Exceptional algebraic structures and applications

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Abstract. These are personal course notes for Math 205, "Exceptional algebraic structures and applications" in Winter 2021 at UCSD.

Contents

1. Introduction

NOTE: This introduction explains what I would have discussed, if time were unlimited. One can see from the table of contents what we actually had time to cover.

There are three parts to these notes:

(1) Explicit models of the exceptional Lie algebras and Lie groups, including E_8;
(2) A treatment of some of Bhargava's "Higher Composition Laws", from the point of view of exceptional algebraic structures;
(3) Modular forms on quaternionic exceptional groups.

The expert reader can glance at the table of contents to see what is covered. We now give an introduction to some of the topics just mentioned.

1.1. The exceptional groups. Classical groups include $\mathrm{GL}_n, \mathrm{Sp}_{2n} = \mathrm{Sp}(W; \langle\, , \,\rangle)$, and $\mathbf{O}(V,q)$. The groups $\mathrm{Sp}(W)$ and $O(V)$ are defined as the subgroups of the general linear group of a vector space that fix a particular bilinear form. If W is a $2n$-dimensional vector space with non-degenerate symplectic form $\langle\, , \,\rangle$, then

$$\mathrm{Sp}_{2n} = \mathrm{Sp}(W) = \{g \in \mathrm{GL}(W) : \langle gw_1, gw_2\rangle = \langle w_1, w_2\rangle \forall w_1, w_2 \in W\}.$$

Similarly, if q is a non-degenerate quadratic form on the vector space V with associated bilinear form $(x,y) = q(x+y) - q(x) - q(y)$, then

$$O(V,q) = \{g \in \mathrm{GL}(V) : (gv_1, gv_2) = (v_1, v_2) \forall v_1, v_2 \in V\}.$$

Other familiar facts about the above classical groups are

(1) A concrete description of their Lie algebras. For example, $\mathfrak{gl}(V) = End(V) = V \otimes V^\vee$; $\mathfrak{so}(V) \simeq \wedge^2 V$, $\mathfrak{sp}(W) \simeq Sym^2(W)$.
(2) Explicit description of some nilpotent elements of these Lie algebras
(3) Explicit description of the flag varieties G/P for G a group as above and P a parabolic subgroup. For example, G is $\mathrm{Sp}(W)$ or $O(V)$ and P a parabolic subgroup of G, the varieties G/P can be identified with flags of isotropic subspaces of W or V.

Our first aim in these notes is to give analogous some of the analogous definitions and constructions for exceptional groups. This means:

(1) A definition of (some form of) each (simply-connected) exceptional group G in terms of stabilizers of concrete tensors;
(2) An explicit description of the Lie algebra $\mathfrak{g}$ of G, including some unipotent elements;
(3) Some results on the flag varieties G/P.

Let us now give a taste of some of the above elements. As this is an introductory book, we always work over ground fields k of characteristic 0.

Example 1.1.1 (The group E_6). There is a certain 27-dimensional vector space J, that comes equipped with a homogenous degree three polynomial "determinant" or "norm" map $N_J : J \to k$. The (simply-connected) group of type E_6 can be defined as

$$E_6 = E_6(J) = \{g \in \mathrm{GL}(J) : N_J(gX) = N_J(X) \forall X \in J\}.$$

Moreover, we will write down an explicit E_6-equivariant map $\Phi : J \otimes J^\vee \to \mathfrak{e}_6$.

Now, there is a quadratic polynomial map $\# : J \to J^\vee$. Call an element $x \in J$ *singular* if $x^\# = 0$. If P is the D_5 parabolic in E_6, then the flag variety G/P can be identified with the singular lines in J. Below, we will make all of this explicit, and more.

Let us also give a quick tour of the group E_7.

Example 1.1.2 (The group E_7). There is a particular 56-dimensional vector space $W_J = k \oplus J \oplus J^\vee \oplus k$. The space W_J comes equipped with a symplectic form $\langle\, , \rangle$ and a homogeneous degree four polynimial $q : W_J \to k$. The simply-connected group of type E_7 can be defined as

$$E_7 = E_7(J) = \{g \in \mathrm{Sp}(W_J; \langle\, , \rangle) : q(gw) = q(w) \forall w \in W_J\}.$$

We'll define the quartic form q explicitly and do calculations with it.

Another familiar aspect for the classical group $\mathrm{SL}_2(\mathbf{R})$ is that it acts on the upper half plane $\mathfrak{h} = \{z = x + iy : x, y \in \mathbf{R}, y > 0\}$ by linear fractional transformations: If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ and $z \in \mathfrak{h}$, then $\gamma z = \frac{az+b}{cz+d}$. Using this action, one can define holomorphic modular forms for SL_2.

There is a parallel story for a group $E_7(J)(\mathbf{R})$ for a particular J. Namely, there is an upper half space $\mathfrak{h}_J = \{z = x + iy : x, y \in J, y > 0\}$ for a notion of positive-definiteness $y > 0$ in J, and the group $E_7(J)$ acts on $\mathfrak{h}_J$ by a sort of exceptional linear fractional transformations. We'll define this action and the corresponding notion of holomorphic modular forms for $E_7(J)$.

We'll also explicitly define $E_8, \mathfrak{e}_8$, check the Jacobi identity on $\mathfrak{e}_8$ and compute the Killing form.

1.2. Bhargava's Higher Composition Laws. The second part of these notes concerns some results of Bhargava [**Bha04a, Bha04b**]. We give now some flavor of what we'll discuss.

To set the stage, recall Gauss composition, which is a bijection between $\mathrm{SL}_2(\mathbf{Z})$-equivalence classes of integral primitive binary quadratic forms of discriminant $D < 0$ and the class group $\mathrm{Cl}(R_D)$ where $R_D = \mathbf{Z} \oplus \mathbf{Z}\frac{D+\sqrt{D}}{2}$ is the quadratic order of discriminant D:

$$\mathrm{SL}_2(\mathbf{Z})\backslash \left\{ax^2 + bxy + cy^2 : b^2 - 4ac = D < 0, \;\; a, b, c \in \mathbf{Z} \text{ and } gcd(a, b, c) = 1\right\} \leftrightarrow \mathrm{Cl}(R_D).$$

Bhargava found many generalizations of this classical result of Gauss. The prototypical examples is essentially a bijection between $\mathrm{SL}_2(\mathbf{Z})^3$ orbits of "cubes" of discriminant D, i.e., elements of the 8-dimension $\mathbf{Z}$-modular $\mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2$ where a certain degree four "discriminant" polynomial equals D, and triples of ideal classes in $\mathrm{Cl}(R_D)$ whose product is trivial:

$$\mathrm{SL}_2(\mathbf{Z})^3\backslash(\mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2)^{\mathrm{disc}=D} \overset{ess.}{\leftrightarrow} (R_D, I_1, I_2, I_3 : I_1I_2I_3 = (1))$$

Write $R_D = \mathbf{Z} \oplus \mathbf{Z}\tau$, and suppose $I_1 = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_2$, $I_2 = \mathbf{Z}\beta_1 \oplus \mathbf{Z}\beta_2$, $I_3 = \mathbf{Z}\gamma_1 \oplus \mathbf{Z}\gamma_2$. Because the product $I_1I_2I_3 = (1)$, the product

$$\alpha_i\beta_j\gamma_k = c_{ijk} + a_{ijk}\tau \tag{1}$$

for certain integers a_{ijk}, c_{ijk}. Here $i, j, k \in \{1, 2\}$. Following [**Bha04a**], the map from triples of ideal classes to $\mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2$ sends (R_D, I_1, I_2, I_3) to the $2 \times 2 \times 2$ "box" (a_{ijk}).

A key step in the proof is that the above map induces the desired bijection is that given a box (a_{ijk}) with discriminant D, there exists an essentially unique set of integers c_{ijk} and ideals I_1, I_2, I_3 so that (1) holds.

We'll explain this proof of this result of Bhargava, with special attention to this key step. And we'll do so from the point of view of exceptional algebra. To give just a hint why there might be some connection, note that the vector space $W_8 = \mathbf{Q}^2 \otimes \mathbf{Q}^2 \otimes \mathbf{Q}^2$ has a $\mathrm{SL}_2(\mathbf{Q})^3$-invariant symplectic form, and the invariant quartic form, the discriminant, already mentioned. This is just like the space W_J that we mentioned can be used to define $E_7(J)$. Indeed, W_8 with its $\mathrm{SL}_2(\mathbf{Q})^3$-action is a sort of degenerate analogue of the space W_J used to define $E_7(J)$.

1.3. Modular forms on exceptional groups. The final part of the notes concerns the so-called modular forms on quaternionic exceptional groups.

Recall that a classical holomorphic level one modular form of weight ℓ for SL_2 is a holomorphic function $f : \mathfrak{h} \to \mathbf{C}$ satisfying

(1) $f(\gamma z) = (cz + d)^\ell f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$

(2) the $\mathrm{SL}_2(\mathbf{Z})$-invariant function $|y^{\ell/2}f(z)|$ has moderate growth.

These modular forms have a classical Fourier expansion

$$f(z) = \sum_{n\geq 0} a_f(n) e^{2\pi i n z} \tag{2}$$

with the Fourier coefficients $a_f(n) \in \mathbf{C}$. Sometimes these Fourier coefficients are even in $\mathbf{Z}, \mathbf{Q}$, or $\overline{\mathbf{Q}}$.

One might reasonably ask if there is any parallel theory of very special automorphic forms on an exceptional group, such as (split) G_2. It turns out that there is, and this notion was singled out by Gan-Gross-Savin [**GGS02**] and Gross-Wallach [**GW96**].

To say a little bit about the definition, first note that one can replace the modular form f with the function $\phi_f : \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R}) \rightarrow \mathbf{C}$ defined as $\phi_f(g) = j(g,i)^{-\ell} f(g \cdot i)$, where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = cz + d$. The function ϕ_f satisfies

(1) $\phi_f(gk_\theta) = e^{-i\theta\ell}\phi_f(g)$ for all $g \in \mathrm{SL}_2(\mathbf{R})$ and $k_\theta = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$

(2) $D_{CR,\ell}\phi_f \equiv 0$, for a certain linear differential operator $D_{CR,\ell}$, that corresponds to the fact that f satisfies the Cauchy-Riemann equations.

The definition of modular forms on G_2–or more generally, a "quaternionic" exceptional group–is similar to the conditions on ϕ_f just enumerated.

Fix an integer $\ell \geq 1$. A modular form on G_2 of weight ℓ is a function of moderate growth $\phi : G_2(\mathbf{Z})\backslash G_2(\mathbf{R}) \rightarrow Sym^{2\ell}(\mathbf{C}^2)$ satisfying

(1) $\phi(gk) = k^{-1} \cdot \phi(g)$ for all $g \in G_2(\mathbf{R})$ and $k \in K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$ the maximal compact subgroup of $G_2(\mathbf{R})$. Here, on the right of this equation, K is acting on $Sym^{2\ell}(\mathbf{C}^2)$ by projection to the first (long-root) factor.

(2) $D_\ell \phi \equiv 0$ for a certain first-order linear differential operator D_ℓ.

Unlike the case of SL_2, the symmetric space $G_2(\mathbf{R})/K$ does not have a complex structure. So, even though one can make the above definition, there is no *a priori* reason to believe that these modular forms might behave analogously to the classical holomorphic modular forms recalled above. Nevertheless, [**GGS02**],[**GW96**] and [**Wal03**] defined these modular forms and defined a notion of some Fourier coefficients of them.

One theorem that we'll discuss [**Pol20a**] is that these modular forms have a robust, $\mathbf{Z}$-normalized Fourier expansion similar to the expansion (2) above. We'll also prove that there exist cusp forms on G_2 all of whose Fourier coefficients are in $\mathbf{Z}$ or $\overline{\mathbf{Q}}$ [**Pol20d**]. Moreover, we'll define modular forms on general quaternionic exceptional groups–including the real Lie group known as $E_{8,4}$–and explicitly construct some modular forms on this group [**Pol20b, Pol20c**].

Part 1

The exceptional Lie algebras and Lie groups

CHAPTER 1

The octonions and G_2

1. Quaternion algebras

Some references:

(1) [**GS17**] Chapter 1
(2) [**GL09**]

We begin by defining quaternion algebras.

Suppose k is a field.

DEFINITION 1.0.1. A quaternion algebra of k is rank four associative k-algebra B with unit that satisfies the following properties: There exists a k-linear order-reversing involution $u \mapsto u^*$

(1) such that $u = u^*$ if and only if $x \in k \cdot 1$
(2) $\mathrm{tr}(u) = u + u^*$ and $n(u) = uu^*$ are in k for all $u \in B$
(3) the quadratic form $n : B \to k$ is non-degenerate.

Note that the first condition implies the second.

When working over rings, instead of fields, and when one drops the condition that the quadratic form n be non-degenerate, then one adds the condition

- left multiplication by u on B has trace $2\,\mathrm{tr}(u)$

EXAMPLE 1.0.2. The two-by-two matrix algebra $M_2(k)$ is a quaternion algebra, with $*$ as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

CLAIM 1.0.3. *The norm on a quaternion algebra B is multiplicative, i.e., $n(xy) = n(x)n(y)$.*

The proof is immediate.

There is another definition that is more concrete.

DEFINITION 1.0.4. Given $a, b \in k^\times$, define $B_{a,b}$ as the four-dimensional associative k-algebra with basis $1, i, j, k = ij$ satisfying $i^2 = a$, $j^2 = b$, $ij = -ji$. An involution $*$ on $B_{a,b}$ is defined as $u = w + xi + yj + zk \mapsto u^* = w - xi - yj - zk$, where $w, x, y, z \in k$.

THEOREM 1.0.5. *The four-dimensional algebra $B_{a,b}$ is a quaternion algebra, and every quaternion algebra arises in this way.*

PROOF. We first explain that $B_{a,b}$ satisfies the axioms stated above. To do this, we must compute uu^*. One obtains

$$\begin{aligned} uu^* &= (w + xi + yj + zk)(w - xi - yj - zk) \\ &= w^2 - x^2(i^2) - y^2(j^2) - z^2(k^2) - xy(ij + ji) - yz(jk + kj) - zx(ki + ik) \\ &= w^2 - ax^2 - by^2 + abz^2. \end{aligned}$$

One must also verify that this map $*$ satisfies $(xy)^* = y^* x^*$.

For the converse direction, let $(\ )$ be the bilinear form associated to the norm form n. Thus

$$(u, v) = n(u + v) - n(u) - n(v) = uv^* + vu^*.$$

By assumption this is non-degenerate. Let $1, i, j, k'$ be an orthogonal basis of B with respect to this bilinear form, and set $k = ij$.

We claim $ij = -ji$, $i^2 = a$, $j^2 = b$ form some $a, b \in k^\times$, and that $1, i, j, k$ is a basis of B. For the first part, note that $0 = (1, i) = i^* + i$ so that $i^* = -i$ and similarly for j. Then $i^2 = -ii^* = -n(i) \in k^\times$ and simiarly for j. Finally, to see that $1, i, j, ij$ is a basis, we check that $k = ij$ is orthogonal to $1, i, j$. We leave that as an exercise. $\square$

Exercise 1.0.6. Complete the above proof by checking the following:

- The map $*$ on $B_{a,b}$ satisfies $(uv)^* = v^*u^*$.
- If $1, i, j, k'$ is an orthogonal basis of a quaternion algebra B, then $k = ij$ is orthogonal to $1, i, j$.

Another way of constructing quaternion algebras: This is called the "doubling" or the Cayley-Dickson construction. Suppose E/k is a quadratic etale algebra with nontrivial involution σ. define $B_{E,\gamma} = E \oplus Ej$ with multiplication induced by $jyj^{-1} = \sigma(y)$ for $y \in E$ and $j^2 = \gamma \in k^\times$. In other words, we define a multiplication on E^2 as

$$(x_1 + y_1 j)(x_2 + y_2 j) = (x_1 x_2 + \gamma\sigma(y_2)y_1) + (y_2 x_1 + y_1\sigma(x_2))j.$$

Define an involution on $B_{E,\gamma}$ as $(x + yj)^* = \sigma(x) - j\sigma(y) = \sigma(x) - yj$.

Proposition 1.0.7. *The algebra just constructed is a quaternion algebra, and every quaternion algebra arises in this way.*

Proof. We first verify that $B_{E,\gamma}$ is a quaternion algebra. For this, let $1, i$ be an orthogonal basis of E. Then $\sigma(i) = -i$ and $i^2 = a$ for some $a \in k^\times$. We have $ji = \sigma(i)j = -ij$. Finally, $j^2 = \gamma$. Thus $B_{E,\gamma} \simeq B_{a,\gamma}$ so is a quaternion algebra.

For the converse direction, suppose $B = B_{a,b}$. Set $E = k \oplus ki = k[x]/(x^2 - a)$. Let $\gamma = b$. Then one verifies quickly that $B_{a,b} = B_{E,\gamma}$. $\square$

2. Octonion algebras

References:

(1) [**SV00**]

The first exceptional algebraic structure we encounter is the octonions.

We begin with a definition. Suppose k is a field.

Definition 2.0.1. Suppose C is a not-neccessarily-associative k algebra with unit 1, and that C comes equipped with non-degenerate quadratic form $n_C : C \to k$. Then C is said to be a composition algebra if n_C is multiplicative, i.e., $n_C(xy) = n_C(x)n_C(y)$ for all $x, y \in C$.

Composition algebras can be classified, and in fact are always dimension $1, 2, 4$ or 8 over the ground field. Every dimension four composition algebra is a quaternion algebra.

There is a way of defining an involution $*$ on a composition algebra, as follows. Let $(x, y) = n_C(x + y) - n_C(x) - n_C(y)$ be the non-degenerate bilinear form associated to n_C. Note that 1 satisfies $(1, 1) = 2 \neq 0$. Let C^0 be the perpendicular space to 1 under the bilinear form. Define $*$ on C as $(x1 + y)^* = x - y$ if $x \in k$ and $y \in C^0$. In other words,

$$z^* = (z, 1)1 - z$$

for $z \in C$.

Note that $z + z^* \in k \cdot 1$ for all $z \in C$. Also note that $n_C(z) = n_C(z^*)$ for all $z \in C$.

Theorem 2.0.2. *The map $*$ satisfies*

(1) $z^*z = n_C(z)$ *for all* $z \in C$.
(2) *Moreover,* $*$ *is an algebra involution, i.e.,* $(xy)^* = y^*x^*$ *for all* $x, y \in C$.

We'll prove this theorem below.

DEFINITION 2.0.3. An octonion algebra Θ is an eight-dimensional composition algebra.

Octonion algebras exist. We give two different constructions, called the Zorn model and the Cayley-Dickson construction.

DEFINITION 2.0.4 (The Zorn model). Denote by V_3 the three-dimensional defining representation of SL_3 and by V_3^* its dual representation. Denote by Θ the set of two-by-two matrices $\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)$ with $a, d \in k$, $v \in V_3$ and $\phi \in V_3^*$ with multiplication

$$\begin{pmatrix} a & v \\ \phi & d \end{pmatrix} \begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix} = \begin{pmatrix} aa' + \phi'(v) & av' + d'v - \phi \wedge \phi' \\ a'\phi + d\phi' + v \wedge v' & \phi(v') + dd' \end{pmatrix}.$$

The involution $*$ is $\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}^* = \begin{pmatrix} d & -v \\ -\phi & a \end{pmatrix}$ and the norm is $n_\Theta(\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}) = ad - \phi(v)$.

The Cayley-Dickson construction starts with a quaternion algebra B and an element $\gamma \in k^\times$, and defines $\Theta = B \oplus B$ with multiplication as follows.

DEFINITION 2.0.5. Let $*$ denote the involution on the quaternion algebra B. Then the multiplicaton on $\Theta = B \oplus B$ is

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 + \gamma y_2^* y_1, y_2x_1 + y_1x_2^*).$$

The involution $*$ on Θ is $(x, y)^* = (x^*, -y)$ and the norm is $n_\Theta((x, y)) = n_B(x) - \gamma n_B(y)$.

We'll check below that the Cayley-Dickson construction and the Zorn model produces composition algebras.

PROPOSITION 2.0.6. *The Zorn model is a special case of the Cayley-Dickson construction, with* $B = M_2(k)$ *and* $\gamma = 1$.

PROOF. The following map induces an isomorphism:

$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\right) \mapsto \begin{pmatrix} a_{11} & a_{12}e_1 + m_{11}e_2 - m_{21}e_3 \\ a_{21}e_1^* + m_{22}e_2^* + m_{12}e_3^* & a_{22} \end{pmatrix}.$$

We have

$$(a, m)(a', m') = \left(\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)\begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix} + \begin{pmatrix} m'_{22} & -m'_{12} \\ -m'_{21} & m'_{11} \end{pmatrix}\left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right), \begin{pmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{pmatrix}\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) + \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right)\begin{pmatrix} a'_{22} & -a'_{12} \\ -a'_{21} & a'_{11} \end{pmatrix}\right).$$

It is a simple but tedious check that the above map induces an isomorphism. ☐

2.1. Proof of Theorem 2.0.2. We follow Springer-Veldkamp [**SV00**] to prove this theorem. We also prove various useful facts and identities along the way.

To prove Theorem 2.0.2, we'll require several lemmas.

LEMMA 2.1.1. *Suppose* C *is a composition algebra. Then*
(1) $(x_1y, x_2y) = n_C(y)(x_1, x_2)$ *for all* $x_1, x_2, y \in C$
(2) $(xy_1, xy_2) = n_C(x)(y_1, y_2)$ *for all* $x, y_2, y_2 \in C$
(3) $(x_1y_1, x_2y_2) + (x_2y_1, x_1y_2) = (x_1, x_2)(y_1, y_2)$ *for all* $x_1, x_2, y_1, y_2 \in C$.

PROOF. Consider $n((x_1+x_2)y)$ to prove that $(x_1y, x_2y) = n(y)(x_1, x_2)$. Similarly, $n(xy_1, xy_2) = n(x)(y_1, y_2)$. Linearize to obtain $(x_1y_1, x_2y_2) + (x_2y_1, x_1y_2) = (x_1, x_2)(y_1, y_2)$. ☐

We can now prove part 1 of Theorem 2.0.2:

Proof of Theorem 2.0.2 (1). . To prove that $zz^* = n(z)$, compute the inner product of both sides with an arbitrary $y \in C$. Then

$$\begin{aligned}
(y, zz^*) &= (y, (1,z)z - z^2) \\
&= (1,z)(y,z) - (y,z^2) \\
&= (y,z^2) + (zy,z) - (y,z^2) \\
&= (zy,z) \\
&= n(z)(y,1).
\end{aligned}$$

□

We therefore have:

Lemma 2.1.2. *For all x, y, z in a composition algebra C:*

1. *The quadratic equation $z^2 - (z,1)z + n_C(z) = 0$ holds;*
2. $xy + yx - (x,1)y - (y,1)x + (x,y) = 0$.

Proof. The first statement is equivalent to $z^*z = n_C(z)$ and the second follows from the first by linearization. □

Proof of Theorem 2.0.2 (2). Let's now compute y^*x^*. One has

$$\begin{aligned}
y^*x^* &= ((1,y) - y)((1,x) - x) \\
&= (1,x)(1,y) - (1,y)x - (1,x)y + yx \\
&= (1,xy) + (x,y) - (1,y)x - (1,x)y + yx \\
&= (1,xy) - xy \\
&= (xy)^*.
\end{aligned}$$

□

We also record some identities we'll need later:

Lemma 2.1.3 ([**SV00**] Lemma 1.3.2). *One has*

1. $(xy, z) = (y, x^*z)$
2. $(xy, z) = (x, zy^*)$
3. $(xy, z^*) = (yz, x^*)$.

Proof. One has

$$\begin{aligned}
(y, x^*z) &= (y, (x,1)z - xz) \\
&= (x,1)(y,z) - (y,xz) \\
&= (xy,z) + (y,xz) - (y,xz) \\
&= (xy,z).
\end{aligned}$$

The other identities are left as an exercise. □

Corollary 2.1.4. *One has $z^*(zy) = n(z)y$.*

Proof. Pair the LHS with an arbitrary $w \in C$. Then

$$(w, z^*(zy)) = (zw, zy) = n(z)(w,y) = (w, n(z)y).$$

□

2.2. The Cayley-Dickson and Zorn constructions.

PROPOSITION 2.2.1. *The Zorn model and the Cayley-Dickson construction define octonion algebras, i.e., the norms are multiplicative.*

PROOF. It suffices to check that the Cayley-Dickson construction produces a composition algebra; however, we check both the Zorn model and the CD construction anyway. First, consider the Cayley-Dickson construction. One has

$$n_\Theta((x_1,y_1)(x_2,y_2)) = n_B(x_1x_2 + \gamma y_2^* y_1) - \gamma n_B(y_2x_1 + y_1x_2^*).$$

Expanding the RHS gives

$$n_B(x_1x_2) + \gamma(x_1x_2, y_2^*y_1) + \gamma^2 n_B(y_1y_2) - \gamma n_B(y_2x_1) - \gamma(y_2x_1, y_1x_2^*) - \gamma n_B(y_1x_2).$$

We claim that $(x_1x_2, y_2^*y_1) = (y_2x_1, y_2x_2^*)$. Given this, the multiplicativity follows. For this latter identity, note that we already know that B is a composition algebra, so we can apply identities proved for C. We have

$$(x_1x_2, y_2^*y_1) = (y_2x_1x_2, y_1) = (y_2x_1, y_1x_2^*)$$

as desired.

Let's also check that directly that the Zorn model gives a composition algebra. The norm of the product of u and u' is

$$(aa' + \phi'(v))(dd' + \phi(v')) - (a'\phi + d\phi' + v\wedge v', av' + d'v - \phi\wedge\phi').$$

This is

$$ada'd' + aa'\phi(v') + dd'\phi'(v) + \phi'(v)\phi(v') - \left(aa'\phi(v') + a'd'\phi(v) + ad\phi'(v') + dd'\phi'(v) - (v\wedge v', \phi\wedge\phi')\right).$$

Cancelling gives

$$\begin{aligned} n_\Theta(uu') &= ada'd' + \phi'(v)\phi(v') - a'd'\phi(v) - ad\phi'(v') + (v\wedge v', \phi\wedge\phi') \\ &= (ad - \phi(v))(a'd' - \phi'(v')) + (v\wedge v', \phi\wedge\phi') - \phi(v)\phi'(v') + \phi'(v)\phi(v'). \end{aligned}$$

□

3. The group G_2

Suppose Θ is an octonion algebra. The group G_2 is defined as the automorphisms of Θ. The exact linear algebraic group one gets depends upon Θ.

Octonion algebras posess lots of automorphisms. It is clear that SL_3 acts on the Zorn model. One can also define an action of $B^1 \times B^1$ on the Cayley-Dickson construction:

Suppose $g, h \in B^1$. Define

$$(g,h)\cdot(x,y) = (gxg^{-1}, hyg^{-1}).$$

CLAIM 3.0.1. *This action defines a map $B^1 \times B^1/\mu_2 \to G_2$, i.e., the action preserves the multiplication and the conjugation.*

The proof is a simple direct check.

So, we have a definition of G_2, and we know the definition of G_2 produces something nontrivial, as $\mathrm{SL}_3 \subseteq G_2$ and $(\mathrm{SL}_2\times\mathrm{SL}_2)/\mu_2 \subseteq G_2$, working with the Zorn model.

We will eventually check that G_2 has dimension 14 and give very concrete models for its Lie algebra.

LEMMA 3.0.2. *The group G_2 preserves the quadratic form on Θ.*

3.1. The dimension of G_2. The dimension is 14. The idea is that G_2 acts transitively on $V_7^{n_C=-1}$ with stabilizer of $u_0 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ being SL_3.

PROPOSITION 3.1.1. *The group G_2 acts transitively on $V_7^{n=-1}$.*

We first require the following lemma.

LEMMA 3.1.2. SL_2 *acts (by conjugation) transitively on the matrices with trace* 0 *and determinant* -1.

PROOF. We know GL_2 acts transitively. So, given a matrix m as in the statement of the lemma, there exists $g \in \mathrm{GL}_2$ with $gmg^{-1} = \mathrm{diag}(1,-1)$. But now write $g = dg'$ with d diagonal and $g' \in \mathrm{SL}_2$. Then $m' := g'm(g')^{-1}$ is diagonal with trace 0 and determinant -1. Thus $m' = \mathrm{diag}(1,-1)$ or $\mathrm{diag}(-1,1)$. Now apply $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ to finish the proof. $\square$

PROOF OF PROPOSITION 3.1.1. The idea is to go back and forth between using the SL_3 and $\mathrm{SL}_2 \times \mathrm{SL}_2$-action. We handle the most difficult case first: $u = \left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)$ with $v, \phi \neq 0$ but $\phi(v) = 0$. Use the SL_3 action to make $v = e_2$ and $\phi = e_3^*$. Then, in the CD model, the first component $x = \mathrm{diag}(a,d)$ with $\mathrm{tr}(x) = 0$. Use the SL_2 action (and the lemma) to make this anti-diagonal, e.g. $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now we have x with $a, d = 0$ and $\phi(v) \neq 1$ (because the norm is still -1). Thus from the SL_3 action we can assume $v = e_1$ and $\phi = e_1^*$, and we're done by the lemma.

The other cases are easier, and left to the reader: One first uses the SL_3 action to reduce to an octonion which has the shape $(x, 0)$ in the CD model, then one applies the lemma to finish. $\square$

LEMMA 3.1.3. *The stabilizer of $u_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Theta$ is* SL_3.

PROOF. It is clear that SL_3 is contained in the stabilizer, so we must check the other direction. For this, let S be the stabilizer, and suppose $g \in S$. Then g fixes $\epsilon_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\epsilon_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ because g must fix 1.

Let $Ann_R(\epsilon_i)$, $Ann_L(\epsilon_i)$ be the right and left annihilators of the ϵ_i, $i = 1, 2$. Then S must stabilizer these subspaces of Θ and their intersection. However, $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right) = Ann_L(\epsilon_1) \cap Ann_R(\epsilon_2)$, and similarly $\left(\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right) = Ann_R(\epsilon_1) \cap Ann_L(\epsilon_2)$. This shows that S preserves the components in the Zorn model, and then $S \hookrightarrow \mathrm{GL}_3$ because S must act on the ϕ's as the dual of how it acts on the v's, because the quadratic form is preserved.

Finally, to see that $S \simeq \mathrm{SL}_3$, note that the trilinear form $\mathrm{tr}(x_1(x_2x_3))$ is preserved by G_2 and thus by S. However, on the v's, this trilinear form is the determinant map $(v_1, v_2, v_3) = v_1 \wedge v_2 \wedge v_3$. This completes the proof. $\square$

COROLLARY 3.1.4. *The dimension of G_2 is* 14.

PROPOSITION 3.1.5. *The representation V_7 of G_2 is irreducible.*

PROOF. Suppose $V \subseteq V_7$ is a G_2-module. Then it is an SL_3 module, so is a direct sum of pieces in $V_7 = 1 \oplus V_3 \oplus V_3^*$. None of the nonzero subsums are stable under the $\mathrm{SL}_2 \times \mathrm{SL}_2$ action, however. So $V = V_7$, as desired. $\square$

LEMMA 3.1.6. *The center of G_2 consists only of the identity.*

PROOF. Inspired by SV Lemma 2.3.2. If you commute with SL_3, and preserve the quadratic form, you must be of the form $\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right) \mapsto \begin{pmatrix} \lambda a & \mu v \\ \mu^{-1}\phi & \lambda^{-1} d \end{pmatrix}$. If you also commute with $\mathrm{SL}_2 \times \mathrm{SL}_2$, we see $\lambda = \mu = \mu^{-1} = \lambda^{-1} = \pm 1$. But we cannot have -1 because 1 must be fixed. $\square$

One concludes that G_2 is a semisimple group of dimension 14.

3.2. The orthogonal model of the Lie algebra. The Lie algebra $\mathfrak{so}(V)$ is $\wedge^2 V$. It acts on V as

$$(u \wedge v) \cdot w = (v,w)u - (u,w)v.$$

The Lie bracket is

$$[u_1 \wedge v_1, u_2 \wedge v_2] = (v_1,u_2)u_1 \wedge v_2 - (v_1,v_2)u_1 \wedge u_2 - (u_1,u_2)v_1 \wedge v_2 + (u_1,v_2)v_1 \wedge u_2.$$

The map $V_7 \otimes V_7 \to V_7$ given by $Im(v_1 v_2)$ is alternating. It thus induces a map $\wedge^2 V_7 \to V_7$. We denote by $\mathfrak{g}$ the kernel of this map.

LEMMA 3.2.1. *The map $\wedge^2 V_7 \to V_7$ is surjective, and thus $\mathfrak{g}$ has dimension* 14.

PROOF. The map is nonzero, and the image is G_2 stable. □

THEOREM 3.2.2. *The subspace $\mathfrak{g}$ of $\wedge^2 V_7$ is closed under the Lie bracket of $\wedge^2 V_7 = \mathfrak{so}(V_7)$. It is the Lie algebra of G_2.*

Before proving this theorem, we give an explicit basis for $\mathfrak{g}$, and make some special notation which we will use throughout. First, denote by e_1, e_2, e_3 a fixed basis of V_3, and write e_1^*, e_2^*, e_3^* for the basis of $V_3^\vee$ dual to the e_i. Set $u_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in V_7$. We will abuse notation and also let e_i, e_j^* denote elements in V_7. Thus $(e_i, e_j^*) = -\delta_{ij}$, $(u_0,u_0) = -2$, and $(u_0, e_i) = (u_0, e_j^*) = 0$ for all i,j.

We set $E_{kj} = e_j^* \wedge e_k$, $v_j = u_0 \wedge e_j + e_{j+1}^* \wedge e_{j+2}^*$, and $\delta_j = u_0 \wedge e_j^* + e_{j+1} \wedge e_{j+2}$ (indices taken modulo three). One checks immediately from the definition of multiplication in Θ that the elements v_j and δ_j are in the kernel of $\wedge^2 V_7 \to V_7$, and thus in $\mathfrak{g}$. The same goes for E_{kj} so long as $j \neq k$. A sum $\alpha_1 E_{11} + \alpha_2 E_{22} + \alpha_3 E_{33}$ is in $\mathfrak{g}$ if and only if $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

The above elements span $\mathfrak{g}$. We write

$$\mathfrak{h} = \{\alpha_1 E_{11} + \alpha_2 E_{22} + \alpha_3 E_{33} : \alpha_1 + \alpha_2 + \alpha_3 = 0\}.$$

LEMMA 3.2.3. *The subspace $\mathfrak{g}$ is irreducible as a G_2-representation.*

PROOF. As an SL_3 representation, $\mathfrak{g}$ is

$$\wedge^2(1 + V_3 + V_3^\vee) - (1 + V_3 + V_3^\vee) = (V_3 \otimes V_3)^0 + V_3 + V_3^\vee.$$

As an $\mathrm{SL}_2 \times \mathrm{SL}_2$-representation,

$$\begin{aligned}\wedge^2(S^2(V_2) \boxtimes 1 + V_2 \boxtimes V_2) &= \wedge^2(S^2(V_2)) \boxtimes 1 + \wedge^2(V_2 \boxtimes V_2) + (S^2(V_2) \otimes V_2) \boxtimes V_2 \\ &= S^2(V_2) \boxtimes 1 + S^2(V_2) \boxtimes 1 + 1 \boxtimes S^2(V_2) + S^3(V_2) \boxtimes V_2 + V_2 \boxtimes V_2.\end{aligned}$$

Consequently,

$$\mathfrak{g} = S^3(V_2) \boxtimes V_2 + S^2(V_2) \boxtimes 1 + 1 \boxtimes S^2(V_2).$$

By using the action subspaces written down above, one can verify that none of the $\mathrm{SL}_2 \times \mathrm{SL}_2$ subsums is actually SL_3-invariant. We omit this, however. □

PROOF OF THEOREM. $\wedge^2 V_7$ is $\mathfrak{g} \oplus V_7$ as a direct sum of irreducible G_2-representations. However, it must contain the G_2-stable, 14-dimensional subspace $Lie(G_2)$. Thus, $Lie(G_2) = \mathfrak{g}$ and is irreducible as a G_2-representation. Consequently, G_2 is simple. □

The subalgebra $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ acts on E_{kj} by $\alpha_k - \alpha_j$, i.e.,

$$[\sum_i \alpha_i E_{ii}, E_{kj}] = (\alpha_k - \alpha_j) E_{kj}.$$

These are the long roots for $\mathfrak{h}$. Together with $\mathfrak{h}$, the E_{jk} span the Lie algebra $\mathfrak{sl}_3$. The Cartan $\mathfrak{h}$ acts on v_j via α_j and δ_j via $-\alpha_j$. These are the short roots.

One has the following Lie bracket relations, which can be checked easily.

- $[\delta_{j-1}, v_j] = 3E_{j,j-1}$
- $[v_{j-1}, \delta_j] = -3E_{j-1,j}$
- $[\delta_{j-1}, \delta_j] = 2v_{j+1}$
- $[v_{j-1}, v_j] = 2\delta_{j+1}$
- $[\delta_j, v_j] = 3E_{jj} - (E_{11} + E_{22} + E_{33})$.

All indices here are taken modulo 3. We will choose a positive system on $\mathfrak{g}$ by letting E_{12} and v_2 be the positive simple roots.

3.3. The $\mathbf{Z}/3$-graded model of the Lie algebra. Abstractly, $\mathfrak{g} = \mathfrak{g}_2 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee$, and this is a $\mathbf{Z}/3$-grading. In fact, all the (split) exceptional Lie algebras have $\mathbf{Z}/3$-grading that generalizes this one. See [**Rum97**]. In this $\mathbf{Z}/3$-model, the Lie bracket is given as follows:

(1) the commutators $[\mathfrak{sl}_3, V_3]$ and $[\mathfrak{sl}_3, V_3^\vee]$ are given by the action of $\mathfrak{sl}_3$ on V_3 and $V_3^\vee$;
(2) the commutators $[V_3, V_3]$ and $[V_3^\vee, V_3^\vee]$ are given by[1] $[x, y] = 2x \wedge y \in V_3^\vee$ for $x, y \in V_3$ and $[\gamma, \delta] = 2\gamma \wedge \delta \in V_3$ for $\gamma, \delta \in V_3^\vee$;
(3) if $x \in V_3$ and $\gamma \in V_3^\vee$ then[2] $[\gamma, x] = 3x \otimes \gamma - (x, \gamma)1_3$, which is in $\mathfrak{sl}_3$.

The elements E_{ij}, v_j, δ_k of $\mathfrak{g}$ defined above are simply the standard basis vectors for $\mathfrak{sl}_3$, V_3 and $V_3^\vee$ in the decomposition $\mathfrak{g} = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee$.

4. Triality and the group Spin_8

Use the norm form on Θ to define $\mathrm{SO}(\Theta)$.
Define the group Spin_8. Let $(x, y, z) = \mathrm{tr}(x(yz))$ be the trilinear form on Θ.

Definition 4.0.1. The group Spin_8: The set of triples $(g_1, g_2, g_3) \in \mathrm{SO}(\Theta)^3$ such that $(g_1x_1, g_2x_2, g_3x_3) = (x_1, x_2, x_3)$ for all $x_j \in \Theta$.

Reference: [**KPS94**, Proposition 4.8].
We omit a proof that Spin_8 is connected.
A warm up:

Lemma 4.0.2. *The kernel of the map* $\mathrm{Spin}_8 \to \mathrm{SO}(\Theta)$ *is* μ_2.

Proof. Say $g_1 = 1$. Then we quickly obtain $(g_2y)(g_3z) = yz$ for all $y, z \in \Theta$. Now taking $y = 1$, $z = 1$, one gets $(g_21)(g_31) = 1$. Now $(g_2y)(g_31) = y$, from which one obtains $g_2y = y(g_21)$. Similarly, $g_3z = (g_31)z$. Let $w = g_21$, $w^{-1} = g_31$. Then we have $(yw)(w^{-1}z) = yz$ for all $y, z \in \Theta$. We want to check that this cannot happen unless w is in the center of Θ. But replacing z by wz we obtain $(yw)z = y(wz)$. The lemma now follows from Exercise 4.0.3. □

Exercise 4.0.3. Prove that if $w \in \Theta$ satisfies $(yw)z = y(wz)$ for all $y, z \in \Theta$, then $w \in k \cdot 1$. For this, suppose $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ in the Cayley-Dickson model. Compute that $\{z_1, z_2, z_3\} := z_1(z_2z_3) - (z_1z_2)z_3 = (W_1, W_2)$ satisfies

$$W_1 = \gamma[x_1, y_3^*y_2] + \gamma[x_2^*, y_3^*y_1] + \gamma[x_3, y_2^*y_1]$$

and

$$W_2 = y_3[x_2, x_1] + y_2[x_3^*, x_1] + y_1[x_3^*, x_2^*] + \gamma(y_1y_2^*y_3 - y_3y_2^*y_1).$$

We now prove the "Principle of Local Triality".

[1] The 2 here is $1_J \times 1_J = 21_J$
[2] The 3 here is $\mathrm{tr}_J(1_J)$

THEOREM 4.0.4. *Given $X \in \wedge^2(\Theta) = \mathfrak{so}(\Theta)$, there exists $Y, Z \in \mathfrak{so}(\Theta)$ so that*

$$X(ab) = (Ya)b + a(Zb)$$

for all $a, b \in \Theta$.

For the proof, we paraphrase the argument in [**Jac71**, pages 8-9].

COROLLARY 4.0.5. *The map $\mathrm{Spin}_8 \to \mathrm{SO}(\Theta)$ induces an isomorphism on Lie algebras.*

PROOF. The principle of local triality, as stated above, implies the surjectivity: If (X, Y, Z) is a triple as above, then $(*X*, Y, Z)$ is in the Lie algebra $Lie(\mathrm{Spin}_8)$. This follows from the fact that the trilinear form $(c, a, b) = (c^*, ab)$.

The injectivity follows from the computation of the kernel above. $\square$

LEMMA 4.0.6. *The alternative identity holds:*

$$c(ab) + (ab)c = (ca)b + a(bc).$$

PROOF. This follows from the first Moufang identity (SV Proposition 1.4.1), which states

$$(ax)(ya) = a((xy)a).$$

Indeed, one linearizes this with $z, 1$ in place of a, a. To prove this Moufang identity, we follow SV exactly: We take the inner product of each side with an aribitrary elements $z \in C$, and one obtains

$$\begin{aligned}((ax)(ya), z) &= (ya, (x^*a^*)z)\\ &= (y, x^*a^*)(a, z) - (yz, (x^*a^*)a)\\ &= (y, x^*a^*)(a, z) - N(a)(yz, x^*).\end{aligned}$$

Similarly,

$$\begin{aligned}(a((xy)a), z) &= ((xy)a, a^*z)\\ &= (xy, a^*)(a, z) - ((xy)z, a^*a)\\ &= (xy, a^*)(a, z) - N(a)(xy, z^*).\end{aligned}$$

$\square$

PROOF OF PLT. Note that we have $\wedge^2(\Theta) = \wedge^2(1 + V_7) = 1 \otimes V_7 + \wedge^2 V_7$. The alternative identity implies the PLT holds for $1 \otimes V_7$. Indeed, if $c \in V_7$, then

$$\begin{aligned}(c \wedge 1)(x) &= c(1, x) - (c, x)1\\ &= c(x + x^*) - (cx^* + xc^*)1\\ &= cx + xc.\end{aligned}$$

(Note that left and right multiplication by an element $c \in V_7$ preserves the bilinear form.)

One checks that the subspace of $\mathfrak{so}(\Theta)$ for which the PLT holds is closed under the Lie bracket. But the bracket induces a surjection $(1 \otimes V_7) \otimes (1 \otimes V_7) \to \wedge^2 V_7$, so the PLT holds for all of $\wedge^2(\Theta)$. $\square$

EXERCISE 4.0.7. [**SV00**, Theorem 3.5.5] Suppose $X = a \wedge b \in \wedge^2\Theta$, so that $X(x) = (b, x)a - (a, x)b$. Prove with $Y = \frac{1}{2}(\ell_a \ell_{b^*} - \ell_b \ell_{a^*})$ and $Z = \frac{1}{2}(r_a r_{b^*} - r_b r_{a^*})$ that the triple X, Y, Z satisfies local triality. Hint: Check that this formula is correct on $V_7 \otimes 1$ and on $\wedge^2 V_7$ individually. We essentially checked it on $V_7 \otimes 1$ in the proof above. To check it on $a \wedge b = \frac{1}{2}[b \otimes 1, a \otimes 1]$, compute the commutator of the triples $(b \otimes 1, \ell_b, r_b)$ and $(a \otimes 1, \ell_a, r_a)$.

CHAPTER 2

Cubic norm structures, F_4 and E_6

1. Preamble

Suppose C is a composition algebra. Define

$$J = H_3(C) = \left\{ X = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} : c_1, c_2, c_3 \in k, x_1, x_2, x_3 \in C \right\}.$$

We define a cubic norm N_J on J as

$$(3) \quad N_J(X) = N_J \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} = c_1c_2c_3 - c_1n_C(x_1) - c_2n_C(x_2) - c_3n_C(x_3) + \mathrm{tr}_C(x_1x_2x_3).$$

Suppose $C = \Theta$ is the octonions. Then the group E_6 is defined as

$$E_6 = E_6(H_3(\Theta)) = \{g \in GL(H_3(\Theta)) : N_J(gX) = N_J(X) \, \forall X \in H_3(\Theta)\}.$$

The group F_4 is defined as the subgroup of E_6 that fixes $1_J = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Note that we have an embedding $\mathrm{Spin}_8 = \mathrm{Spin}(\Theta) \to F_4 \subseteq E_6$ as follows: If $g = (g_1, g_2, g_3) \in \mathrm{Spin}(\Theta)$, then define

$$gX = g \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} = \begin{pmatrix} c_1 & g_3x_3 & (g_2x_2)^* \\ (g_3x_3)^* & c_2 & g_1x_1 \\ g_2x_2 & (g_1x_1)^* & c_3 \end{pmatrix}.$$

It is immediate from (3) that this action of $\mathrm{Spin}(\Theta)$ fixes the norm on $J = H_3(\Theta)$ and fixes the element 1_J.

2. Cubic norm structures

Suppose k is a field of characteristic 0 and J is a finite dimensional k vector space. That J is a cubic norm structure means that it comes equipped with a cubic polynomial map $N : J \to k$, a quadratic polynomial map $\# : J \to J$, an element $1_J \in J$, and a non-degenerate symmetric bilinear pairing $(\, , \,) : J \otimes J \to k$, called the trace pairing, that satisfy the following properties. For $x, y \in J$, set $x \times y = (x+y)^\# - x^\# - y^\#$ and denote $(\, , \, , \,) : J \otimes J \otimes J \to k$ the unique symmetric trilinear form satisfying $(x, x, x) = 6N(x)$ for all $x \in J$. Then

1. $N(1_J) = 1$, $1_J^\# = 1_J$, and $1_J \times x = (1_J, x) - x$ for all $x \in J$.
2. $(x^\#)^\# = N(x)x$ for all $x \in J$.
3. The pairing $(x, y) = \frac{1}{4}(1_J, 1_J, x)(1_J, 1_J, y) - (1_J, x, y)$.
4. One has $N(x+y) = N(x) + (x^\#, y) + (x, y^\#) + N(y)$ for all $x, y \in J$.

One should see [**McC04**] for a thorough treatment of cubic norm structures.

There is a weaker notion of a cubic norm pair. In this case, the pairing $(\, , \,)$ is between J and $J^\vee$, the linear dual of J, the adjoint map $\#$ takes $J \to J^\vee$ and $J^\vee \to J$, and each $J, J^\vee$ have a norm map $N_J : J \to F$ and $N_{J^\vee} : J^\vee \to F$. The adjoints and norms on J and $J^\vee$ satisfy the same compatibilities as above in items (2) and (4).

If J is a cubic norm structure, we define the group

$$M_J = \{(\lambda, g) \in \mathrm{GL}_1 \times \mathrm{GL}(J) : N(gX) = \lambda N(X) \text{ for all } X \in J\},$$

the group of all linear automorphisms of J that preserve the norm N up to scaling. Thus if $x \in J$ with $N(x) \neq 0$, the map U_x defines an element of M_J. We set M_J^1 the subgroup of M_J consisting of those g with $\lambda(g) = 1$ and we set A_J the subgroup of M_J^1 that also stabilzes the element $1_J \in J$. It follows that A_J preserves the bilinear pairing $(\, , \,)$: if $a \in A_J$, then $(ax, ay) = (x, y)$ for all $x, y \in J$. The group A_J is the automorphism group of J. If $a \in A_J$, then one also has $(ax) \times (ay) = a(x \times y)$ for all $x, y \in J$.

2.1. Examples. We make $J = H_3(C)$ into a cubic norm structure, with the following choice of data:

(1) $N_J(X) = N_J \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} = c_1c_2c_3 - c_1n_C(x_1) - c_2n_C(x_2) - c_3n_C(x_3) + \mathrm{tr}_C(x_1x_2x_3).$

(2) $X^\# = \begin{pmatrix} c_2c_3 - n_C(x_1) & x_2^*x_1^* - c_3x_3 & x_3x_1 - c_2x_2^* \\ x_1x_2 - c_3x_3^* & c_1c_3 - n_C(x_2) & x_3^*x_2^* - c_1x_1 \\ x_1^*x_3^* - c_2x_2 & x_2x_3 - c_1x_1^* & c_1c_2 - n_C(x_3) \end{pmatrix}$

(3) The pairing (X, X'), in obvious notation, is
$$(X, X') = c_1c_1' + c_2c_2' + c_3c_3' + (x_1, x_1') + (x_2, x_2') + (x_3, x_3').$$

Theorem 2.1.1. *With data described above, $J = H_3(C)$ is a cubic norm structure.*

Proof. One immediately check that $1^\# = 1$.

We now check that $(X^\#)^\# = N(X)X$. To see this, we compute the c_1 and x_1 coefficient of $(X^\#)^\#$, the other coefficients being similar. The c_1 coefficient is

$$(c_3c_1 - n(x_2))(c_1c_2 - n(x_3)) - n(x_2x_3 - c_1x_1^*)$$

which one quickly computes to be $c_1N(x)$. The x_1 coefficient gives

$$(x_1x_2 - c_3x_3^*)(x_3x_1 - c_2x_2^*) - (c_2c_3 - n(x_1))(x_3^*x_2^* - c_1x_1).$$

Using the identity $(ba)a^* = n(a)b$ this becomes

$$(c_1c_2c_3 - c_1n(x_1) - c_2n(x_2) - c_3n(x_3))x_1 + (x_1x_2)(x_3x_1) + n(x_1)x_3^*x_2^*.$$

But now the first Moufang identity gives

$$(x_1x_2)(x_3x_1) + n(x_1)x_3^*x_2^* = x_1((x_2x_3)x_1) + x_1(x_1^*(x_3^*x_2^*)) = x_1(\mathrm{tr}((x_2x_3)x_1))$$

yielding the result.

To verify $N(x + x') = N(x) + (x^\#, x') + (x, (x')^\#) + N(x')$, one reduces quickly to computing the coefficient of ϵ in $N(x + \epsilon x')$. One obtains

$$\begin{aligned}\frac{1}{2}(x, x, x') = {} & c_1'c_2c_3 + c_1c_2'c_3 + c_1c_2c_3' - c_1(x_1, x_1') - c_1'n(x_1) - c_2(x_2, x_2') - c_2'n(x_2) - c_3(x_3, x_3') - c_3'n(x_3) \\ & + (x_1', x_2, x_3) + (x_1, x_2', x_3) + (x_1, x_2, x_3').\end{aligned}$$

On the other hand

$$\begin{aligned}(x^\#, x') = {} & c_1'(c_2c_3 - n(x_1)) + c_2'(c_3c_1 - n(x_2)) + c_3'(c_1c_2 - n(x_3)) \\ & + (x_1', x_3^*x_2^* - c_1x_1) + (x_2', x_1^*x_3^* - c_2x_2) + (x_3', x_2^*x_1^* - c_3x_3).\end{aligned}$$

Comparing quickly gives equality.

Linearizing the identity just proved gives $(x,y,z) = (x \times y, z)$. Using this, to check that $(x,x') = \frac{1}{4}(1,1,x)(1,1,x') - (1,x,x')$, it is equivalent to check $(x,x') = (1,x)(1,x') - (1, x \times x')$. Now

$$x \times x' = \begin{pmatrix} c_2c_3' + c_2'c_3 - (x_1,x_1') & * & * \\ * & c_3c_1' + c_3'c_1 - (x_2,x_2') & * \\ * & * & c_1c_2' + c_1'c_2 - (x_3,x_3') \end{pmatrix}.$$

The desired equality now follows easily. □

Example 2.1.2. The basic examples: $J = k$, $J = E$ an etale cubic algebra, $J = k \times C$ with C an associative composition algebra.

More generally,

Example 2.1.3. $J = k \times S$ with S a pointed quadratic space. In more detail, take $1_S \in S$ with $q(1_S) = 1$. Define an involution ι on S fixing 1_S and acting as minus the identity on $(1_S)^{\perp}$. The norm on J is $N_J(\beta, s) = \beta q_S(s)$, one has $1_J = (1, 1_S)$, and the adjoint map is $(\beta, s)^{\#} = (q_S(s), \beta\iota(s))$. Finally, the pairing is $((\beta,t),(\beta',t')) = \beta\beta' + (t, \iota(t'))$.

Proposition 2.1.4. *With data as defined above $J = k \times S$ is a cubic norm structure.*

Proof. It is clear that $N_J(1_J) = 1$ and $1_J^{\#} = 1_J$. Note that ι preserves the quadratic form q_S,i.e., $q_S(\iota(s)) = q_S(s)$ for all $s \in S$. One computes

$$\begin{aligned} ((\beta,s)^{\#})^{\#} &= (q_S(s), \beta\iota(s))^{\#} \\ &= (\beta^2 q_S(\iota(s)), q_S(s)\beta s) \\ &= \beta q_S(s)(\beta, s). \end{aligned}$$

We compute

$$N_J((\beta_1 + \epsilon\beta_2, s_1 + \epsilon s_2) = (\beta_1 + \epsilon\beta_2)q_S(s_1 + \epsilon s_2) = \beta_1 q_S(s_1) + \epsilon(\beta_2 q_S(s_1) + \beta_1(s_1,s_2)) + O(\epsilon^2).$$

On the other hand,

$$\begin{aligned} ((\beta_1,s_1)^{\#}, (\beta_2,s_2)) &= ((q_S(s_1), \beta_1\iota(s_1)), (\beta_2, s_2)) \\ &= \beta_2 q_S(s_1) + \beta_1(s_1,s_2). \end{aligned}$$

This proves that $N_J(x+y) = N_J(x) + (x^{\#}, y) + (y^{\#}, x) + N_J(y)$.

Finally, we must verify $(x,y) = (1_J,x)(1_J,y) - (1_J, x \times y)$. So suppose $x = (\beta_1, s_1), y = (\beta_2, s_2)$. Then $(1_J, x) = \beta_1 + (1_S, s_1)$ and $(1_J, y) = \beta_2 + (1_S, s_2)$. Moreover,

$$x \times y = ((s_1,s_2), \beta_1\iota(s_2) + \beta_2\iota(s_1))$$

so that $(1_J, x \times y) = (s_1,s_2) + \beta_1(1_S,s_2) + \beta_2(1_S,s_1)$. Combining gives

$$(1_J,x)(1_J,y) - (1_J, x \times y) = \beta_1\beta_2 + (1_S,s_1)(1_S,s_2) - (s_1,s_2).$$

On the other hand, one readily verifies that $(1_S,s_1)(1_S,s_2) - (s_1,s_2) = (s_1, \iota(s_2))$. The proposition follows. □

3. The group E_6

The group E_6 is defined as M_J^1 for $J = H_3(\Theta)$. Note that the group $\mathrm{Spin}_8 = \mathrm{Spin}(\Theta)$ embeds in M_J^1 for this J. In this section, we give some results on the Lie algebra $\mathfrak{m}(J)$.

We begin by defining an M_J-equivariant map $J \otimes J^{\vee} \to \mathfrak{m}(J)$. See [**Spr62**] and [**Rum97**].

For $\gamma \in J^{\vee}$ and $x \in J$, define the element $\Phi_{\gamma,x} \in End(J)$ as

$$\Phi_{\gamma,x}(z) = -\gamma \times (x \times z) + (\gamma, z)x + (\gamma, x)z.$$

Proposition 3.0.1. [**Rum97**, Equation (9)] *One has*

$$(\Phi_{\gamma,x}(z_1), z_2, z_3) + (z_1, \Phi_{\gamma,x}(z_2), z_3) + (z_1, z_2, \Phi_{\gamma,x}(z_3)) = 2(\gamma, x)(z_1, z_2, z_3)$$

for all z_1, z_2, z_3 in J. In particular, $\Phi_{\gamma,x} \in \mathfrak{m}(J)$.

Lemma 3.0.2. *In a cubic norm pair, one has $z^{\#} \times (z \times w) = n(z)w + (z^{\#}, w)z$.*

Proof. One starts with the identity $(z^{\#})^{\#} = N(z)z$. Then, replacing z by $z + \epsilon w$ and taking the coefficient of ϵ, one gets the result. □

Note that $\Phi_{\gamma,x}(z) = \Phi_{\gamma,z}(x)$. One sets $\Phi'_{\gamma,x} = \Phi_{\gamma,x} - \frac{2}{3}(\gamma, x)$. Then $\Phi'_{\gamma,x} \in \mathfrak{m}(J)^0$.

Proof of Proposition. By linearization, it suffices to evaluate $(z^{\#}, \Phi_{\gamma,x}(z))$. Applying Lemma 3.0.2, one obtains $2(\gamma, x)N(z)$. The proposition follows. □

Suppose $g \in M_J$. Let $\widetilde{g}$ denote the action of $g \in M_J$ on $J^{\vee}$. Recall $\lambda(g) \in \mathrm{GL}_1$ so that $N_J(gx) = \lambda(g)N_J(x)$ for all $x \in J$.

Proposition 3.0.3. *For all $x, y \in J$ and $\gamma, \mu \in J^{\vee}$ one has*

(1) $\widetilde{g}(x \times y) = \lambda(g)^{-1}(gx) \times (gy)$
(2) $N_{J^{\vee}}(\widetilde{g}\gamma) = \lambda(g)^{-1}N_{J^{\vee}}(\gamma)$
(3) $g(\gamma \times \mu) = \lambda(g)\widetilde{g}(\gamma) \times \widetilde{g}(\mu)$.

If J is a cubic norm structure and $g(1_J) = 1_J$ so that g commutes with the isomorphism $J \simeq J^{\vee}$, then $g(x \times y) = (gx) \times (gy)$ for all $x, y \in J$.

Lemma 3.0.4. *In a cubic norm pair, the set of elements $\mu x^{\#}$, $\mu \in k$ and $x \in J$, are Zariski dense in $J^{\vee}$.*

Proof. Indeed, if $\gamma \in J^{\vee}$, set $x = \gamma^{\#}$ then $x^{\#} = N_{J^{\vee}}(\gamma)\gamma$. So every element of $J^{\vee}$ with nonzero norm is of the form $\mu x^{\#}$. Because $N_{J^{\vee}}$ is not zero (because its value on $1_{J^{\vee}}$ is 1), the elements of nonzero norm are Zariski dense. □

Proof of Proposition 3.0.3. We prove the statements in turn. For the first statement, pair with an arbitrary element $z \in J$ to obtain

$$(z, \widetilde{g}(x \times y)) = (g^{-1}z, x \times y) = (g^{-1}z, x, y)_J = \lambda(g)^{-1}(z, gx, gy)_J = (z, \lambda(g)^{-1}(gx) \times (gy)).$$

For the second statement, applying Lemma 3.0.4, it suffices to verify it for $\gamma = x^{\#}$. Now

$$3N_{J^{\vee}}(y^{\#}) = (y^{\#}, (y^{\#})^{\#}) = (y^{\#}, N_J(y)y) = 3N_J(y)^2$$

so $N_{J^{\vee}}(y^{\#}) = N_J(y)^2$ for all $y \in J$. One now has

$$N_J(\widetilde{g}x^{\#}) = N_J(\lambda(g)^{-1}(gx)^{\#}) = \lambda(g)^{-3}N(gx)^2 = \lambda(g)^{-1}N(x)$$

proving statement two of the proposition.

Statement three of the proposition now follows as the proof of part one, using part two. The final statement follows from statement one. □

Exercise 3.0.5. Suppose $\delta \in \mathfrak{m}(J)$ and $\mu(\delta)$ is the scalar so that

$$(\delta(z_1), z_2, z_3) + (z_1, \delta(z_2), z_3) + (z_1, z_2, \delta(z_3)) = \mu(\delta)(z_1, z_2, z_3)$$

for all $z_1, z_2, z_3 \in J$. Prove that $\widetilde{\delta}(x \times y) = -\mu(\delta)(x \times y) + \delta(x) \times y + x \times \delta(y)$ for all $x, y \in J$.

Exercise 3.0.6. Prove that the map $\Phi : J \otimes J^{\vee} \to \mathfrak{m}(J)$ is equivariant, i.e., if $g \in M_J$ then $Ad(g)\Phi_{\gamma,x} = \Phi_{\widetilde{g}(\gamma),g(x)}$.

Proposition 3.0.7. [**Spr62**, Lemma 1 and Proposition 5]. *If $\gamma^\# = 0$ and $(\gamma, v) = 0$, then $\Phi_{\gamma,v} = \Phi'_{\gamma,v}$ satisfies $\Phi^2_{\gamma,v}(z) = -2(\gamma, z)\gamma \times v^\#$ and $\Phi^3_{\gamma,v} = 0$. Consequently, $\Phi_{\gamma,v}$ is nilpotent. Similarly, if $v^\# = 0$ and $(\gamma, v) = 0$, then $\Phi_{\gamma,v} = \Phi'_{\gamma,v}$ satisfies $\Phi^2_{\gamma,v}(z) = -2(\gamma^\# \times v, z)v$ and $\Phi^3_{\gamma,v} = 0$. Consequently, $\Phi_{\gamma,v}$ is nilpotent.*

When $J = H_3(C)$ with C an associative composition algebra, one has a map $\mathrm{GL}_3(C) \to M_J$ given by g acting on X is gXg^*. Here $X \in H_3(C)$ and $g \in \mathrm{GL}_3(C)$. This gives us a heuristic that when $C = \Theta$ is the octonions, we can think of $GE_6 = M_{H_3(\Theta)}$ as related to "$\mathrm{GL}_3(\Theta)$". We'll make this relation precise in the rest of this subsection.

For notation, denote $\epsilon_1 = \mathrm{diag}(1, 0, 0)$, $\epsilon_2 = \mathrm{diag}(0, 1, 0)$ and $\epsilon_3 = (0, 0, 1)$ as elements of $H_3(C)$.

Denote by $\mathrm{Spin}(\Theta)'$ the group

$$\mathrm{Spin}(\Theta)' = \{(g_1, g_2, g_3) \in O(\Theta)^3 : (g_1x_1, g_2x_2, g_3x_3) = (x_1, x_2, x_3) \forall x_j \in \Theta\}.$$

Note that we only require the g_j to be in $O(\Theta)$ as opposed to $\mathrm{SO}(\Theta)$. Clearly $\mathrm{Spin}(\Theta) \subseteq \mathrm{Spin}(\Theta)'$ and the groups have the same Lie algebra. I don't know if $\mathrm{Spin}(\Theta) = \mathrm{Spin}(\Theta)'$.

Lemma 3.0.8. *Suppose $J = H_3(\Theta)$. The simultaneous stabilizer of $\epsilon_1, \epsilon_2, \epsilon_3$ is* $\mathrm{Spin}(\Theta)'$.

Proof. It is clear that $\mathrm{Spin}(\Theta)'$ is contained in this stabilizer. For the converse, suppose g stabilizes the ϵ_i. Then g fixes 1_J and thus g is an automorphism of J. We must check that g preserves the spaces $x_i(\Theta)$. To see this, note that $x_1(\Theta)$'s can be characterized as the image of the map $\epsilon_1 \times z$ for $z \in (k\epsilon_1 + k\epsilon_2 + k\epsilon_3)^\perp$. The lemma follows. $\square$

We now make some explicit calculations, as given in the following proposition. For $x_1, x_2, x_3 \in \Theta$, denote by $V(x_1, x_2, x_3)$ the of J as

$$V(x_1, x_2, x_3) = \begin{pmatrix} 0 & x_3 & x_2^* \\ x_3^* & 0 & x_1 \\ x_2 & x_1^* & 0 \end{pmatrix}.$$

Proposition 3.0.9. *One has*

(1) $\Phi_{\epsilon_1, V(x_1,x_2,x_3)}(\epsilon_1) = V(0, x_2, x_3)$, $\Phi_{\epsilon_1, V(x_1,x_2,x_3)}(\epsilon_2) = 0$ *and* $\Phi_{\epsilon_1, V(x_1,x_2,x_3)}(\epsilon_3) = 0$.
(2) $\Phi_{V(x_1,x_2,x_3),\epsilon_1}(\epsilon_1) = 0$, $\Phi_{V(x_1,x_2,x_3),\epsilon_1}(\epsilon_2) = V(0, 0, x_3)$ *and* $\Phi_{V(x_1,x_2,x_3),\epsilon_1}(\epsilon_3) = V(0, x_2, 0)$.

Proof. Note the identity $\epsilon_1 \times V(x_1, x_2, x_3) = V(-x_1, 0, 0)$ and similarly for ϵ_2, ϵ_3. Also $\epsilon_1 \times \epsilon_2 = \epsilon_3$ and similarly for other permutations. From these identities the proposition is a simple calculation. $\square$

Let T be the torus of M_J that acts as $X \mapsto \mathrm{diag}(t_1, t_2, t_3) X \,\mathrm{diag}(t_1, t_2, t_3)$.

We can arrange the maps of Proposition 3.0.9 into an A_2 root system for T as follows:

$$\left(\begin{array}{c|c|c} & \Phi_{V(0,0,x_3),\epsilon_1}, \Phi_{\epsilon_2,V(0,0,x_3)} & \Phi_{V(0,x_2,0),\epsilon_1}, \Phi_{\epsilon_3,V(0,x_2,0)} \\ \Phi_{V(0,0,x_3),\epsilon_2}, \Phi_{\epsilon_1,V(0,0,x_3)} & & \Phi_{V(x_1,0,0),\epsilon_2}, \Phi_{\epsilon_3,V(x_1,0,0)} \\ \Phi_{V(0,x_2,0),\epsilon_3}, \Phi_{\epsilon_1,V(0,x_2,0)} & \Phi_{V(x_1,0,0),\epsilon_3}, \Phi_{\epsilon_2,V(x_1,0,0)} & \end{array}\right)$$

Note that the elements in the same box are the same map: For example, in the $(1, 2)$ box, both elements send ϵ_2 to $V(0, 0, x_3)$ and annihilate ϵ_1, ϵ_3. Then, the different annihilates all three ϵ_k, so is in $\mathfrak{spin}(\Theta)$. However, T acts on it by t_1/t_2, so the difference must be 0.

Let $\mathfrak{t}$ be the Lie algebra of T, so that a triple $(t_1, t_2, t_3) \in \mathfrak{t}$ acts on $H_3(C)$ as

$$\begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} \mapsto \begin{pmatrix} 2t_1c_1 & (t_1 + t_2)x_3 & (t_3 + t_1)x_2^* \\ (t_1 + t_2)x_3^* & 2t_2c_2 & (t_2 + t_3)x_1 \\ (t_3 + t_1)x_2 & (t_2 + t_3)x_1^* & 2t_3c_3 \end{pmatrix}.$$

Define $n_+(\Theta) \simeq \Theta^3$ as the sum of the maps in the $(1,2)$, $(1,3)$ and $(2,3)$ positions, and define $n_-(\Theta) \simeq \Theta^3$ as the sum of the maps in the $(2,1)$, $(3,1)$ and $(3,2)$ positions.

Proposition 3.0.10. *Suppose $J = H_3(\Theta)$. The map $n_-(\Theta) \oplus \mathfrak{t} \oplus \mathfrak{spin}(\Theta) \oplus n_+(\Theta) \to \mathfrak{m}(J)$ is a linear isomorphism.*

Proof. For the injectivity, note that T acts by different weights on the $n_-(\Theta)$, $n_+(\Theta)$, $\mathfrak{t} \oplus \mathfrak{spin}(\Theta)$, and that it acts by 1 on this last piece. So, we just must check that the map $\mathfrak{t} \oplus \mathfrak{spin}(\Theta) \to \mathfrak{m}(J)$ is injective. But for this, one uses the action on the diagonal elements.

For the surjectivity, suppose given $\Phi \in \mathfrak{m}(J)$. We will find $X \in n_+(\Theta)$, $Y \in n_-(\Theta)$ and $t \in \mathfrak{t}$ so that $\Phi - X - Y - t$ annihilates $\epsilon_1, \epsilon_2, \epsilon_3$. Then the proposition follows from Lemma 3.0.8. To find X, Y and t, first note that because $\epsilon_j^\# = 0$, one has $\epsilon_j \times \Phi(\epsilon_j) = 0$. It follows that $\Phi(\epsilon_1) = \mu_1\epsilon_1 + V(0, x_2^1, x_3^1)$ for some $\mu_1 \in k$ and $x_2^1, x_3^1 \in \Theta$. Similarly for ϵ_2, ϵ_3. The proposition now follows easily from Proposition 3.0.9, using that $(t_1, t_2, t_3) \cdot \epsilon_1 = 2t_1\epsilon_1$. $\square$

Corollary 3.0.11. *With $J = H_3(\Theta)$, $\dim_k \mathfrak{m}(J) = 79$.*

See [**PWZ19**, section 1.5] for the exponential of the $n(\Theta)$ action.

4. The group F_4

Recall that the group F_4 is defined as the group A_J with $J = H_3(\Theta)$. In this section, we give some results about the Lie algebra $\mathfrak{a}(J)$ for general CNS's J and also some results in the special case $J = H_3(\Theta)$.

For a cubic norm structure J, we have an isomorphism $J \to J^\vee$. We let ι denote this isomorphism.

The Jordan product is defined as $X \cdot Y = \frac{1}{2}\{X, Y\}$ where $\{X, Y\} = \Phi_{\iota(1),X}(Y) = \Phi_{\iota(1),Y}(X)$. A derivation of J is a linear map $\delta : J \to J$ satisfying $\delta(A \cdot B) = \delta(A) \cdot B + A \cdot \delta(B)$ for all $A, B \in J$. Note that $\Phi_{\iota(1),X}(Y) = \Phi_{\iota(1),Y}(X)$ so that $\{X, Y\} = \{Y, X\}$.

Exercise 4.0.1. Verify that $\{1_J, X\} = 2X$ for all $X \in J$. Check that

$$\{X, Y\} = X \times Y + (1, Y)X + (1, X)Y - (1, X, Y)1.$$

Use Lemma 4.0.2 to do the computation.

Lemma 4.0.2. *For a cubic norm structure J, one has $1_J \times x = (1_J, x)1_J - x$ for all $x \in J$.*

Proof. Pair both sides with an arbitrary $y \in J$. $\square$

For $J = H_3(C)$, the Jordan product is expressible in terms of the ordinary matrix multiplication. Namely, $\{X, Y\} = XY + YX$ where the products on the right are taken in terms of ordinary matrix multiplication.

Proposition 4.0.3. *Suppose $J = H_3(C)$. Then $\{X, Y\} = XY + YX$. In particular, the element on the RHS is in J.*

Proof. By linearization, it suffices to verify that $\frac{1}{2}\{X, X\} = X^2$ is in $H_3(C)$. One computes the RHS directly to be

$$X^2 = \begin{pmatrix} c_1^2 + n_C(x_3) + n_C(x_2) & * & * \\ * & * & x_3^*x_2^* + c_2x_1 + c_3x_1 \\ * & x_2x_3 + c_2x_1^* + c_3x_1^* & \end{pmatrix}$$

where the $*$'s are determined by cyclically permuting the indices for the written-out quantities. In particular, $X^2 \in H_3(C)$.

Now, by Lemma 4.0.2, $\frac{1}{2}\{X,X\} = -1_J \times X^{\#} + (1_J,X)X = -(1_J,X^{\#})1_J + X^{\#} + (1_J,X)X$. The LHS is thus

$$\frac{1}{2}\{X,X\} = (n_C(x_1)+n_C(x_2)+n_C(x_3)-c_1c_2-c_2c_3-c_3c_1)\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$$
$$+(c_1+c_2+c_3)\begin{pmatrix}c_1&x_3&x_2^*\\x_3^*&c_2&x_1\\x_2&x_1^*&c_3\end{pmatrix}+\begin{pmatrix}c_2c_3-n_C(x_1)&*&*\\ *&*&x_3^*x_2^*-c_1x_1\\ *&*&*\end{pmatrix}.$$

Comparing with X^2 gives the result. □

PROPOSITION 4.0.4. *Suppose $\delta \in \mathfrak{m}^0(J) = Lie(M_J^1)$. Then the following are equivalent:*

(1) *δ is a derivation of J*
(2) *δ annihilates 1_J*
(3) *δ preserves the pairing $(\ ,\ )$.*

PROOF. Because $1 \cdot 1 = 1$, (1) implies (2). To see that (2) implies (3), use that $(x,y) = \frac{1}{4}(1,1,x)(1,1,y) - (1,x,y)$.

We now check that (3) implies (2). Thus suppose δ preserves the pairing. Then δ commutes with the isomorphism $\iota : J \to J^\vee$. Now $1_J \times 1_J = 2 1_{J^\vee} = 2\iota(1_J)$, which is an axiom. Applying δ gives $\delta(1_{J^\vee}) = 1_J \times \delta(1_J)$. Recall the relation $1 \times x = (1_{J^\vee}, x)1_{J^\vee} - \iota(x)$, which can be checked by pairing both sides with an arbitrary element $z \in J$. Thus

$$\delta(1_{J^\vee}) = 1_J \times \delta(1_J) = (1_{J^\vee}, \delta(1_J))1_{J^\vee} - \iota(\delta(1_J)) = -\delta(1_J^\vee)$$

because $(1,\delta(1)) = 0$. Consequently δ annihilates 1.

We can now check that (3) implies (1). Indeed, assuming (3), we know that δ annihilates 1 and commutes with ι. Now use that $\{A,B\} = \Phi_{\iota(1_J),A}(B)$, and it follows that δ is a derivation. □

We define an A_J-equivariant map $\wedge^2 J \to \mathfrak{a}(J)$ as $\Phi_{X\wedge Y} = \Phi_{\iota X,Y} - \Phi_{\iota Y,X}$.

LEMMA 4.0.5. *The element $\Phi_{X\wedge Y} \in \mathfrak{m}(J)$ in fact is in $\mathfrak{a}(J)$.*

PROOF. One calculates that $\Phi_{X\wedge Y}$ annihilates 1_J. In more detail

$$\begin{aligned}\Phi_{\iota(X),Y}(1) &= -X \times (Y \times 1) + (X,1)Y + (X,Y)1\\ &= -X \times ((1,Y)1 - Y) + (X,1)Y + (X,Y)1\\ &= -(1,Y)((1,X)1 - X) + X \times Y + (X,1)Y + (X,Y)1\\ &= -(1,X)(1,Y)1 + (1,Y)X + (1,X)Y + (X,Y)1 + X \times Y.\end{aligned}$$

This is symmetric in X,Y so $\Phi_{X\wedge Y}$ annihilates 1. □

PROPOSITION 4.0.6. *With $J = H_3(\Theta)$, one has a decomposition $\mathfrak{a}(J) = \mathfrak{spin}(\Theta) \oplus \Theta^3$. Here $\Theta^3 \subseteq a(J)$ is defined as the sum of elements of the form $\Phi_{\epsilon_2\wedge V(x_1,0,0)}, \Phi_{\epsilon_3\wedge V(0,x_2,0)}, \Phi_{\epsilon_1\wedge V(0,0,x_3)}$.*

PROOF. Inspired by [**Jac71**, page 20]. The proof is similar to the proof of Proposition 3.0.10. Let $D = \Phi_{\epsilon_2\wedge V(x_1,0,0)} + \Phi_{\epsilon_3\wedge V(0,x_2,0)} + \Phi_{\epsilon_1\wedge V(0,0,x_3)}$. Then

- $D(\epsilon_1) = V(0,-x_2,x_3)$
- $D(\epsilon_2) = V(x_1,0,-x_3)$
- $D(\epsilon_3) = V(-x_1,x_2,0)$.

The injectivity of the map $\mathfrak{spin}(\Theta) \oplus \Theta^3 \to \mathfrak{a}(J)$ follows.

For the surjectivity, suppose $\delta \in \mathfrak{a}(J)$. Then as in the proof of Proposition 3.0.10, $\delta(\epsilon_1) = \mu_1\epsilon_1 + V(0,-x_2^1,x_3^1)$ and similarly for $\delta(\epsilon_2),\delta(\epsilon_3)$. Because $\delta(\epsilon_1+\epsilon_2+\epsilon_3) = 0$, one concludes $\mu_1 = \mu_2 = \mu_3 = 0$ and $x_2^1 = x_2^3$ etc. The surjectivity follows. □

EXERCISE 4.0.7. This is inspired by [**Jac71**, page 34, Proposition 11]. The point of this exercise is to compute the roots of F_4.

(1) Suppose V is an even-dimensional orthogonal space with isotropic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. Verify that $\mathfrak{h} = Span\{e_1 \wedge f_1, \ldots, e_n \wedge f_n\} \subseteq \wedge^2 V = \mathfrak{so}(V)$ is a Cartan subalgebra of $\mathfrak{so}(V)$. Hint: Compute the weights of this Cartan subalgebra on V, and deduce its weights on $\wedge^2 V$. Observe that the weight spaces outside of $\mathfrak{h}$ are one-dimensional and nonzero.

(2) For $V = \Theta$, consider $\mathfrak{h}_\Theta = Span\{\epsilon_1 \wedge \epsilon_2, -e_1 \wedge e_1^*, -e_2 \wedge e_2^*, -e_3 \wedge e_3^*\}$. Using the formulas for the action of $\wedge^2\Theta$ on its triality representation (i.e., Exercise 4.0.7), compute the weights of $\mathfrak{h}_\Theta$ on the triality representation. One should get the following answer:

(a) For the "X" representation, the weights

$$\pm(1,0,0,0), \pm(0,1,0,0), \pm(0,0,1,0), \pm(0,0,0,1).$$

(b) For the "Y" representation, the weights

$$\pm\frac{1}{2}(1,1,1,1), \pm\frac{1}{2}(1,1,-1,-1), \pm\frac{1}{2}(1,-1,1-1), \pm\frac{1}{2}(1,-1,-1,1).$$

(c) For the "Z" representation, the weights

$$\pm\frac{1}{2}(1,1,1,-1), \pm\frac{1}{2}(1,1,-1,1), \pm\frac{1}{2}(1,-1,1,1), \pm\frac{1}{2}(-1,1,1,1).$$

(3) Using the decomposition $\mathfrak{a}(J) = \mathfrak{f}_4 = \mathfrak{spin}(\Theta) \oplus \Theta^3$, compute the weights of $\mathfrak{h}_\Theta$ on $\mathfrak{a}(J)$. One should obtain that outside of $\mathfrak{h}_\Theta$, the weights are the 24 elements $(\pm1, \pm1, 0, 0)$ and permutations, the eight elements $(\pm1, 0, 0, 0)$ and permutations, and the 16 elements $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. Because these are nonzero and one-dimensional, deduce that $\mathfrak{h}_\Theta$ is a Cartan subalgebra of $\mathfrak{a}(J)$ with roots given as just listed.

Our next task is to give an A_J-decomposition of the Lie algebra $\mathfrak{m}(J)$. Identity $X \in J$ with the Lie element $\Phi_{\iota 1, X} \in \mathfrak{m}(J)$. We will prove the following theorem.

THEOREM 4.0.8. *The map* $\mathfrak{a}(J) \oplus J \to \mathfrak{m}(J)$ *is a linear isomorphism. Moreover,* $[\Phi_{\iota 1, X}, \Phi_{\iota(1), Y}] = \Phi_{Y \wedge X}$.

PROOF. Suppose $\delta \in \mathfrak{m}(J)$ is given. If $Z = \delta(1_J)$, then $\delta - \frac{1}{2}\Phi_{1,Z}$ annihilates 1_J. Thus $\mathfrak{a}(J) + J \to \mathfrak{m}(J)$ is surjective. The injectivity is just as easy, by evaluating on 1_J. For the second part of the theorem, we first require some other results. ☐

If $\gamma \in J^\vee$ and $x \in J$, we have defined an element $\Phi_{\gamma,x} \in \mathfrak{m}(J)$. The action of $\Phi_{\gamma,x}$ on $J^\vee$ is given by (as one can check)

$$\Phi_{\gamma,x}(\mu) = x \times (\gamma \times \mu) - (x,\gamma)\mu - (x,\mu)\gamma.$$

LEMMA 4.0.9. *One has*

(1) $[\delta, \Phi_{\gamma,x}] = \Phi_{\delta(\gamma),x} + \Phi_{\gamma,\delta(x)}$
(2) $\Phi_{\iota 1, X}(\iota(Z)) = -\iota\{X, Z\}$.

PROOF. The first identity is the Lie-theoretic version of the group-theoretic identity proved above.

For the second identity,

$$\begin{aligned} -\Phi_{\iota(1),X}(\iota(Z)) &= -X \times (1 \times Z) + (X,1)Z + (X,Z)1 \\ &= X \times Z - (1,Z)((1,X)1 - X) + (X,1)Z + (X,Z)1 \\ &= X \times Z + (1,Z)X + (1,X)Z + (X,Z)1 - (1,Z)(1,X)1 \\ &= X \times Z + (1,Z)X + (1,X)Z - (1,X,Z)1 \end{aligned}$$

$$= \{X, Z\}$$

□

THEOREM 4.0.10. *One has the identity* $\Phi_{\iota(X),Y} + \Phi_{\iota(Y),X} = \{\{X, Y\}, \bullet\} = \Phi_{\iota(1),\{X,Y\}}$.

PROOF. The identity is [**Jac69**, QJ27,page 25]. □

REMAINDER OF PROOF OF THEOREM 4.0.8. We must prove $[\Phi_{\iota(1),X}, \Phi_{\iota(1),Y}] = \Phi_{Y\wedge X}$. From the previous lemma and theorem we obtain

$$\begin{aligned}[\Phi_{\iota(1),X}, \Phi_{\iota(1),Y}] &= \Phi_{\Phi_{\iota(1),X}(\iota(1)),Y} + \Phi_{\iota(1),\Phi_{\iota(1),X}(Y)} \\ &= -2\Phi_{\iota(X),Y} + \Phi_{\iota 1,X,Y} \\ &= \Phi_{\iota(Y),X} - \Phi_{\iota(X),Y}.\end{aligned}$$

□

Note the equality

$$\begin{aligned}\Phi_{\iota(X),Y} &= \frac{1}{2}(\Phi_{\iota X,Y} + \Phi_{\iota Y,X}) + \frac{1}{2}(\Phi_{\iota X,Y} - \Phi_{\iota Y,X}) \\ &= \frac{1}{2}\Phi_{\iota 1,\{X,Y\}} + \frac{1}{2}\Phi_{X\wedge Y}.\end{aligned}$$

This is how an arbitrary $\Phi_{\iota(X),Y}$ decomposes in the direct sum $\mathfrak{m}(J) \simeq \mathfrak{a}(J) \oplus J$ of Theorem 4.0.8.

CHAPTER 3

The Freudenthal construction and E_7

1. The Freudenthal construction

Suppose that J is a cubic norm structure, or that $J, J^\vee$ is a cubic norm pair. Define a vector space $W_J = k \oplus J \oplus J^\vee \oplus k$. The space W_J comes equipped with a symplectic pairing $\langle\, , \rangle$ and a quartic form q, which are defined as follows. We write a typical element in W_J as $v = (a, b, c, d)$, so that $a, d \in k$, $b \in J$ and $c \in J^\vee$. Then

$$\langle (a, b, c, d), (a', b', c', d')\rangle = ad' - (b, c') + (c, b') - da'$$

and

$$q((a, b, c, d)) = (ad - (b, c))^2 + 4aN(c) + 4dN(b) - 4(b^\#, c^\#).$$

The definition of this algebraic data goes back to Freudenthal.

We now define a group

$$H_J = \{(g, \nu) \in \mathrm{GL}(W_J) \times \mathrm{GL}_1 : \langle gv, gv'\rangle = \nu\langle v, v'\rangle \,\forall v, v' \in W_J \text{ and } q(gv) = \nu^2 q(v) \,\forall v \in W_J\}.$$

We set $H_J^1 = \ker \nu : H_J \to \mathrm{GL}_1$. The element ν is called the similitude. The group GE_7 is defined as H_J with $J = H_3(\Theta)$ and E_7 is H_J^1.

We now construct explicit elements of H_J.

First, if $\lambda \in \mathrm{GL}_1$, then one checks immediately that the map $(a, b, c, d) \mapsto (\lambda^2 a, \lambda b, c, \lambda^{-1}d)$ is in H_J with similitude equal to λ.

Next, if $m \in M_J$ with $N_J(mX) = \lambda N_J(X)$ for all $X \in J$, the map $(a, b, c, d) \mapsto (\lambda a, m(b), \widetilde{m}(c), \lambda^{-1}d)$ is in H_J^1. Indeed, for this, one uses that $N_{J^\vee}(\widetilde{m}(c)) = \lambda^{-1}N_{J^\vee}(c)$ and

$$4(m(b)^\#, \widetilde{m}(c)^\#) = (\lambda\widetilde{m}(b \times b), \lambda^{-1}m(c \times c)) = (b \times b, c \times c) = 4(b^\#, c^\#).$$

If J is a cubic norm structure, so that we have an identification $\iota : J \leftrightarrow J^\vee$, we have a map

$$(a, b, c, d) \mapsto (-d, \iota(c), -\iota(b), a)$$

and one checks quickly that this map is in H_J^1.

The final elements we write down are more complicated. For $x \in J$ define

$$n_J(x)(a, b, c, d) = (a, b + ax, c + b \times x + ax^\#, d + (c, x) + (b, x^\#) + aN_J(x))$$

and for $\gamma \in J^\vee$ define

$$n_{J^\vee}(\gamma)(a, b, c, d) = (a + (b, \gamma) + (c, \gamma^\#) + dN_{J^\vee}(\gamma), b + c \times \gamma + d\gamma^\#, c + d\gamma, d).$$

We will prove the following theorem.

THEOREM 1.0.1. *The linear maps $n_J(x)$ and $n_{J^\vee}(\gamma)$ are in H_J^1 for all $x \in J$ and $\gamma \in J^\vee$. Moreover, $n_J(x)n_J(y) = n_J(x + y)$ for all $x, y \in J$ and similarly for $n_{J^\vee}$.*

To prove the theorem, we compute with the Lie algebra versions of these maps. Namely, define $n_{Lie,J}(x) \in End(W_J)$ as

$$n_{Lie,J}(x)(a, b, c, d) = (0, ax, b \times x, (c, x))$$

and similarly for $n_{Lie,J^\vee}(\gamma)$ for $\gamma \in J^\vee$. One verifies the following lemma.

Lemma 1.0.2. *One has $n_{Lie,J}(x)^4 = 0$, so that $n_{Lie,J}(x)$ is nilpotent. Moreover,*

$$n_J(x) = \exp(n_{Lie,J}(x)) = 1 + n_{Lie,J}(x) + \frac{1}{2}n_{Lie,J}(x)^2 + \frac{1}{6}n_{Lie,J}(x)^3$$

and similarly $n_{J^\vee}(\gamma) = \exp(n_{Lie,J^\vee}(\gamma))$.

Proof. The computation is straightforward. One has $\exp(n_{Lie,J}(x))(a,b,c,d)$

$$\begin{aligned}&= (a,b,c,d) + (0, ax, b\times x, (c,x)) + \frac{1}{2}(0,0,ax\times x, (b\times x, x)) + \frac{1}{6}(0,0,0,a(x\times x, x))\\ &= n_J(x)(a,b,c,d).\end{aligned}$$

$\square$

Denote by $(\ ,\ ,\ ,\ )_{W_J}$ the unique symmetric four-linear form normalized so that $(v,v,v,v)_{W_J} = 2q(v)$. Define $t : W_J \times W_J \times W_J \to W_J$ as $(w,x,y,z) = \langle w, t(x,y,z)\rangle$ and set $v^\flat = t(v,v,v)$.

Thus to prove Theorem 1.0.1, it suffices to verify

(1) $\langle v_1, n_{Lie,J}(x)v_2\rangle$ is symmetric in v_1, v_2;
(2) $\langle n_{Lie,J}(x)(v), v^\flat\rangle = 0$ for all $v \in W_J$.

We use the following proposition, which computes $v^\flat$ in coordinates.

Proposition 1.0.3. *Suppose $v = (a,b,c,d)$. Then $v^\flat = (a^\flat, b^\flat, c^\flat, d^\flat)$ with*

- $a^\flat = -a(ad - (b,c)) - 2N(b)$;
- $b^\flat = -2c\times b^\# + 2ac^\# - (ad-(b,c))b$;
- $c^\flat = 2b\times c^\# - 2db^\# + (ad - (b,c))c$;
- $d^\flat = d(ad-(b,c)) + 2N(c)$.

Proof of Theorem 1.0.1. Suppose $v = (a,b,c,d)$ and $v^\flat = (a^\flat, b^\flat, c^\flat, d^\flat)$. We have

$$\begin{aligned}\langle n_{Lie,J}(x)(v), v^\flat\rangle &= \langle (0, ax, b\times x, (c,x)), (a^\flat, b^\flat, c^\flat, d^\flat)\rangle\\ &= (x, -ac^\flat + b\times b^\flat - a^\flat c).\end{aligned}$$

The quantity paired with x is

$$\begin{aligned}-ac^\flat + b\times b^\flat - a^\flat c = &-2ab\times c^\# + 2adb^\# - (ad-(b,c))ac\\ &- 2b\times(b^\#\times c) + 2ab\times c^\# - (ad-(b,c))b\times b\\ &+ a(ad-(b,c))c + 2N(b)c.\end{aligned}$$

Most terms cancel, and then the proof follows from the identity $b\times(b^\#\times c) = N(b)c + (b,c)b^\#$. $\square$

It remains to prove Proposition 1.0.3.

Proof of Proposition 1.0.3. Set $v = (a,b,c,d)$, $v' = (a',b',c',d')$. Then $\frac{1}{2}q(v+\epsilon v') = \frac{1}{2}q(v) + \epsilon\langle v', v^\flat\rangle + O(\epsilon^2)$, so it suffices to compute the coefficient of ϵ in $\frac{1}{2}q(v+\epsilon v')$. With $\epsilon^2 = 0$, one has

$$\begin{aligned}\frac{1}{2}q(v+\epsilon v') = &\frac{1}{2}(ad - (b,c) + \epsilon(ad' + a'd - (b,c') - (b',c)))^2 + 2(a+\epsilon a')N(c+\epsilon c')\\ &+ 2(d+\epsilon d')N(b+\epsilon b') - 2((b+\epsilon b')^\#, (c+\epsilon c')^\#).\end{aligned}$$

The coefficient of ϵ is thus

$$\begin{aligned}\epsilon \text{ coefficient } &= (ad' + a'd - (b,c') - (b',c))(ad-(b,c)) + 2a'N(c) + 2d'N(b)\\ &+ 2a(c^\#, c') + 2d(b^\#, b') = 2(b', b\times c^\#) - 2(b^\#\times c, c')\\ &= a'((ad-(b,c))d + 2N(c)) + (b', -(ad-(b,c))c + 2db^\# - 2b\times c^\#)\end{aligned}$$

$$+ (c', -(ad-(b,c))b + 2ac^{\#} - 2b^{\#} \times c) + d'(a(ad-(b,c)) + 2N(b)).$$

Comparing with the statement of Proposition 1.0.3 gives the result. $\square$

2. The group and Lie algebra E_7

In this section, we describe the Lie algebra $\mathfrak{h}_J^0$ of H_J^1. We give what is called the Koecher-Tits construction of the Lie algebra.

We first define $\mathfrak{h}'_J = J^\vee \oplus \mathfrak{m}_J \oplus J$. We will put a Lie algebra structure on $\mathfrak{h}'_J$, construct an explicit map $\mathfrak{h}'_J \to End(W_J)$ that lands in $\mathfrak{h}_J^0$, and prove that this map is a Lie algebra isomorphism.

To begin, define a bracket $[\,,\,]$ on $\mathfrak{h}'_J$ as $[\gamma, x] = \Phi_{\gamma,x}$ if $\gamma \in J^\vee$ and $x \in J$, together with $[J,J] = 0$, $[J^\vee, J^\vee] = 0$, and $[\phi, x] = \phi(x)$, $[\phi, \gamma] = \widetilde{\phi}\gamma$.

Proposition 2.0.1. *This bracket satisfies the Jacobic identity. Consequently, $\mathfrak{h}'_J$ is a Lie algebra.*

Proof. The identity that must be verified is $\sum_{cyc}[X,[Y,Z]] = 0$. By linearity, it suffices to verify the identity for X, Y, Z in the various graded pieces. If all three are in J, each term in the sum is 0 so that the sum itself is 0. Similarly, if two of X, Y, Z are in J and the third in $\mathfrak{m}_J$, then all terms of the sum are 0 so the sum itself is 0.

If two of X, Y, Z are in J and the third is in $J^\vee$, the Jacobi identity becomes the equality $\Phi_{\gamma,X}(Y) = \Phi_{\gamma,Y}(X)$.

If one term is in J and two are in $\mathfrak{m}_J$, the Jacobi identity becomes the fact that $[\phi_1, \phi_2](x) = \phi_1(\phi_2(x)) - \phi_2(\phi_1(x))$, i.e., that the bracket of $\mathfrak{m}_J$ and J comes from an action of $\mathfrak{m}_J$ on J.

When one element is in J, one in $\mathfrak{m}_J$ and one in $J^\vee$, the Jacobi identity becomes the relation $[\phi, \Phi_{\gamma,x}] = \Phi_{\widetilde{\phi}(\gamma),x} + \Phi_{\gamma,\phi(x)}$.

The other cases follow just as above with the roles of J and $J^\vee$ interchanged. $\square$

We now define a map $\mathfrak{h}'_J \to End(W_J)$. First suppose $\phi \in \mathfrak{m}_J$ with multiplier $t(\phi)$, so that $(\phi(z_1), z_2, z_3) + (z_1, \phi(z_2), z_3) + (z_1, z_2, \phi(z_3)) = t(\phi)(z_1, z_2, z_3)$ for all $z_1, z_2, z_3 \in J$. Define $M(\phi) \in End(W_J)$ as

$$M(\phi)(a,b,c,d) = \left(-\frac{t(\phi)}{2}a, -\frac{t(\phi)}{2}b + \phi(b), \frac{t(\phi)}{2}c + \widetilde{\phi}(c), \frac{t(\phi)}{2}d\right).$$

Now, define $\mathfrak{h}'_J \to End(W_J)$ as

$$(\gamma, \phi, x) \mapsto n_{Lie,J^\vee}(\gamma) + M(\phi) + n_{Lie,J}(-x).$$

To see that the map lands in $\mathfrak{h}_J^0$, we just must check that $M(\phi)$ is in $\mathfrak{h}_J^0$. To see that it is, we work on the group level, and note that $(a,b,c,d) \mapsto (\lambda^{-3}a, \lambda^{-1}b, \lambda c, \lambda^3 d)$ is in H_J^1 so that

$$(a,b,c,d) \mapsto (t(\phi)a, \phi(b), \widetilde{\phi}(c), -t(\phi)d) + \frac{t(\phi)}{2}(-3a, -b, c, 3d) = M(\phi)(a,b,c,d)$$

is in $\mathfrak{h}_J^0$.

Proposition 2.0.2. *The map $\mathfrak{h}'_J \to \mathfrak{h}_J^0$ is a Lie algebra homomorphism.*

Proof. We compute

$$\begin{aligned}
[n_{Lie,J^\vee}(\gamma), n_{Lie,J}(-x)](a,b,c,d) &= [n_{Lie,J}(x), n_{Lie,J^\vee}(\gamma)](a,b,c,d) \\
&= n_{Lie,J}(x)((b,\gamma), c\times\gamma, d\gamma, 0) - n_{Lie,J^\vee}(\gamma)(0, ax, b\times x, (c,x)) \\
&= (-a(x,\gamma), -\gamma\times(b\times x) + (b,\gamma)x, x\times(\gamma\times c) - (c,x)\gamma, d(\gamma,x)) \\
&= M(\Phi_{\gamma,x})(a,b,c,d)
\end{aligned}$$

because $t(\Phi_{\gamma,x}) = 2(\gamma, x)$.

We also compute

$$\begin{aligned}
[M(\phi), n_{Lie,J}(x)](a,b,c,d) &= M(\phi)(0, ax, b \times x, (c,x)) - n_{Lie,J}(x)(-ta/2, -tb/2 + \phi(b), tc/2 + \widetilde{\phi}(c), td/2) \\
&= (0, -tax/2 + a\phi(x), t/2(b \times x) + \widetilde{\phi}(b \times x), t(c,x)/2) \\
&\quad + (0, tax/2 + t/2(b \times x) - x \times \phi(b), -t(c,x)/2 - (x, \widetilde{\phi}(c))) \\
&= (0, a\phi(x), b \times \phi(x), (c, \phi(x))) \\
&= n_{Lie,J}(\phi(x))(a,b,c,d).
\end{aligned}$$

The computation for $n_{Lie,J^\vee}(\widetilde{\phi}(\gamma))$ is similar. $\square$

To prove that the map $\mathfrak{h}'_J \to \mathfrak{h}^0_J$ is an isomorphism, we first take a detour into a special set of elements of W_J, called *rank one* elements.

3. Rank one elements

We begin with the notion of rank for elements of J.

Definition 3.0.1. All elements of J are of rank at most 3. If $N(x) = 0$, then x has rank at most 2. If $x^\# = 0$, then x has rank at most one. If $x = 0$, then x has rank 0.

Here is the definition of rank for elements of W_J.

Definition 3.0.2. All elements of W_J are of rank at most 4. If $q(v) = 0$, then v has rank at most 3. If $v^\flat = 0$, then v has rank at most two. If $(v, v, w', w) = 0$ for all $w' \in (kv)^\perp$, then v has rank at most one. If $v = 0$, then v has rank 0.

We will later prove that v has rank at most one if and only if $3t(v,v,x) + \langle v, x\rangle v = 0$ for all $x \in W_J$. (Note that this condition implies the rank one condition of the definition, but the converse is not at all obvious.)

We now give the key computation that we will need, and then give the results that follow from this computation.

Proposition 3.0.3. *Suppose $v = (a,b,c,d)$ and $x = (\alpha, \beta, \gamma, \delta)$. Then $\frac{1}{2}\Phi_{v,v}(x) = 3t(v,v,x) + \langle v, x\rangle v = (a'', b'', c'', d'')$ with*

$$\begin{aligned}
a'' &= \alpha((b,c) - 3ad) + 2(\beta, ac - b^\#) \\
b'' &= 2\alpha(c^\# - db) + \frac{1}{3}((b,c) - 3ad)\beta + 2\Phi'_{c,b}(\beta) + 2(ac - b^\#) \times \gamma \\
c'' &= 2(c^\# - db) \times \beta + \frac{1}{3}(3ad - (b,c))\gamma + 2\widetilde{\Phi'_{c,b}}(\gamma) + 2\delta(ac - b^\#) \\
d'' &= 2(c^\# - db, \gamma) + (3ad - (b,c))\delta.
\end{aligned}$$

Here recall that for $\gamma \in J^\vee$, $x, z \in J$

$$\Phi'_{\gamma,x}(z) = -\gamma \times (x \times z) + (\gamma, z)x + \frac{1}{3}(x, \gamma)z.$$

Proof. We begin with formulas for $v^\flat$ and $3t(v,v,x)$ for $v, x \in W_J$. Then one has $v^\flat = (a^\flat, b^\flat, c^\flat, d^\flat)$ with

- $a^\flat = -a^2 d + a(b,c) - 2n(b)$;
- $b^\flat = -2c \times b^\# + 2ac^\# - (ad - (b,c))b$;
- $c^\flat = 2b \times c^\# - 2db^\# + (ad - (b,c))c$;

- $d^\flat = ad^2 - d(b,c) + 2n(c)$.

Symmetrizing this formula for $v^\flat$, one finds that $3t(v,v,x) = (a',b',c',d')$, with

$a' = \alpha((b,c) - 2ad) + (\beta, ac - 2b^\#) + (\gamma, ab) - \delta a^2$

$b' = \alpha(2c^\# - bd) + ((b,c) - ad)\beta - 2c \times (b \times \beta) + (\beta, c)b + 2(ac - b^\#) \times \gamma + (b,\gamma)b - \delta ab$

$c' = \alpha(dc) + 2(c^\# - db) \times \beta + (ad - (b,c))\gamma + 2b \times (c \times \gamma) - (b,\gamma)c - (\beta,c)c + (ac - 2b^\#)\delta$

$d' = \alpha d^2 + (\beta, -dc) + (2c^\# - db, \gamma) + (2ad - (b,c))\delta.$

The element $3t(v,v,x) + \langle v, x\rangle v$ is then computed from the above quantity. $\square$

PROPOSITION 3.0.4. *The element $e = (1,0,0,0)$ of W_J is rank one. In fact, $\frac{1}{2}\Phi_{e,e}(x) = 0$ for all $x \in W_J$.*

PROOF. This follows directly from the proposition above. $\square$

LEMMA 3.0.5. *If $v = (1,0,c,d)$ is rank one, then $c = 0$ and $d = 0$.*

PROOF. From the fact that $v^\flat = 0$, we obtain $d = 0$ and $c^\# = 0$. Now, note that

$$\langle (1,0,c,0), (\alpha,\beta,\gamma,\delta)\rangle = \delta + (\beta, c).$$

However, in the notation of the proposition above, $a'' = 2(\beta,c)$. By taking β arbitrary and $\delta = -(c,\beta)$, we see that $x \in (kv)^\perp$ but $t(v,v,x) = 0$ implies $(c,\beta) = 0$. Since β is arbitrary, we get $c = 0$. $\square$

It is clear that the set of rank one elements is an H_J-set. In fact,

PROPOSITION 3.0.6. *There is one H_J^1-orbit of rank one lines. In fact, denote by H_J' the subgroup of H_J generated by the explicit elements described above. Then there is one H_J' orbit of rank one lines.*

PROOF. We first claim that we can use the elements of H_J' to move an arbitrary nonzero $v = (a,b,c,d)$ to one of the form $(1,b',c',d')$. To see this, note that $n_{J^\vee}(t\gamma)(a,b,c,d) = (a + t(b,\gamma) + t^2(c,\gamma^\#) + t^3 dN(\gamma), *, *, *)$. If this first component is 0 for all $t \in k$ then $a = (b,\gamma) = (c,\gamma^\#) = dN(\gamma) = 0$. If $(b,\gamma) = 0$ for all $\gamma \in J^\vee$, then $b = 0$. Similarly, if $dN(\gamma) = 0$ for all $\gamma \in J^\vee$, then $d = 0$. Finally, because the elements $\mu\gamma^\#$ are Zariski dense in J, if $(c,\gamma^\#) = 0$ for all $\gamma \in J^\vee$, then $c = 0$. This proves that we can make the first term nonzero. Then, by applying the map $(a,b,c,d) \mapsto (\lambda^{-1}a, b, \lambda c, \lambda^2 d)$ we see that we can assume $a = 1$.

Now, we have an element of the form $(1,b',c',d')$. Then apply $n_J(-b')$ to obtain an element of the form $(1,0,c,d)$. The proposition thus follows from the lemma. $\square$

THEOREM 3.0.7. *If v is rank one, then $3t(v,v,x) = \langle x, v\rangle v$ for all $x \in W_J$.*

PROOF. It is true for $e = (1,0,0,0)$ and thus it is true for every rank one element by H_J-equivariance. $\square$

Thus, we could have equivalently defined rank one elements in terms of the equality of this theorem.

COROLLARY 3.0.8. *An element $v = (a,b,c,d) \in W_J$ is rank at most one if and only if*

(1) $b^\# - ac = 0$,
(2) $c^\# - db = 0$,
(3) $3ad - (b,c) = 0$,
(4) *and* $\Phi'_{b,c} = 0$.

PROOF. We computed above explicitly the value $3t(v,v,x)+\langle v,x\rangle v$ for general v and $x=(\alpha,\beta,\gamma,\delta)$. Taking the coefficients of $\alpha,\beta,\gamma,\delta$ in this explicit computation gives the corollary. $\square$

For a rank one element e, define $X_e=\{x\in W_J : (e,x,v,v')=0\,\forall v,v'\in (ke)^\perp\}$.

LEMMA 3.0.9. *If $e=(1,0,0,0)$ then $X_e=\{(a,b,0,0)\in W_J\}$.*

PROOF. Set $v=(1,\mu b+\lambda b',\mu c+\lambda c',0)$. It suffices to compute the compute the coefficient of $\lambda\mu$ in $v^\flat$. This coefficient is immediately computed to be $((b,c')+(b',c),2c\times c',0,0)$. The lemma follows. $\square$

PROPOSITION 3.0.10. *Set $e=(1,0,0,0)$ and $f=(0,0,0,1)$. The simultaneous stabilizer of the lines ke and kf in H_J^1 is M_J.*

PROOF. Suppose g stabilizes the lines ke and kf. Then g also stabilizes $(ke)^\perp$, $(kf)^\perp$, X_e and X_f. Taking intersections, one sees that g stabilizes the spaces $(0,*,0,0)$ and $(0,0,*,0)$. The proposition now follows without much effort. $\square$

EXERCISE 3.0.11. Finish the proof of the above proposition.

COROLLARY 3.0.12. *Suppose $\delta\in\mathfrak{h}_J^0$ stabilizes the lines $k(1,0,0,0)=ke$ and $k(0,0,0,1)=kf$. Then $\delta=M(\phi)$ for some $\phi\in\mathfrak{m}_J$.*

PROOF. From the previous proposition, we have

$$\delta(a,b,c,d)=(t(\phi')a,\phi'(b),\widetilde{\phi}'(c),-t(\phi')d)$$

for some $\phi'\in\mathfrak{m}_J$. Now set $\phi=-t(\phi')Id+\phi'$. One has $\phi\in\mathfrak{m}_J$ with $t(\phi)=-2t(\phi')$. The corollary follows. $\square$

We can prove that the map $\mathfrak{h}_J'\to\mathfrak{h}_J^0$ is an isomorphism.

PROPOSITION 3.0.13. *The Lie algebra homomorphism $\mathfrak{h}_J'\to\mathfrak{h}_J^0$ is an isomorphism.*

PROOF. We first prove injectivity. Thus suppose $\delta=n_{Lie,J^\vee}(\gamma)+M(\phi)-n_{Lie,J}(x)=0$. Then, evaluating δ on $e=(1,0,0,0)$, we find $x=0$. Evaluating on $f=(0,0,0,1)$, we find $\gamma=0$. Evaluating $M(\phi)$ on (a,b,c,d), we find $\phi=0$. Thus our map is injective.

For the surjectivity, suppose $\delta\in\mathfrak{h}_J^0$ is given. We claim $\delta(e)\in X_e$. Indeed, if $v,v'\in(ke)^\perp$, then $(e,e,v,v')=0$. Applying δ gives $(e,\delta(e),v,v')=0$, as $(e,e,\delta(v),v')$ and $(e,e,v,\delta(v'))$ are each 0, because e is rank one.

Because $\delta(e)\in X_e$, there exists $x\in J$ so that $\delta+n_{Lie,J}(x)$ stabilizes the line ke. Similarly, there exists $\gamma\in J^\vee$ so that $\delta+n_{Lie,J}(x)-n_{Lie,J^\vee}(\gamma)$ stabilizes both the lines ke and kf. The surjectivity now follows from Corollary 3.0.12. $\square$

4. More on rank one elements

In this section, we develop more of the structure theory of the space W_J and of the Lie algebra $\mathfrak{h}_J^0$.

We begin with the following important theorem.

THEOREM 4.0.1. *Suppose $v\in W_J$ has $q(v)\neq 0$, and let ω be the image of x in $E_v:=k[x]/(x^2-q(v))$. Then $\omega v+v^\flat$ is rank one in $W_J\otimes E_v$.*

PROOF. By H_J-equivariance, extension of scalars, and a Zariski density argument, we are reduced to considering the case $v=(1,0,c,d)$ with $N(c)\neq 0$ and $\omega\in k$ satisfies $\omega^2=q(v)=d^2+4N(c)$. Then one computes

$$\omega v+v^\flat=(\omega-d,2c^\#,(\omega+d)c,\omega d+d^2+2N(c))$$

$$= (\omega - d)\left(1, \frac{(\omega+d)c^{\#}}{2N(c)}, \frac{\omega+d}{\omega-d}c, \frac{(\omega+d)d+2N(c)}{\omega-d}\right).$$

But $\frac{\omega+d}{\omega-d} = \frac{(\omega+d)^2}{4N(c)}$ and $\frac{(\omega+d)d+2N(c)}{\omega-d} = \frac{(\omega+d)^3}{8N(c)}$. The theorem follows. $\square$

Corollary 4.0.2. *One has the following identities:*

$$3t(v^\flat, v^\flat, x) + 3q(v)t(v, v, x) = \langle x, v^\flat\rangle v^\flat + q(v)\langle x, v\rangle v \tag{4}$$

$$6t(v, v^\flat, x) = \langle x, v\rangle v^\flat + \langle x, v^\flat\rangle v. \tag{5}$$

Proof. One separates the "real" and "imaginary" parts of the identity

$$3t(\omega v + v^\flat, \omega v + v^\flat, x) = \langle x, \omega v + v^\flat\rangle(\omega v + v^\flat). \tag{6}$$

$\square$

See [**Pol18**, Theorem 5.1.1.] for some context regarding (6).

With these normalizations, for $w, w' \in W_J$ define $\Phi_{w,w'} \in \mathrm{End}(W_J)$ as follows:

$$\Phi_{w,w'}(x) = 6t(w, w', x) + \langle w', x\rangle w + \langle w, x\rangle w'.$$

We have the following fact.

Proposition 4.0.3. *For $w, w' \in W_J$, the endomorphism $\Phi_{w,w'}$ is in $\mathfrak{h}(J)^0$, i.e., it preserves the symplectic and quartic form on W_J. Furthermore, if $\phi \in \mathfrak{h}(J)^0$, then $[\phi, \Phi_{w,w'}] = \Phi_{\phi(w),w'} + \Phi_{w,\phi(w')}$.*

Proof. The fact that $\Phi_{w,w'}$ preserves the symplectic form follows immediately from the definitions. To check that is preserves the quartic form, one must evaluate $\langle \phi_{w,w'}(v), v^\flat\rangle$. To do this, one uses (5), and obtains 0, as desired. The equivariance statement $[\phi, \Phi_{w,w'}] = \Phi_{\phi(w),w'} + \Phi_{w,\phi(w')}$ is easily checked. $\square$

5. The exceptional upper half-space

When the ground field $k = \mathbf{R}$ and the pairing $(\ ,\ )$ on J is positive definite, the group H_J has a Hermitian symmetric space. This space is $\mathcal{H}_J = \{Z = X + iY : Y > 0\}$. Here $Y > 0$ means that $Y = U_y 1_J$ for some $y \in J$ with $N(y) \neq 0$. Here $U_y x = -y^\# \times x + (y, x)y$, and $U_y 1_J = y^2 = \frac{1}{2}\{y, y\}$. We will prove that $Y > 0$ is equivalent to the conditions $\mathrm{tr}(Y) > 0, \mathrm{tr}(Y^\#) > 0$ and $N(Y) > 0$.

We will need the following theorem.

Theorem 5.0.1. *One has $N(U_y x) = N(y)^2 N(x)$. In particular, if $N(y) \neq 0$, then $U_y \in M_J$.*

Proof. For a proof of this theorem, see McCrimmon, "A tast of Jordan algebras", Theorem C.2.4. $\square$

If $m \in M_J$ with $N(mX) = \delta^2 N(X)$ for all $X \in J$, define $M(\delta, m) \in H_J^1$ as $M(\delta, m)(a, b, c, d) = (\delta^{-1}a, \delta^{-1}m(b), \delta\widetilde{m}(c), \delta d)$. In particular $M(N(y), U_y) \in H_J^1$ for all y with $N(y) \neq 0$.

5.1. The positive definite cone. We first sketch the proof of the key theorem, that every element of J is diagonalizable by an element of A_J.

Theorem 5.1.1. *Suppose $X \in J$. Then there exists $a \in A_J$ with $a(X)$ diagonal.*

Proof. Set $J^0 = (\mathbf{R}1_J)^\perp$. By writing $X = \mu 1_J + X_0$, with $X_0 \in J^0$, it suffices to assume $X \in J^0$. Now, one uses that $\mathfrak{m}_J^0 = \mathfrak{a}_J \oplus J^0$ is the Cartan decomposition of $\mathfrak{m}_J^0$, and the trace 0 diagonal elements make up a maximal abelian subalgebra of J^0. The result then follows from the fact that $\mathfrak{p} = \cup_{k\in K}(Ad(k)\mathfrak{a})$; see Knapp, "Lie groups: Beyond an introduction", Theorem 6.51. $\square$

Corollary 5.1.2. *The following statement are equivalent:*

(1) *$Y \in J$ is positive-definite, i.e., $Y = y^2$ for some $y \in J$ with $N(y) \neq 0$.*
(2) *There exists $a \in A_J$ with aY diagonal with positive entries*
(3) *$\mathrm{tr}(Y), \mathrm{tr}(Y^{\#})$ and $N(Y)$ are all positive.*

Proof. To see that (1) implies (2), apply the theorem to find $a \in A_J$ with $ay = \mathrm{diag}(t_1, t_2, t_3)$. Now $Y = y^2$ implies $a(Y) = (ay)^2 = \mathrm{diag}(t_1^2, t_2^2, t_3^2)$, as desired. Conversely, it is clear that (2) implies (1), because if $a(Y) = t^2$ with t diagonal, then $Y = (a^{-}t)^2$.

Because a commutes with $\#$, fixes 1_J and preserves the trace pairing, (2) implies (3). Finally, suppose given (3). By the theorem, we can assume $Y = \mathrm{diag}(t_1, t_2, t_3)$ is diagonal. Now, it is easy to see that the inequalities $t_1 + t_2 + t_3 > 0$, $t_1t_2 + t_2t_3 + t_3t_1 > 0$ and $t_1t_2t_3 > 0$ force each $t_j > 0$. Thus (3) implies (2), and we are done. ☐

We require one additional corollary.

Corollary 5.1.3. *Let D denote the connected component of the set $\{Y \in J : N(Y) > 0\}$ containing 1_J. Then D is contained in the set $Y > 0$.*

Proof. Let $\gamma : [0, 1] \to D$ be a path, with $\gamma(0) = 1_J$. Then the polynomial $p_t(u) = u^3 - \mathrm{tr}(\gamma(t))u^2 + \mathrm{tr}(\gamma(t)^{\#})u - N(\gamma(t))$ has three real roots, by the Theorem. Moreover, $N(\gamma(t)) > 0$ for all t. It follows that the three real roots of $p_1(u)$ are positive. The corollary follows. ☐

Exercise 5.1.4. Let C denote the set of Y in J with $Y > 0$. The point of this exercise is to prove that C is connected and convex.

(1) Suppose Y_0, Y_1 are positive definite. Prove that the trace pairing $(Y_0, Y_1) > 0$. **Hint**: First note that the diagonal entries of a square are positive, from the formula for X^2. Now apply an $a \in A_J$ so that $a(Y_0)$ is diagonal with positive entries.
(2) Prove that if $Y > 0$ then $Y^{\#} > 0$. **Hint**: Reduce to the diagonal case.
(3) Prove that if $Y_0, Y_1 > 0$ then $N(tY_0 + (1-t)Y_1)) > 0$ for $t \in [0, 1]$.
(4) Argue as in the Corollary above that Y is convex (and in particular, connected.)

Exercise 5.1.5. The point of this exercise is to work out the shape of the positive definite cone in case $J = \mathbf{R} \times S$ with S a quadratic space of signature $(1, *)$. Specifically, we have $S = \mathbf{R}1_S \oplus S_0$ where $S_0 = (\mathbf{R}1_S)^{\perp}$ and q is 1 on 1_S and negative definite on S_0. Let $C = \{y^2 : y \in J, N(y) \neq 0\}$.

(1) Recall that if $y \in J$, then $y^2 = -1 \times y^{\#} + (y, 1_J)y$. Prove that if $y = (\beta, s)$, then $y^2 = (\beta^2, -q(s)1_S + (1_S, s)s)$. If $s = \mu 1_S + s_0$, then this is $y^2 = (\beta^2, (\mu^2 - q(s_0))1_s + 2\mu s_0)$.
(2) Deduce that if $Y = (\beta', v) \in C$, then $\beta' > 0$, $(1_S, v) > 0$ and $q(v) > 0$.
(3) Prove conversely that if $Y = (\beta', v)$ satisfies $\beta' > 0$, $(1_S, v) > 0$ and $q(v) > 0$, then $Y = y^2$ for some $y \in J$ with $N(y) \neq 0$. **Hint**: Write $v = r1_S + r_0$ with $r \in \mathbf{R}_{>0}$ and $r_0 \in S_0$. Define μ to be the positive square root of $(r + q(v)^{1/2})/2$, $s_0 = \frac{1}{2\mu}r_0$ and $\beta = (\beta')^{1/2}$. Check that $y^2 = Y$ with $y = (\beta, \mu 1_S + s_0)$.
(4) Prove that if $Y \in C$, then $\mathrm{tr}(Y) > 0, \mathrm{tr}(Y^{\#}) > 0$ and $N(Y) > 0$.
(5) Prove conversely that if $Y \in J$ satisfies $\mathrm{tr}(Y) > 0, \mathrm{tr}(Y^{\#}) > 0$ and $N(Y) > 0$, then $Y \in C$. **Hint**: Suppose $Y = (\beta, s)$. Then $\mathrm{tr}(Y) = \beta + (1_S, s)$, $\mathrm{tr}(Y^{\#}) = q(s) + \beta(1_S, s)$, and $N(Y) = \beta q(s)$. Now, using that q is negative definite on S_0, prove that there all real numbers β' and β'' with $(1_S, s) = \beta' + \beta''$ and $q(s) = \beta'\beta''$. The inequalities are thus $\beta + \beta' + \beta'' > 0$, $\beta\beta' + \beta'\beta'' + \beta''\beta > 0$ and $\beta\beta'\beta'' > 0$. Conclude that $\beta, \beta', \beta'' > 0$ and thus $Y \in C$.
(6) Prove that C is convex, and in particular, and connected. **Hint**: You may find it helpful to use Cauchy-Schwartz on S_0.

5.2. The upper half space. We now define how $H_J(\mathbf{R})^0$ acts on $\mathcal{H}_J$.

To do this, suppose $Z \in J_{\mathbf{C}}$. Define $r_0(Z) = (1, -Z, Z^\#, -N(Z)) = n(-Z)(1,0,0,0)$. Then one has the following proposition, which must be well-known.

PROPOSITION 5.2.1. *Suppose $Z \in \mathcal{H}_J$, so that $Im(Z)$ is positive definite. Suppose moreover that $g \in H_J(\mathbf{R})^0$. Then there is $j(g,Z) \in \mathbf{C}^\times$ and $gZ \in \mathcal{H}_J$ so that $gr_0(Z) = j(g,Z)r_0(gZ)$. This equality defines the factor of automorphy $j(g,Z)$ and the action of $H_J(\mathbf{R})^0$ simultaneously.*

Both for this proposition, and below, we will need the following lemma.

Suppose J is a cubic norm structure. Let $\iota : J \leftrightarrow J^\vee$ be the identification given by the symmetric pairing on J. Define $J_2 : W_J \to W_J$ as $J_2(a,b,c,d) = (d, -\iota(c), \iota(b), -a)$. One checks that $J_2 \in H_J^1$. Define a pairing on W_J via $(v,w) = \langle J_2 v, w\rangle$.

LEMMA 5.2.2. *The pairing (v,w) is symmetric, with $(v,v) = a^2 + (b,b) + (c,c) + d^2$ if $v = (a,b,c,d)$. Thus if the trace pairing on J is positive-definite, the pairing $(\ ,\ )$ on W_J is as well. Furthermore, one has $|\langle r_0(i), v\rangle|^2 = (v,v) + 2\operatorname{tr}(b^\# - ac) + 2\operatorname{tr}(c^\# - db)$. Thus if v is rank one, $|\langle r_0(i), v\rangle|^2 = (v,v)$.*

PROOF. The first part of the lemma is immediate from the definitions. For the second part, suppose $v = (a,b,c,d)$ and recall $r(i) = (1, -i, -1, i)$. Then $\langle v, r(i)\rangle = (\operatorname{tr}(b) - d) + i(a - \operatorname{tr}(c))$, and thus

$$\begin{aligned}
|\langle v, r(i)\rangle|^2 &= (\operatorname{tr}(b) - d)^2 + (a - \operatorname{tr}(c))^2 \\
&= d^2 - 2d\operatorname{tr}(b) + \operatorname{tr}(b)^2 + a^2 - 2a\operatorname{tr}(c) + \operatorname{tr}(c)^2 \\
&= d^2 + (b,b) + a^2 + (c,c) + 2\operatorname{tr}(b^\# - ac) + 2\operatorname{tr}(c^\# - db),
\end{aligned}$$

as desired. Here we have used the identity $\operatorname{tr}(x)^2 = (x,x) + 2\operatorname{tr}(x^\#)$ for $x \in J$, which follows immediately from the definition of the pairing (x,x). If v is rank one, then $b^\# - ac = 0$ and $c^\# - db = 0$, and the lemma follows. $\square$

PROOF OF PROPOSITION 5.2.1. We recall the argument sketched in [**Pol17**, Section 6.2.1] and [**Pol20a**]. First, if $g \in H_J^1(\mathbf{R})$ one has $\langle gr_0(i), (0,0,0,1)\rangle \neq 0$, since $\langle gr_0(i), (0,0,0,1)\rangle = \langle r_0(i), g^{-1}(0,0,0,1)\rangle$ and $|\langle r_0(i), g^{-1}(0,0,0,1)\rangle|^2 \neq 0$ by Lemma 5.2.2 since $g^{-1}(0,0,0,1)$ is rank one. Thus, there is $j(g,i) \in \mathbf{C}^\times$ and $Z \in J_{\mathbf{C}}$ so that $gr_0(i) = j(g,i)r_0(Z)$. We claim that $N(Im(Z)) > 0$.

To see this, first note the general identity $\langle r_0(Z), r_0(W)\rangle = N(Z - W)$ for $Z, W \in J_{\mathbf{C}}$. Thus

$$\nu(g)N(2i) = \nu(g)\langle r_0(i), r_0(-i)\rangle = \langle gr_0(i), gr_0(-i)\rangle = \langle gr_0(i), \overline{gr_0(i)}\rangle = |j(g,i)|^2 N(Z - \overline{Z}).$$

Thus if $\nu(g) > 0$, then $N(Y) > 0$.

Now, if $g \in H_J(\mathbf{R})^0$, then $N(Im(gi)) > 0$, and thus by continuity $Im(gi)$ is positive definite. Hence the proposition is proved when $Z = i1_J$.

The general case follows from the fact that the subgroup generated by the $M(N(y), U_y)$ and $n(X)$ acts transitively on $\mathcal{H}_J$, and that these elements can be taken to be in $H_J(\mathbf{R})^0$. Indeed, first note that if $Y > 0$, there exists $y > 0$ with $y^2 = Y$. To see this, one can diagonalize Y via the action of A_J then take a positive square root of a diagonal element. It follows that for such a y, $M(N(y), U_y)$ is in $H_J^1(\mathbf{R})^0$, as desired.

Now, if $g \in H_J^(\mathbf{R})^0$ and $Z \in \mathcal{H}_J$, say $Z = g_1(i1_J)$, then

$$gr_0(Z) = gr_0(g_1 i) = j(g_1,i)^{-1} g g_1 r_0(i) = j(g_1,i)^{-1} j(gg_1,i) r_0(gg_1 i) = j(g,Z) r_0(gZ)$$

with $gZ = gg_1 i \in \mathcal{H}_J$, as required. $\square$

We will now work to understand the stabilizer of $i1_J$ inside of H_J^1. We prove the following proposition.

PROPOSITION 5.2.3. *Suppose $g \in H_J^1(\mathbf{R})$ stabilizers $i1_J \in \mathfrak{H}_J$. Then g commutes with J_2.*

To prove this proposition, we introduce the Cayley transform c as follows. Define $c \in H_J^1(\mathbf{C})$ as $c = n_{J^\vee}(i1_J/2)n_J(i1_J)$. We have the following useful lemma.

Define $W^+ = \{(a, ic, c, ia)\}$ to be the i-eigenspace of J_2 on $W_J \otimes \mathbf{C}$ and similarly define $W^- = \{(a, -ic, c, -ia)\}$ to be the $(-i)$-eigenspace of J_2 on $W_J \otimes \mathbf{C}$.

LEMMA 5.2.4. *One has*

(1) $cr_0(i) = (1, 0, 0, 0)$
(2) $cr_0(-i) = -8i(0, 0, 0, 1)$
(3) $cW^+ = (*, 0, *, 0)$
(4) $cW^- = (0, *, 0, *)$

PROOF. These are tedious but straightforward calculations; we omit them. □

Using the lemma, we can now prove the proposition.

PROOF OF PROPOSITION 5.2.3. If g stabilizes $i1_J$, then g fixes the lines $\mathbf{C}r_0(i)$ and $\mathbf{C}r_0(-i)$. Consequently $cgc^{-1} \in H_J^1(\mathbf{C})$ fixes the lines $(*, 0, 0, 0)$ and $(0, 0, 0, *)$. It follows (we've proved this) that then cgc^{-1} also fixes the spaces $(0, *, 0, 0)$ and $(0, 0, *, 0)$. Consequently, cgc^{-1} stabilizes cW^+ and cW^-, from which we obtain that g stabilizes W^+ and W^-. As these are the eigenspaces of J_2, we conclude that g commutes with J_2, as desired. □

5.3. Modular forms. Let $J = H_3(\Theta)$ be as above, where Θ has positive definite norm form. Define $G = H_J^1(\mathbf{R})$. It turns out that the group G is connected, so it acts (transitively) on $\mathcal{H}_J$.

Conjugation by J_2 is a Cartan involution Θ on G (see [**Pol20a**, section 3.4.5]). Define a norm on G as $||g||^2 = \mathrm{tr}(Ad(g)Ad(\Theta(g)^{-1}))$. A function $\phi : G \to \mathbf{C}$ is said to be of moderate growth if $||\phi(g)|| < C||g||^N$ for some $C, N > 0$.

Following Baily, "An exceptional arithmetic group and its Eisenstein series", a discrete subgroup $\Gamma \subseteq G$ is defined as follows. Let $\Theta_0 \subseteq \Theta$ be Coxeter's ring of integral octonions; see, e.g., *loc cit.* Define $J_0 \subseteq J$ to be the integral lattice consisting of matrices $X = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix}$ with $c_1, c_2, c_3 \in \mathbf{Z}$ and $x_1, x_2, x_3 \in \Theta_0$. Define $W_{J_0} \subseteq W_J$ to be the lattice $W_{J_0} = \mathbf{Z} \oplus J_0 \oplus J_0^\vee \oplus \mathbf{Z}$. Then Γ is defined to be the subgroup of $H_J^1(\mathbf{Q})$ that preserves W_{J_0}.

A modular form for Γ of weight $\ell > 0$ is a holomorphic function $f : \mathcal{H}_J \to \mathbf{C}$ satisfying

(1) $f(\gamma Z) = j(\gamma, Z)^\ell f(Z)$ for all $\gamma \in \Gamma$ and
(2) the function $\phi_f : \Gamma\backslash G \to \mathbf{C}$ defined by $\phi_f(g) = j(g, i)^{-\ell} f(g \cdot i)$ is of moderate growth.

Some results about modular forms on G can be found in:

(1) Baily, "An exceptional arithmetic group and its Eisenstein series"
(2) Kim, "Exceptional modular form of weight 4 on an exceptional domain contained in $\mathbf{C}^{27}$
(3) Gan and Loke, "Modular forms of level p on the exceptional tube domain"
(4) Kim an Yamauchi, "Cusp forms on the exceptional group of type E_7"

CHAPTER 4

The Lie algebra and group E_8

Suppose $J, J^\vee$ is a cubic norm pair. Associated to this pair, we define a Lie algebra $\mathfrak{g}(J)$. When $J = H_3(\Theta)$, $\mathfrak{g}(J)$ turns out to be the Lie algebra of type E_8. One can take this as a definition of the Lie algebra e_8.

1. The $\mathbf{Z}/2$ grading on $\mathfrak{g}(J)$

I believe the construction of the Lie algebra $\mathfrak{g}(J)$ in this section essentially goes back to Freudenthal.

Denote by V_2 the defining two-dimensional representation of $\mathfrak{sl}_2 = \mathfrak{sp}_2$. Recall that we have an identification $Sym^2(V_2) \simeq \mathfrak{sl}_2$ as $(v \cdot v')(x) = \langle v', x\rangle v + \langle v, x\rangle v'$. Here $\langle\, , \,\rangle$ is the standard symplectic pairing on V_2:

$$\langle (a,b)^t, (c,d)^t\rangle = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = ad - bc.$$

We define

$$\mathfrak{g}(J) = \mathfrak{g}(J)_0 \oplus \mathfrak{g}(J)_1 := \left(\mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0\right) \oplus (V_2 \otimes W_J).$$

Here $\mathfrak{g}(J)_0 = \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$ is the zeroth graded piece of $\mathfrak{g}(J)$, and $\mathfrak{g}(J)_1 = V_2 \otimes W_J$ is the first graded piece of $\mathfrak{g}(J)$.

1.0.1. *The bracket.* We define a map $[\, , \,] : \mathfrak{g}(J) \otimes \mathfrak{g}(J) \to \mathfrak{g}(J)$ as follows: If $\phi, \phi' \in \mathfrak{g}(J)_0 = \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$, $v, v' \in V_2$, and $w, w' \in W_J$, then

$$[(\phi, v\otimes w), (\phi', v' \otimes w')] = \left([\phi, \phi'] + \frac{1}{2}\langle w, w'\rangle (v \cdot v') + \frac{1}{2}\langle v, v'\rangle \Phi_{w,w'}, \phi(v' \otimes w') - \phi'(v \otimes w)\right).$$

With this definition, we have the following fact.

Proposition 1.0.1. *The bracket $[\, , \,]$ on $\mathfrak{g}(J)$ satisfies the Jacobi identity.*

Proof. To check the Jacobi identity $\sum_{cyc}[X,[Y,Z]] = 0$, by linearity it suffices to check it on the various $\mathbf{Z}/2$-graded pieces. Then there are four types identities that must be checked. Namely, if $0, 1, 2$ or 3 of the elements X, Y, Z are in $\mathfrak{g}(J)_1 = V_2 \otimes W_J$. If all three of X, Y, Z are in $\mathfrak{g}(J)_0 = \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$, then the Jacobi identity is of course satisfied. If two of X, Y, Z are in $\mathfrak{g}(J)_0$, then the Jacobi identity is satisfied. This fact is equivalent to the fact that the bracket $[\, , \,]_\alpha$ defines a Lie algebra action of $\mathfrak{g}(J)_0$ on $\mathfrak{g}(J)_1$: $[\phi, \phi'](x) = \phi(\phi'(x)) - \phi'(\phi(x))$ for $x \in \mathfrak{g}(J)_1$ and $\phi, \phi' \in \mathfrak{g}(J)_0$. If one of X, Y, Z is in $\mathfrak{g}(J)_0$, then the Jacobi identity is satisfied by the equivariance of the map $\mathfrak{g}(J)_1 \otimes \mathfrak{g}(J)_1 \to \mathfrak{g}(J)_0$. Finally, when X, Y, Z are all in $\mathfrak{g}(J)_1$, a simple direct computation shows that $\sum_{cyc}[X,[Y,Z]] = 0$.

In more detail, suppose $X_1 = v_1 \otimes w_1$, $X_2 = v_2 \otimes w_2$ and $X_3 = v_3 \otimes w_3$. We must evaluate:

$$\begin{aligned} -2\sum_{cyc}[X_1,[X_2,X_3]] &= 2\sum_{cyc}[v_2\otimes w_2, v_3 \otimes w_3](v_1 \otimes w_1) \\ &= \sum_{cyc}(\langle v_2, v_3\rangle \Phi_{w_2,w_3} + \langle w_2, w_3\rangle v_2 \cdot v_3)(v_1 \otimes w_1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{cyc} \langle v_2, v_3\rangle v_1 \otimes (6t(w_1,w_2,w_3) + \langle w_3, w_1\rangle w_2 + \langle w_2, w_1\rangle w_3) \\
&+ \sum_{cyc} \langle w_2, w_3\rangle (\langle v_3, v_1\rangle v_2 + \langle v_2, v_1\rangle v_3) \otimes w_1.
\end{aligned}
$$

The term $t(w,w',w'')$ drops out right away because it is symmetric by applying the identity $\sum_{cyc} \langle v_2, v_3\rangle v_1 = 0$ for $v_1, v_2, v_3 \in V_2$. The other cyclic sums cancel in pairs. □

2. The $\mathbf{Z}/3$ grading

2.1. The $\mathbf{Z}/3$ grading on $\mathfrak{g}(J)$. In this subsection we recall elements from the paper [**Rum97**] (in different notation). Rumelhart constructed the Lie algebra $\mathfrak{g}(J)$ through a $\mathbf{Z}/3$-grading, as opposed to a $\mathbf{Z}/2$-grading as we have described above.

Denote by V_3 the defining representation of $\mathfrak{sl}_3$, and by $V_3^\vee$ the dual representation. In the $\mathbf{Z}/3$-graded picture, one defines

$$\mathfrak{g}(J) = \mathfrak{sl}_3 \oplus \mathfrak{m}(J)^0 \oplus V_3 \otimes J \oplus V_3^\vee \oplus J^\vee.$$

We consider $V_3, V_3^\vee$ as left modules for $\mathfrak{sl}_3$, and $J, J^\vee$ as left modules for $\mathfrak{m}(J)^0$.

2.1.1. *The bracket.* Following [**Rum97**], the Lie bracket is given as follows. First, because V_3 is considered as a representation of $\mathfrak{sl}_3$, there is an identification $\wedge^2 V_3 \simeq V_3^\vee$, and similarly $\wedge^3 V_3^\vee \simeq V_3$. If v_1, v_2, v_3 denotes the standard basis of V_3, and $\delta_1, \delta_2, \delta_3$ the dual basis of $V_3^\vee$, then $v_1 \wedge v_2 = \delta_3$, $\delta_1 \wedge \delta_2 = v_3$, and cyclic permutations of these two identifications.

Take $\phi_3 \in \mathfrak{sl}_3$, $\phi_J \in \mathfrak{m}(J)^0$, $v, v' \in V_3$, $\delta, \delta' \in V_3^\vee$, $X, X' \in J$ and $\gamma, \gamma' \in J^\vee$. Then

$$
\begin{aligned}
[\phi_3, v \otimes X + \delta \otimes \gamma] &= \phi_3(v) \otimes X + \phi_3(\delta) \otimes \gamma. \\
[\phi_J, v \otimes X + \delta \otimes \gamma] &= v \otimes \phi_J(X) + \delta \otimes \phi_J(\gamma) \\
[v \otimes X, v' \otimes X'] &= (v \wedge v') \otimes (X \times X') \\
[\delta \otimes \gamma, \delta' \otimes \gamma'] &= (\delta \wedge \delta') \otimes (\gamma \times \gamma') \\
[\delta \otimes \gamma, v \otimes X] &= (X,\gamma) v \otimes \delta + \delta(v)\Phi_{\gamma,X} - \delta(v)(X,\gamma) \\
&= (X,\gamma)\left(v \otimes \delta - \frac{1}{3}\delta(v)\right) + \delta(v)\left(\Phi_{\gamma,X} - \frac{2}{3}(X,\gamma)\right).
\end{aligned}
$$

Note that $v \otimes \delta - \frac{1}{3}\delta(v) \in \mathfrak{sl}_3$ and $\Phi_{\gamma,X} - \frac{2}{3}(X,\gamma) = \Phi'_{\gamma,X} \in \mathfrak{m}(J)^0$. Also recall that $\Phi_{\gamma,X} \in \mathfrak{m}(J)$ acts on J via

$$\Phi_{\gamma,X}(Z) = -\gamma \times (X \times Z) + (\gamma, Z)X + (\gamma, X)Z.$$

Furthermore, the action of $\mathfrak{sl}_3$ and $\mathfrak{m}(J)^0$ on $V_3^\vee$ and $J^\vee$ is determined by the equalities $(\phi_3(v),\delta) + (v, \phi_3(\delta)) = 0$ and $(\phi_J(X),\gamma) + (X, \phi_J(\gamma)) = 0$.

One can give an explicit identification between the Lie algebra $\mathfrak{g}(J)$ defined in this section and the one defined in the previous section, which is why we have given both Lie algebras the same name. See [**Pol20a**, Proposition 4.2.1].

3. The Killing forms on the Lie algebras

We define symmetric, non-degenerate, invariant bilinear forms on the Lie algebras we've discussed.

3.1. The form on $\mathfrak{m}(J)$. Suppose J is a cubic norm structure, or $J, J^\vee$ is a cubic norm pair. We have defined the Lie algebra $\mathfrak{m}(J)$. To define a bilinear form on $\mathfrak{m}(J)$, we begin as follows. Suppose $\phi, \phi' \in \mathfrak{m}(J)$ and $\phi = \sum_j \Phi_{\gamma_j, x_j}$, as we can assume because $\Phi : J^\vee \otimes J \to \mathfrak{m}(J)$ is surjective. (We haven't proved this surjectivity, but it is true.) Then we define $B_{\mathfrak{m}}(\phi', \phi) = \sum_j (\phi'(x_j), \gamma_j)$.

Lemma 3.1.1. *The bilinear form $B_{\mathfrak{m}}$ is well-defined and symmetric on $\mathfrak{m}(J)$.*

Proof. First one checks easily that

$$B_{\mathfrak{m}}(\Phi_{\gamma', x'}, \Phi_{\gamma, x}) = (\gamma', x')(\gamma, x) + (\gamma', x)(\gamma, x') - (x \times x', \gamma \times \gamma').$$

It is clear that this is symmetric in the two elements of $\mathfrak{m}(J)$, and thus

$$B_{\mathfrak{m}}(\Phi_{\gamma', x'}, \Phi_{\gamma, x}) = (\Phi_{\gamma', x'}(x), \gamma) = (\Phi_{\gamma, x}(x'), \gamma').$$

Because the pairing can be expressed in terms of the action of $\Phi_{\gamma, x}$ on J, it is well-defined. I.e., if $\phi = \sum_j \Phi_{\gamma_j, x_j} = 0 \in \mathfrak{m}(J)$, then $B_{\mathfrak{m}}(\phi', \phi) = \sum_j (\phi'(x_j), \gamma_j) = 0$.

This completes the proof. $\square$

Proposition 3.1.2. *The bilinear form $B_{\mathfrak{m}}$ is invariant, i.e., $B_{\mathfrak{m}}([\phi, \phi'], \phi'') = B_{\mathfrak{m}}(\phi, [\phi', \phi''])$ for all $\phi, \phi', \phi'' \in \mathfrak{m}(J)$.*

Proof. We can assume $\phi'' = \Phi_{\gamma, x}$. Then the LHS of our desired equality is $([\phi, \phi'](x), \gamma) = (\phi(\phi'(x)), \gamma) + (\phi(x), \phi'(\gamma))$. The RHS is $B_{\mathfrak{m}}(\phi, \Phi_{\phi'(\gamma), x} + \Phi_{\gamma, \phi'(x)}) = (\phi(x), \phi'(\gamma)) + (\phi(\phi'(x)), \gamma)$. $\square$

3.2. The form on $\mathfrak{h}(J)^0$. Define an invariant pairing $B_{\mathfrak{h}}$ on $\mathfrak{h}(J)^0$ by

$$B_{\mathfrak{h}}((\phi, a, \gamma), (\phi', a', \gamma')) = B_{\mathfrak{m}}(\phi, \phi') - (a, \gamma') - (a', \gamma).$$

Here $\phi, \phi' \in \mathfrak{m}(J)$, $a, a' \in J$ and $\gamma, \gamma' \in J^\vee$.

Thinking of $\mathfrak{h}(J)^0$ as defined via its action on W_J, as opposed to its description via the 3-step $\mathbf{Z}$-grading, it is natural to ask how the pairing $B_{\mathfrak{h}}$ looks like on elements of the form $\Phi_{w, w'}$. One has the following:

$$B_{\mathfrak{h}}(\Phi_{w_1, w_1'}, \Phi_{w_2, w_2'}) = -2\left(\langle w_1, w_2' \rangle \langle w_1', w_2 \rangle + \langle w_1, w_2 \rangle \langle w_1', w_2' \rangle\right) + 12(w_1, w_1', w_2, w_2')$$

for $w_1, w_1', w_2, w_2' \in W_J$. The pairing $B_{\mathfrak{h}}$ satisfies

$$B_{\mathfrak{h}}(\Phi_{w, w'}, \phi) = 2\langle w, \phi(w') \rangle.$$

That the map $\Phi : Sym^2(W_J) \to \mathfrak{h}(J)^0$ is surjective follows from the surjectivity of $J^\vee \otimes J \to \mathfrak{m}(J)$ and [**Pol20a**, Proposition 3.4.4].

3.3. The form on $\mathfrak{g}(J)$. Define a symmetric pairing $B_{\mathfrak{g}} : \mathfrak{g}(J) \otimes \mathfrak{g}(J) \to k$ via

$$B_{\mathfrak{g}}(\phi + v \otimes w, \phi' + v' \otimes w') = B_0(\phi, \phi') - \langle v, v' \rangle \langle w, w' \rangle.$$

Here B_0 is the invariant symmetric pairing on $\mathfrak{g}(J)_0$ defined by

$$B_0(\phi_2 + \phi_J, \phi_2' + \phi_J') = B_{\mathfrak{sl}(V_2)}(\phi_2, \phi_2') + B_{\mathfrak{h}}(\phi_J, \phi_J')$$

for $\phi_2, \phi_2' \in \mathfrak{sl}_2$ and $\phi_J, \phi_J' \in \mathfrak{h}(J)^0$.

The pairing

$$B_{\mathfrak{sp}(W)}(w_1 w_1', w_2 w_2') = -2(\langle w_1, w_2 \rangle \langle w_1', w_2' \rangle + \langle w_1', w_2 \rangle \langle w_1, w_2' \rangle).$$

On $\mathfrak{sl}_2$, this is the same as the trace pairing $B_{\mathfrak{sl}_2(V_2)}(X, X') = \mathrm{tr}(XX')$.

Lemma 3.3.1. *The pairing $B_{\mathfrak{g}}$ on $\mathfrak{g}(J)$ is invariant, i.e. $B_{\mathfrak{g}}([x_1, x_3], x_2) = -B_{\mathfrak{g}}([x_2, x_3], x_1)$ for all x_1, x_2, x_3 in $\mathfrak{g}(J)$.*

Proof. Writing out both $B_{\mathfrak{g}}([x_1,x_3],x_2)$ and $-B_{\mathfrak{g}}([x_2,x_3],x_1)$, one finds that the form $B_{\mathfrak{g}}$ is invariant if and only if

$$2\langle w,\phi(w')\rangle = B_0(\Phi_{w,w'},\phi)$$

and

$$2\langle v,\phi(v')\rangle = B_0(v\cdot v',\phi)$$

for all $v,v'\in V_2$, $w,w'\in W_J$, and $\phi\in\mathfrak{g}(J)_0$. These properties of B_0 were discussed above. $\square$

3.4. The form on $\mathfrak{g}(J)$ again. One can also express the invariant bilinear form in terms of the $\mathbf{Z}/3$-grading. If one does this, one gets as follows.

The form $B_{\mathfrak{g}}$ on $\mathfrak{g}(J)$ restricts to the form on $\mathfrak{sl}_3$ given by $B_{\mathfrak{g}}(m_1,m_2)=\mathrm{tr}(m_1m_2)$ for $m_1,m_2\in \mathrm{End}(V_3)$. This form $B_{\mathfrak{g}}$ on $\mathfrak{g}(J)$ is given as follows:

- On $\mathfrak{sl}_3$: $B_{\mathfrak{g}}(v\otimes\phi,v'\otimes\phi')=\phi(v')\phi'(v)$
- On $\mathfrak{m}(J)^0$: $B_{\mathfrak{g}}(\phi_1,\phi_2)=B_{\mathfrak{m}}(\phi_1,\phi_2)$.
- On $V_3\otimes J\oplus V_3^\vee\otimes J^\vee$: $B_{\mathfrak{g}}(v\otimes X,\delta'\otimes\gamma')=-\delta'(v)(X,\gamma')$.

Part 2

Arithmetic invariant theory

CHAPTER 5

Bhargava's Higher Composition Laws I

1. Composition of $2 \times 2 \times 2$ cubes

We make some of the notation and definitions and define the bijection.

Set $A = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$. We let $S = S_D$ be the quadratic ring of discriminant D. Thus $S_D = \mathbf{Z}[y]/(y^2 - Dy + \frac{D^2-D}{4})$. We let τ denote the image of y in S, so that $\tau^2 - D\tau + \frac{D^2-D}{4} = 0$. Thus one should think of τ as $\frac{D+\sqrt{D}}{2}$. We set $E = S \otimes \mathbf{Q}$ and ω the element $\sqrt{D}$ in E.

Set $W_A = \mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2$ (tensor product of row vectors), so that W_A has a right action of $\mathrm{GL}_2(A) = \mathrm{GL}_2(\mathbf{Z})^3$. One writes an element of W_A as a four tuple (a, b, c, d) where $a, d \in \mathbf{Z}$ and $b, c \in A$. We denote by q the quartic form on W_A and note that $q(v)$ is a square modulo 4 for all $v \in W_A$.

As proved by Bhargava [**Bha04a**], we will relate triples of ideal classes for S_D for $D \neq 0$ a square modulo 4 to elements $v \in W_A$ with $q(v) = D$.

We begin with some notation. If $r = (r_1, r_2) \in A^2$, so that $r_1 = (r_{11}, r_{12}, r_{13})$ and $r_2 = (r_{21}, r_{22}, r_{23})$, we define $r^! \in W_A$ as

$$\begin{aligned} r^! &= (r_{11}, r_{21}) \otimes (r_{12}, r_{22}) \otimes (r_{13}, r_{23}) \\ &= (r_{11}r_{12}r_{13}, (r_{21}r_{12}r_{13}, r_{11}r_{22}r_{13}, r_{11}r_{12}r_{23}), (r_{11}r_{22}r_{23}, r_{21}r_{12}r_{23}, r_{21}r_{22}r_{13}), r_{21}r_{22}r_{23}). \end{aligned}$$

Note that $r^!$ is rank one.

Suppose $I = (I_1, I_2, I_3)$ is a triple of fractional S-ideals, and $b = (b_1, b_2)$ is a basis for I so that $b_1 = (b_{11}, b_{12}, b_{13})$ and $b_2 = (b_{21}, b_{22}, b_{23})$ with $b_{ij} \in E$. That b is a basis for I means $I_j = \mathbf{Z}b_{1j} \oplus \mathbf{Z}b_{2j}$ for $j = 1, 2, 3$. Given a choice of $\tau \in S$, which is called an orientation, one defines the norm $N(I; b, \tau)$ as follows. There is a unique $g \in \mathrm{GL}_2(A \otimes \mathbf{Q}) = \mathrm{GL}_2(\mathbf{Q})^3$ so that $(b_1, b_2) = (\tau, 1)g$, i.e., $g = (g_1, g_2, g_3)$ and $(b_{1j}, b_{2j}) = (\tau, 1)g_j$. Now set $N(I; b, \tau) = \det_6(g) := \det(g_1)\det(g_2)\det(g_3)$.

Definition 1.0.1. Suppose I, b, τ as above, and $\beta \in E^\times$. The data (I, b, τ, β) is said to be **balanced** if

1. $\beta^{-1}x_1x_2x_3 \in S$ for all $x_j \in I_j$. Equivalently, $\beta^{-1}b^! \in W_A \otimes S$.
2. $N_{E/\mathbf{Q}}(\beta) = N(I; b, \tau)$.

We now make a definition of equivalence of types of data (I, b, β).

Definition 1.0.2. One says that two triples (I, b, β) and (I', b', β') are *equivalent* if there exists $x = (x_1, x_2, x_3) \in E^\times \times E^\times \times E^\times$ with

- $I' = xI$
- $b' = xb$
- $\beta' = x_1x_2x_3\beta$.

Associated to an equivalence class of balanced data $[I, b, \beta]$, we define an element $v \in W_A$ as follows. One set $X(I, b, \beta) = \beta^{-1}b^! \in W_A \otimes S$ and one lets v be the coefficient of τ. so that $X(I, b, \beta) = \tau v + v'$ for unique $v, v' \in W_A$.

The following theorem is essentially one of the results of [**Bha04a**], and is a special case of one the main results of [**Pol18**].

THEOREM 1.0.3. *Suppose $D \neq 0$ is a square modulo 4 and S is the oriented quadratic ring of discriminant D. The association $(I, b, \beta) \mapsto v$ defines a bijection between equivalence classes of triples of balanced data and elements $v \in W_A$ with $q(v) = D$. Moreover, this bijection is equivariant for the action $\mathrm{SL}_2(\mathbf{Z})^3$ on both sides, with $\mathrm{SL}_2(\mathbf{Z})^3$ acting on the data $[(I,(b_1,b_2),\beta)]$ as $[(I,(b_1,b_2),\beta)] \mapsto [(I,(b_1,b_2)g,\beta)]$.*

We will now describe the inverse map.

To do so, we begin by defining the quadratic covariant $S(v) \in M_2(A \otimes \mathbf{Q}) = M_2(\mathbf{Q})^3$ for an element $v \in W_A$. Namely, if $v = (a,b,c,d)$, then

$$S(v) = \begin{pmatrix} b^{\#} - ac & ad - cb - \mathrm{tr}(ad - bc)/2 \\ ad - bc - \mathrm{tr}(ad - bc)/2 & c^{\#} - db \end{pmatrix}.$$

This is three 2×2 symmetric matrices which are half-integral, or in other words, 3 binary quadratic forms. See [**Pol18**, Example 4.4.2] for this written out.

Let $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

PROPOSITION 1.0.4. *One has $S(v)J_2S(v) = -\frac{q(v)}{4}J_2$. In other words, these three binary quadratic forms have the same discriminant, $q(v)$.*

PROOF. This is proved in a more general situation in [**Pol18**, Proposition 4.4.1]. $\square$

Now set $R_r(v) = 2J_2S(v)$, so that $R_r(v)^2 = q(v)$.

PROPOSITION 1.0.5. *For $g \in \mathrm{GL}_2(A \otimes \mathbf{Q}) = \mathrm{GL}_2(\mathbf{Q})^3$, one has $R_r(v \cdot g) = \det_6(g)g^{-1}R_r(v)g$.*

PROOF. One can check this on generators, as is done in [**Pol18**]. $\square$

Now set $\Omega(v) = \frac{q(v)+R_r(v)}{2}$, which one can check is in $M_2(\mathbf{Z})$. Note that $\Omega^2 - D\Omega + \frac{D^2-D}{4} = 0$. Thus $\Omega(v)$ defines an action of S_D on the column vectors A^2. This will give the triple of S-modules associated to the vector v.

To embed this module into $A_E = E \times E \times E$, we proceed as follows. Set $\epsilon = \epsilon(v) = \frac{1}{2} + \frac{1}{2\omega}R_r(v)$. Moreover, define

- $X(v,\omega) = \frac{\omega v + v^\flat}{2}$
- $X(v,-\omega) = \frac{-\omega v + v^\flat}{2}$.

Now, if $\ell = (\ell_1, \ell_2) \in A_{\mathbf{Q}}^2 = (\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q})^2$ is a row vector, we define

(1) $I(\ell) = \ell\epsilon(v)A^2$
(2) $b(\ell) = \ell\epsilon(v) = (b_1, b_2)$
(3) $\beta(\ell) = \omega^{-3}\langle \ell^!, X(v,-\omega)\rangle$.

PROPOSITION 1.0.6. *The row vector ℓ can be chosen so that $I(\ell)$ is a fractional ideal and $\beta(\ell) \in E^\times$. Different choices of such ℓ yield equivalent data. Moreover, for these ℓ, the data $I(\ell), b(\ell), \beta(\ell)$ is balanced.*

PROOF. This is proved in [**Pol18**]. $\square$

We can now state the inverse bijection to the theorem above.

THEOREM 1.0.7. *The maps $[(I,b,\beta)] \mapsto v$ and $v \mapsto [I(\ell), b(\ell), \beta(\ell)]$ define inverse bijections between the set of balanced data and the $v \in W_A$ with $q(v) = D$. These bijections are equivariant for the action of $\mathrm{SL}_2(\mathbf{Z})^3$.*

Recall that we already know that the element $X(v,\omega)$ is rank one. One of the key steps in the proof of the above theorem is to write this rank one element explicitly in terms of the rank one elements $b^!$ for $b \in A_E^2 = (E \times E \times E)^2$. In fact, one has the following result.

Proposition 1.0.8. *For all $\ell \in A^2_{\mathbf{Q}}$ row vectors, one has*

$$\omega^3(\ell\epsilon(v))^! = \langle \ell^!, X(v, -\omega)\rangle X(v, \omega).$$

The proposition implies that the composition $W_A^{q=D} \mapsto$ balanced data $\mapsto W_A$ is the identity.

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