-type Eisenstein series, see Proposition 5.2.4. We obtain

$$\Theta_f(g,h)_{N_F} = \sum_{B(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{Q})} \overline{f}(\mu \gamma g).$$

But $\overline{f}(\gamma^{-1}x) = \overline{f}(x)$, so the sum above is the N_F-constant term of the Siegel-Weil Eisenstein series.

The case of the Siegel-Weil theorem when F is a field now follows by the argument of Theorem 9.1.1. The case when $F = \mathbf{Q} \times \mathbf{Q}$ also follows, this time using the outer S_3 (symmetric group) action: If $\tau \in S_3 \subseteq H^1_J(\mathbf{Q})$, then

$$\Theta_f(\mathbf{1})(\tau g \tau^{-1}) = \int_{[S_E]} \Theta_f(\tau g \tau^{-1} h)\, dh = \int_{[S_E]} \Theta_f(\tau g h \tau^{-1})\, dh = \Theta_{\tau^{-1} f}(\mathbf{1})(g)$$

where we have changed variables in the integral. □

9.4. **The case of G_E.** We now state and prove the Siegel-Weil theorem for the groups $G_E \times S_E \to G_J$. To setup the result, suppose $f(g,s) \in I_{G_J}(s)$ is a flat section, with vector-valued archimedean component fixed as in subsection 5.3. Let $\overline{f}(g) = Res_{s=24} M(w_0) f(g,s)$. The Siegel-Weil Eisenstein series $E_{G_E}(g, \overline{f})$ is defined in section 8. Restricting $\overline{f}$ to G_E, one obtains an element in $I_{G_E}(s=5)$. Extending this to a flat section $\overline{f}(g,s)$, one can define the Eisenstein series $E_{G_E}(g, \overline{f}, s) = \sum_{\gamma \in P_E(\mathbf{Q}) \backslash G_E(\mathbf{Q})} \overline{f}(\gamma g, s)$, where P_E is the standard Heisenberg parabolic of G_E. The sum converges absolutely for $Re(s) > 5$, and we proved in section 8 that the Eisenstein series is regular at $s = 5$.

Because the Eisenstein series is regular for all flat sections with our fixed special archimedean component, the Eisenstein map gives a $G_E(\mathbf{A}_f)$-intertwining map $I_{G_E,fte}(s=5) \to \mathcal{A}(G_E)$. The Siegel-Weil Eisenstein series is defined as $E_{G_E}(g, \overline{f}) := E_{G_E}(g, \overline{f}, s=5)$.

The theta lift is

$$\Theta_f(\mathbf{1})(g) = \int_{S_E(\mathbf{Q}) \backslash S_E(\mathbf{A})} \Theta_f(g,h)\, dh.$$

Theorem 9.4.1. *With notation as above, one has an identity $\Theta_f(\mathbf{1})(g) = E_{G_E}(g, \overline{f})$.*

Proof. Our first task is to compute the constant terms of $\Theta_f(\mathbf{1})(g)$ along the maximal parabolic subgroups of G_E, so that we may prove that $\Theta_f(\mathbf{1})(g) - E_{G_E}(g, \overline{f})$ is cuspidal.

We begin with:

Claim 9.4.2. *Let $V \in W_J$ be rank at most one, and suppose V has 0 projection to W_E. Then $V = 0$.*

Proof. This claim reduces immediately to Claim 9.3.2. □

It follows that the constant term $\Theta_f(g)_{N_E} = \Theta_f(g)_{N_J}$ where N_E is the unipotent radical of the standard Heisenberg parabolic subgroup of G_E and N_J is the unipotent radical of the standard Heisenberg parabolic subgroup of G_J.

Let $Q_E = L_E V_E$ be the long root GL_2 parabolic subgroup of G_E, in its G_2-root system. Let $Q_J = L_J V_J$ be the standard maximal parabolic subgroup of G_J with simple root α_2 in its unipotent radical, so that the Levi subgroup L_J is isogenous to $\mathrm{GL}_2 \times M^1_J$.

Claim 9.4.3. *The constant term of $\Theta_f(g)_{V_E} = \Theta_f(g)_{V_J}$.*

Proof. Again, one uses Claim 9.3.2 to prove this. Note that we use the fact that Θ_f has only rank 0 and rank 1 Fourier coefficients along N_J and a conjugate of N_J. This can be justified using the identity $\varphi_{(N_J,\chi)}(\gamma g) = \varphi_{(N_J \cdot \gamma, \chi \cdot \gamma)}(g)$ of global Fourier coefficients of an automorphic form φ on G_J. Here $\varphi_{(N_J,\chi)}$ is the χ-Fourier coefficient of φ along N_J. □

In case E is a field, we can now conclude that $\Theta_f(\mathbf{1})(g) - E_{G_E}(g, \overline{f})$ is cuspidal, using our work from subsections 5.3 and 8.1 and Theorem 9.3.1.

Now suppose $E = \mathbf{Q} \times F$ with F quadratic étale. Let $P_4 = M_4 N_4$ be the standard parabolic subgroup of G_J with simple root α_4 in its unipotent radical. One can visualize this parabolic in the F_4 root system, using Remark 9.4.7 below. Let $P_{G_E,1} = M_{G_E,1} N_{G_E,1}$ be the standard parabolic of G_E with the first simple root in its unipotent radical, in the ordering of simple roots given in subsection 8.3. Thus $P_{G_E,1}$ stabilizes an isotropic line in the representation $V_{6,F} = H^2 \oplus F$ and $M_{G_E,1}$ has absolute Dynkin type D_3. One has that $P_{G_E,1} = P_4 \cap G_E$ and $N_{G_E,1} = N_4 \cap G_E$.

Now, one has the identity of constant terms $\Theta_f(g)_{N_{G_E,1}} = \Theta_f(g)_{N_4}$. To prove this, use Claim 9.3.5 and the following two claims.

Claim 9.4.4. *Suppose $V \in J$ is rank one, $c_1(V) = 0$, and $V^\# \in \mathbf{Q}e_{11}$. Then $V \in H_2(\Theta)$.*

Claim 9.4.5. *Suppose φ is a quaternionic modular form on G_J, $\chi : N_J(\mathbf{Q})\backslash N_J(\mathbf{A}) \to \mathbf{C}^\times$ is a character, and $\varphi_\chi(g)$ the corresponding Fourier coefficients. Suppose $u \in H_J^1(\mathbf{A})$ is unipotent and stabilizes χ. Then $\varphi_\chi(ug) = \varphi_\chi(g)$.*

Proof. We leave the proof of the first claim to the reader. For the second claim, one uses the fact that main theorem of [Pol20a] for the formula for the generalized Whittaker function implies that $\varphi_\chi(ug) = \varphi_\chi(g)$ if u is purely archimedean, and then the general case follows by an approximation argument. □

Note that Claim 9.4.5 can be applied to $\varphi = \Theta_f$, because Θ_f is a quaternionic modular form in the sense of [Pol20a]. Indeed, for the vector that is spherical at finite places, this is proved in [Pol20b]; the general cases follows because the map $f \mapsto \Theta_f$ is $G_J(\mathbf{A}_f)$-intertwining. (One can also use [Gan00a, Proposition 6.4] in place of Claim 9.4.5.)

Using Theorem 9.2.1 and the results of subsections 5.3 and 8.3, 8.2, we now have that the constant term of $\Theta_f(\mathbf{1})(g) - E_{G_E}(g, \overline{f})$ along $N_{G_E,1}$ vanishes when $E = \mathbf{Q} \times F$.

Suppose now that F is a field. Then G_E has a maximal parabolic subgroup $P_{G_E,3} = M_{G_E,3}N_{G_E,3}$ with the third simple root (in the numbering of subsection 8.3) in the unipotent radical. This is the standard maximal parabolic with Levi subgroup $M_{G_E,3}$ isogenous to $\mathrm{GL}_3 \times \mathrm{SO}_{2,F}$. The analysis of the constant term $\Theta_f(\mathbf{1})(g)_{N_{G_E,3}}$ is similar to that of $\Theta_f(\mathbf{1})(g)_{N_{G_E,1}}$. To do the computation, it is easiest to first consider E as $E = F \times \mathbf{Q}$ instead of $\mathbf{Q} \times F$, and then use a conjugation argument as in the proof of Theorem 9.3.1.

Finally, we must consider the case where $E = E_{sp} = \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$. But to handle the constant terms of $\Theta_f(\mathbf{1})(g)$ and $E_{G_E}(g, \overline{f})$ in this case, we can use triality to bootstrap off of the constant terms already computed above. Specifically, if $\tau \in C_3 \subseteq S_3 \subseteq G_J(\mathbf{Q})$, then

$$\Theta_f(\mathbf{1})(\tau g \tau^{-1}) = \int_{[S_{E_{sp}}]} \Theta_f(\tau g \tau^{-1} h)\, dh = \int_{[S_{E_{sp}}]} \Theta_f(\tau g h \tau^{-1})\, dh = \Theta_{\tau^{-1}f}(\mathbf{1})(g)$$

where we have used triality for $S_{E_{sp}}$ to make a change of variables in the integral. One also has $E(\tau g \tau^{-1}, \overline{f}) = E(g, \overline{\tau^{-1} f})$, using that the Heisenberg parabolic $P_E(\mathbf{Q})$ is stable by τ and $\overline{f}$ is left-invariant by τ.

Combining the above work, we have now proved that $\Theta_f(\mathbf{1})(g) - E_{G_E}(g, \overline{f})$ is cuspidal in all cases. To finish the proof, we make a slightly different representation-theoretic argument compared to the proofs of the other Siegel-Weil theorems, because this time the induced representation $I_{G_E,p}(s = 5)$ is reducible.

Fix a split place p for E as usual. Note that $f \mapsto \Theta_f$ is an intertwining map, and so is the Siegel-Weil Eisenstein series, as was remarked above. Fix inducing data in $I_{G_J}(s = 24)$ away from p. Let τ_p be the p-adic rerepresentation on G_E coming from the theta lift, and σ_p the p-adic representation coming from the Eisenstein series map $I_{G_E,p}(s = 5) \to \mathcal{A}(G_{4,E})$.

If χ is a non-degenerate unitary character of $N_E(\mathbf{Q}_q)$ for an arbitary finite prime q, recall the functional

$$L_\chi(g_q \overline{f}) = \int_{N_E(\mathbf{Q}_q)} \chi^{-1}(n) \overline{f}(w_0 n g_q)\, dn.$$

By Proposition 6.3.1, L_χ is not identically 0 on $V_{min,q}$. Thus, by the argument of the proof of Theorem 9.1.1, we have an injection $\tau_p \hookrightarrow \sigma_p$, the map given by the Eisenstein projection. We explain this in detail.

Claim 9.4.6. *The Eisenstein projection induces an injection $\tau_p \to \sigma_p$.*

Proof. We have a theta function $\Theta_f(g)$, for which $E(g,\overline{f}) \equiv 0$. We wish to show that $\Theta_f(g) \equiv 0$.

Let S be a sufficiently large finite set of places so that $f_{fte}(g,s) = f^S(g^S)f_S(g_S)$ is a pure tensor and f^S is spherical on $G_J(\mathbf{A}_f^S)$, the finite adeles away from S. We have that $E(g,\overline{f}) \equiv 0$. Taking the constant term of this Eisenstein series, it is not hard to deduce from Proposition 8.3.3 that $\overline{f}(g) \equiv 0$. It follows that $f_S(g_S) \equiv 0$, because $f^S(1) \neq 0$ and $f_\infty(g_\infty) \neq 0$. It follows that

$$\int_{N_E(\mathbf{Q}_S)} f_S(w_0 n g_S)\chi^{-1}(n)\, dn \equiv 0$$

for any non-degenerate unitary character of χ of $N_E(\mathbf{Q}_S)$. But by Proposition 6.2.7 and Proposition 6.3.1, we deduce that the χ-Fourier coefficient $\Theta_{f,\chi}(g) \equiv 0$ for every non-degenerate character $\chi : N_E(\mathbf{Q})\backslash N_E(\mathbf{A}) \to \mathbf{C}^\times$. Finally, it is a consequence of the main theorem of [Pol20a] that any such cusp form must be identically 0. This proves the claim. □

Now, σ_p is a quotient of $I_{G_E,p}(s=5)$, and τ_p is a subquotient of this representation. If τ_p is 0, there is nothing to prove, so we may assume $\tau_p \neq 0$. The representation $I_{G_E,p}(s=5)$ has a nonsplit composition series of length two, with subrepresentation denoted V and uniuqe irreducible quotient the trivial representation. Thus either $\tau_p = V$, $\tau_p = I_{G_E,p}(s=5)$, or τ_p is the trivial representation. In the first case, the representation V is not unitarizable; see [BW00, Chapter XI, section 4]. Thus the cuspidal projection of τ_p is 0 in that case. In the latter two cases, the cuspidal projection of τ_p must be 0 or the trivial representation, because it is semisimple. But by considering Fourier coefficients of cusp forms, one sees immediately that the trivial representation of $G_E(\mathbf{Q}_p)$ cannot appear in the cuspidal spectrum. Thus in all cases, the cuspidal projection of τ_p is 0.

Because the data at the finite places away from p was arbitrary, this proves the theorem. □

Remark 9.4.7. The Lie algebra $\mathfrak{g}(J)$ has a $\mathbf{Z}/3\mathbf{Z}$-grading,

$$\mathfrak{g}(J) = (\mathfrak{sl}_3 \oplus \mathfrak{m}(J)^0) \oplus (V_3 \otimes J) \oplus (V_3 \otimes J)^\vee,$$

where V_3 is the standard representation of $\mathfrak{sl}_3$. To compare this decomposition with the $\mathbf{Z}/2$-grading recalled in subsection 2.3, see [Pol20a, Paragraph 4.2.4]. We express various elements of $\mathfrak{g}(J)$, in the $\mathbf{Z}/3\mathbf{Z}$-grading, in terms of the F_4 root system. Here $[a_1a_2a_3a_4]$ denotes the root $\sum_j a_j\alpha_j$ in the F_4 root system.

- $E_{13} = [2342]$
- $E_{12} = [1000]$
- $v_1 \otimes J = \begin{pmatrix} [1122] & [1121] & [1111] \\ [1121] & [1120] & [1110] \\ [1111] & [1110] & [1100] \end{pmatrix}$
- $\delta_3 \otimes J^\vee = \begin{pmatrix} [1220] & [1221] & [1231] \\ [1221] & [1222] & [1232] \\ [1231] & [1232] & [1242] \end{pmatrix}$
- $E_{23} = [1342]$
- $\mathfrak{m}(J) = \begin{pmatrix} * & [0001] & [0011] \\ -[0001] & * & [0010] \\ -[0011] & -[0010] & * \end{pmatrix}$
- $v_2 \otimes J = \begin{pmatrix} [0122] & [0121] & [0111] \\ [0121] & [0120] & [0110] \\ [0111] & [0110] & [0100] \end{pmatrix}$

We have written v_1, v_2, v_3 for the standard basis of V_3 and $\delta_1, \delta_2, \delta_3$ for the dual basis of $V_3^\vee$.

9.5. **Deduction of Theorem 1.1.1.** Finally, we deduce Theorem 1.1.1 as a corollary of Theorem 9.4.1. Let us make precise the statement of the result.

First, we need to define functions Θ_f for general $K_{G_J,\infty}$-type vectors. Let $V_{min,\infty}$ denote the space of the archimedean minimal representation, as defined by Gross-Wallach [GW94]. Fix a basis $w_{-4}, w_{-3}, \ldots, w_4$ of the minimal K-type of $V_{min,\infty}$, which we identify with $\mathbf{V}_4$, and let $w_j^\vee$ be the dual basis. Suppose $f^\infty(g,s) \in I_{G_J,fte}(s)$ is a flat section. As before, let $f_{\infty,4}(g,s)$ be our specified vector-valued archimedean flat inducing section.

Now, suppose $v \in V_{min,\infty}$, and $v = u \cdot w_0$ for some u in the complexified universal enveloping algebra $\mathcal{U}(\mathfrak{g}(J) \otimes \mathbf{C})$. We set $\Theta_{f^\infty \otimes v}(g) = u\langle \Theta_f(g), w_0^\vee\rangle$. Here $f(g,s) = f^\infty(g,s) f_{\infty,4}(g,s)$. It follows from [GS05, Corollary 12.12] that this association is well-defined, and thus gives an intertwining map $I_{G_J,fte}(s=24) \otimes V_{min,\infty} \to \mathcal{A}(G_J)$. We can therefore define the theta lift $\Theta_{f^\infty \otimes v}(\mathbf{1})(g) = \int_{[S_E]} \Theta_{f^\infty \otimes v}(g,h)\,dh$.

We now define the Siegel-Weil Eisenstein series. Recall that $\overline{f}(g) := Res_{s=24} M(w_0) f(g,s)$.

Lemma 9.5.1. *Suppose $v \in V_{min,\infty}$, $v = u \cdot w_0$ with $u \in \mathcal{U}(\mathfrak{g}(J) \otimes \mathbf{C})$. Then the association $v \mapsto u \cdot \langle \overline{f}(g), w_0^\vee \rangle$ gives a well-defined, $G(\mathbf{A}_f) \times (\mathfrak{g}(J) \otimes \mathbf{C}, K_{G_J,\infty})$-equivariant map $I_f(s=24) \otimes V_{min,\infty} \to I_{G_J}(s=5)$.*

Proof. Consider the constant term map $\varphi \mapsto \varphi_{U_{P_0}}$ from $\mathcal{A}(G_J) \to \mathcal{A}(U_{P_0}(\mathbf{A}) \backslash G_J(\mathbf{A}))$. Here U_{P_0} is the unipotent radical of the minimal standard parabolic of G_J. One sees that on the residues of Eisenstein series $Res_{s=24} E(g,f,s)$ for arbitrary flat sections f, there are most three terms, and they have distinct exponents. One of these terms is the long intertwining operator $Res_{s=24} M(w_0) f(g,s)$. Suppose ξ is its exponent, restricted to $T(\mathbf{R})$. Then the ξ-part of the constant term recovers the long-intertwining operator for every residue $Res_{s=24} E(g,f,s)$. Because it's a constant term, it is an intertwining map. The lemma follows. □

Given $f^\infty \otimes v \in I_{G_J,f}(s=24) \otimes V_{min,\infty}$, our Siegel-Weil Eisenstein series is defined as

$$E_{G_E}(g, u \cdot \langle \overline{f}, w_0^\vee \rangle), s = 5).$$

Here we restrict $u \cdot \langle \overline{f}, w_0^\vee \rangle$ to a flat section of $I_{G_E}(s)$ and then evaluate the corresponding Eisenstein series at $s=5$. The Eisenstein series is regular at $s=5$ because restricting $V_{min,\infty}$ to the maximal compact subgroup $K_{G_E,\infty} \subseteq G_E(\mathbf{R})$, we never see the trivial representation. (In fact, the long root SU_2 never sees the trivial representation.) If we mod out by the trivial representation, the Eisenstein map

$$\mathrm{Eis}_1 : I_{G_J,f}(s=24) \otimes V_{min,\infty} \to \mathcal{A}(G_E)/\mathbf{1}$$

becomes $G_E(\mathbf{A}_f) \times (\mathfrak{g}_E, K_{G_E,\infty})$-intertwining, and factors through the $S_E(\mathbf{A})$-coinvariants.

We now have:

Corollary 9.5.2. *In $\mathcal{A}(G_E)/\mathbf{1}$, we have an identity*

$$\Theta_{f^\infty \otimes v}(\mathbf{1})(g) = E(g, u \cdot \langle \overline{f}, w_0^\vee \rangle, s = 5).$$

Proof. Fix the data away from ∞. Then both sides are $(\mathfrak{g}_E, K_{G_E,\infty})$-equivariant maps on the coinvariant $(V_{min,\infty})_{S_E(\mathbf{R})}$ that agree on the vector $v = w_0$ by Theorem 9.4.1. But by [HPS96], $(V_{min,\infty})_{S_E(\mathbf{R})}$ is an irreducible $(\mathfrak{g}_E, K_{G_E,\infty})$-module. The corollary follows. □

9.6. **Data availability.** There is not available data associated with this manuscript.

9.7. **Statements and declarations.** The author has no competing interests.

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