AUTOMATIC CONVERGENCE FOR SIEGEL MODULAR FORMS

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Abstract. Bruinier and Raum, building on work of Ibukiyama-Poor-Yuen, have studied a notion of *formal Siegel modular forms*. These objects are formal sums that have the symmetry properties of the Fourier expansion of a holomorphic Siegel modular form. These authors proved that formal Siegel modular forms necessarily converge absolutely on the Siegel half-space, and thus are the Fourier expansion of an honest Siegel modular form. The purpose of this note is to give a new proof of the cuspidal case of this "automatic convergence" theorem of Bruinier-Raum. We use the same basic ideas in a separate paper to prove an automatic convergence theorem for cuspidal quaternionic modular forms on exceptional groups.

Contents

1. Introduction

In the recent preprint [Pol24], we proved that the cuspidal quaternionic modular forms on the groups of type F_4 and E_n, $n = 6, 7, 8$ have an algebraic structure, defined in terms of Fourier coefficients. One key step in the proof is to prove what can be called an *automatic convergence theorem*, which asserts that certain infinite series that "look" like the Fourier expansion of a modular form necessarily *are* the Fourier expansion of a modular form. The proof in [Pol24] is rather technical. In this note, we illustrate the basic method of [Pol24] in the case of Siegel modular forms. While some key pieces of the argument in [Pol24] have no analogue in the Siegel modular case, we believe that the main flavor of the argument remains.

Automatic convergence theorems have a history, going back to Ibukiyama-Poor-Yuen [IPY13], and then developed in work of Bruinier [Bru15], Raum [WR15], Bruinier-Raum [BWR15, BR24] and Xia [Xia22]. The automatic convergence theorem we give here is the cuspidal case of a result of [BWR15], but the argument we give is new.

2. Fourier-Jacobi expansion of Siegel modular forms

In this section, we review the Fourier and Fourier-Jacobi expansions of Siegel modular forms.

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2.1. **The Fourier expansion.** Let $\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbf{C}) : Z = Z^t, Y > 0\}$ be the Siegel upper half space of degree n. The group $\mathrm{Sp}_{2n}(\mathbf{R})$ acts on $\mathcal{H}_n$ by fractional linear transformations, $g \cdot Z = (aZ+b)(cZ+d)^{-1}$ if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let K_∞ denote the stabilizer of $i1_n$ for this action. Then $K_\infty \approx U(n)$, the unitary group. If $Z \in \mathcal{H}_n$ and $g \in \mathrm{Sp}_{2n}(\mathbf{R})$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, let $J(g,Z) = cZ + d \in \mathrm{GL}_n(\mathbf{C})$ and $j(g,Z) = \det(J(g,Z))$.

Fix an integer $\ell \geq 0$. If $\varphi : \mathrm{Sp}_{2n}(\mathbf{Q})\backslash \mathrm{Sp}_{2n}(\mathbf{A}) \to \mathbf{C}$ is an automorphic form, we say that φ corresponds to a Siegel modular form of weight ℓ if, for every $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$, the function $f_{\varphi,g_f} : \mathrm{Sp}_{2n}(\mathbf{R}) \to \mathbf{C}$ given by $f_{\varphi,g_f}(g_\infty) = j(g_\infty, i)^\ell \varphi(g_f g_\infty)$ descends to $\mathcal{H}_n$ and gives a holomorphic function there.

Let $S_n(\mathbf{Z})$, $S_n(\mathbf{Q})$, etc denote the $n \times n$ symmetric matrices with integer or rational entries etc. Let $S_n(\mathbf{Z})^\vee$ denote the half-integral symmetric matrices. If $T \in S_n(\mathbf{R})$ is positive semi-definite, define $\mathcal{W}_{\ell,T} : \mathrm{Sp}_{2n}(\mathbf{R}) \to \mathbf{C}$ as $\mathcal{W}_{\ell,T}(g) = j(g,i)^{-\ell} e^{2\pi i (g \cdot i)}$.

Suppose φ is a cuspidal automorphic form, corresponding to a Siegel modular form of weight ℓ. Then φ has a Fourier expansion

$$\varphi(g_f g_\infty) = \sum_{T \in S_n(\mathbf{Q}), T > 0} a_T(g_f) \mathcal{W}_{\ell,T}(g_\infty). \tag{1}$$

Here $a_T : \mathrm{Sp}_{2n}(\mathbf{A}_f) \to \mathbf{C}$ is a locally constant function, called the T-Fourier coefficient of φ. Let $\psi : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}^\times$ denote the standard additive character. If $X \in S_n$, let $n(X) = \left(\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}$. The Fourier coefficients a_T satisfy $a_T(n(X)g_f) = \psi((T,X))a_T(g_f)$ for every $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$ and every $X \in S_n(\mathbf{A}_f)$. Here $(T,X) = \frac{1}{2}\operatorname{tr}(TX + XT) = \operatorname{tr}(TX)$.

Definition 2.1. Suppose $T \geq 0$ and $a_T : G(\mathbf{A}_f) \to \mathbf{C}$ is a locally constant function satisfying

1. $a_T(n(X)g_f) = \psi((T,X))a_T(g_f)$ for every $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$ and every $X \in S_n(\mathbf{A}_f)$;
2. there exists a compact open subgroup U so that a_T is right-invariant by U.

In this case, we say that a_T is Siegel-Whittaker function, or T-Siegel Whittaker function if we need to specify the element T.

If $r \in \mathrm{GL}_n$, let $m_n(r) = \mathrm{diag}(r, {}^t r^{-1}) \in \mathrm{Sp}_{2n}$. The group GL_n acts on S_n on the right as $T \cdot r = r^t T r$. The automorphy of φ implies that $a_T(m_n(\gamma)g_f) = \det(\gamma)^{-\ell} a_{T\cdot\gamma}(g_f)$ for every $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$ and every $\gamma \in \mathrm{GL}_n(\mathbf{Q})$.

Let $\widetilde{\mathrm{Sp}}_{2n}(\mathbf{R})$ be the double cover of $\mathrm{Sp}_{2n}(\mathbf{R})$. For $g \in \widetilde{\mathrm{Sp}}_{2n}(\mathbf{R})$, let $j_{1/2}(g,Z)$ denote the canonical squareroot of $j(\overline{g}, Z)$, where $\overline{g}$ is the image of g in $\mathrm{Sp}_{2n}(\mathbf{R})$. If $\ell \in 2^{-1}\mathbf{Z}$ and $g \in \widetilde{\mathrm{Sp}}_{2n}(\mathbf{R})$, one can define $\mathcal{W}_{\ell,T}(g) = j_{1/2}(g,i)^{-2\ell} e^{2\pi i(\overline{g}\cdot i)}$.

Let now $\widetilde{\mathrm{Sp}}_{2n}(\mathbf{A})$ denote the metaplectic double cover of $\mathrm{Sp}_{2n}(\mathbf{A})$. The group $\mathrm{Sp}_{2n}(\mathbf{Q})$ splits uniquely into $\widetilde{\mathrm{Sp}}_{2n}(\mathbf{A})$. One can define automorphic forms on the metaplectic double cover. We say that φ corresponds to a Siegel modular form of weight $\ell \in 2^{-1}\mathbf{Z}$ if, for every $g_f \in \widetilde{\mathrm{Sp}}_{2n}(\mathbf{A}_f)$, the function $f_{\varphi,g_f} : \widetilde{\mathrm{Sp}}_{2n}(\mathbf{R}) \to \mathbf{C}$ given by $f_{\varphi,g_f}(g_\infty) = j_{1/2}(g_\infty, i)^{2\ell}\varphi(g_f g_\infty)$ descends to $\mathcal{H}_n$ and gives a holomorphic function there. These half-integral weight automorphic forms again have a Fourier expansion of the form (1).

2.2. **The Fourier-Jacobi expansion.** We will recall, without proof, the Fourier-Jacobi expansion of Siegel modular forms along the Klingen parabolic subgroup $Q = M_Q N_Q$ of Sp_{2n}. Here N_Q is the unipotent radical of Q and M_Q is its standard Levi subgroup. Let $X = N_Q \cap M_P$, where $M_P \simeq \mathrm{GL}_n$ is the Levi of the Siegel parabolic subgroup. For every nonzero rational number t, there is a Weil representation ω_t of $J_{n-1}(\mathbf{A}) := N_Q(\mathbf{A}) \rtimes \widetilde{\mathrm{Sp}}_{2n-2}(\mathbf{A})$ on $S(X(\mathbf{A}))$, the Schwartz-Bruhat space of $X(\mathbf{A})$. This representation has the center $Z(N_Q)$ of N_Q acting by $\psi(tz)$, for $z \in Z(N_Q)(\mathbf{A})$.

If $\phi \in S(X(\mathbf{A}_f))$, $T' \in S_{n-1}(\mathbf{Q})$, $a_{\mathrm{diag}(t,T')}$ is a $T = \mathrm{diag}(t,T')$ Siegel Whittaker function, $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$ and $r_f \in \widetilde{\mathrm{Sp}}_{2n-2}(\mathbf{A}_f)$, define

$$a_{t,T'}(r_f, g_f; \phi) = \int_{X(\mathbf{A}_f)} a_{\mathrm{diag}(t,T')}(x\overline{r_f}g_f)(\omega_{-t}(r_f)\phi)(x)\, dx$$

where Sp_{2n-2} is embedded in Sp_{2n} inside of M_Q.

Proposition 2.2. *Suppose φ corresponds to a cuspidal Siegel modular form of weight ℓ, $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$, and $\phi \in S(X(\mathbf{A}_f))$. If $t > 0$, then the $a_{t,T'}(r_f, g_f; \phi)$ are the Fourier coefficients of a cuspidal weight $\ell - \frac{1}{2}$ modular form on $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbf{A})$ as T' varies over the positive-definite elements of $S_{n-1}(\mathbf{Q})$.*

The proof of the proposition is to first define a theta function $\Theta_\phi(nr)$ on $J_{n-1}(\mathbf{A})$,

$$\Theta_\phi(nr) = \sum_{\xi \in X(\mathbf{Q})} \omega_{-t}(nr)(\phi \otimes \phi_\infty)(\xi).$$

Here ϕ_∞ is an appropriately chosen Gaussian on $X(\mathbf{R})$. Then, one obtains an automorphic form on $\widetilde{\mathrm{Sp}}_{2n-2}$ as

$$\mathrm{FJ}_\phi(r_f, g_f) = \int_{[N_Q]} \Theta_\phi(nr)\varphi(nrg_f)\, dn.$$

One calculates the Fourier expansion of this automorphic form, and finds that it corresponds to a holomorphic modular form of weight $\ell - \frac{1}{2}$ and has Fourier coefficients the $a_{t,T'}(r_f, g_f; \phi)$.

Definition 2.3. Suppose $\{a_T\}_T$ are a collection of Siegel-Whittaker functions. We say that the collection satisfies the P**-symmetries** if $a_T(m_n(\gamma)g_f) = \det(\gamma)^{-\ell} a_{T\cdot\gamma}(g_f)$ for every $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$ and every $\gamma \in \mathrm{GL}_n(\mathbf{Q})$. We say the collection satisfies the Q**-symmetries** if the conclusion of Proposition 2.2 holds. That is, for every $t > 0$ and $\phi \in S(X(\mathbf{A}_f))$, the $a_{t,T'}(r_f, g_f; \phi)$ are the Fourier coefficients of a cuspidal weight $\ell - \frac{1}{2}$ modular form on $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbf{A})$ as T' varies over the positive-definite elements of $S_{n-1}(\mathbf{Q})$.

3. Converse theorem

If φ is corresponds to a cuspidal Siegel modular form on Sp_{2n} of weight ℓ, then its Fourier coefficients a_T satisfy the P and Q symmetries. Additionally, the Fourier coefficients are uniformly smooth, in the sense there is an open compact subgroup U of $\mathrm{Sp}_{2n}(\mathbf{A}_f)$ so that the a_T are right U invariant for every T. Finally, the $a_T(g_f)$ grow polynomially in the norm $||T|| = (T,T)^{1/2}$ for every $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$. (In fact, they satisfy more specific, tighter bounds.)

The converse is also true.

Proposition 3.1. *Suppose the $\{a_T\}_T$ are a collection of Siegel Whittaker functions that satisfy the P and Q symmeties, are uniformly smooth, and grow polynomially with $||T||$. Then the sum*

$$\Psi(g_f g_\infty) = \sum_{T \in S_n(\mathbf{Q}), T>0} a_T(g_f) W_{\ell,T}(g_\infty)$$

converges absolutely. It defines a cuspidal automorphic form on Sp_{2n} that corresponds to a Siegel modular form of weight ℓ.

Proof sketch. One can show that the sum converges absolutely by the fact that the a_T grow polynomially. The result is a smooth, moderate growth, $\mathcal{Z}(\mathfrak{sp}_{2n})$-finite function on $\mathrm{Sp}_{2n}(\mathbf{A})$. The fact that the a_T satisfies the P symmetries implies that $\Psi(g)$ is left $P(\mathbf{Q})$-invariant.

Let Q^1 denote the derived group of Q. The fact that the a_T satisfies the Q symmetries implies that $\Psi(g)$ is left $Q^1(\mathbf{Q})$-invariant on $\mathrm{Sp}_{2n}(\mathbf{A}_f) \times (Q^1(\mathbf{R})K_\infty)$. Indeed, suppose $t > 0$. Let

$\Psi_t(g) = \int_{[Z(N_Q)]} \psi(-tz)\Psi(zg)\,dz$. Then one can reconstruct $\Psi_t(g)$ from the $a_{t,T'}(r_f, g_f, \phi)$ and theta functions. Specifically, set

$$\Psi_{t,\phi}(r; g_f) = \sum_{T' \in S_{n-1}(\mathbf{Q}), T'>0} a_{t,T'}(r_f, g_f; \phi)\mathcal{W}_{\ell-\frac{1}{2},T'}(r_\infty)$$

which is a cuspidal automorphic form on $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbf{A})$ by the fact that the a_T satisfy the Q-symmetries. Then

$$\Psi_t(nrg_f) = \sum_\alpha \Psi_{t,\phi_\alpha}(r; g_f)\Theta_{\phi_\alpha^\vee}(nr).$$

Here ϕ_α is a basis of $S(X(\mathbf{A}_f))$ and $\phi_\alpha^\vee$ is the dual basis. This expansion shows that $\Psi_t(qkg_f)$ is left-invariant under $Q^1(\mathbf{Q})$ for every $t \in \mathbf{Q}_{>0}^\times$, $q \in Q^1(\mathbf{Q})$, $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$ and $k \in K_\infty$. As $\Psi = \sum_t \Psi_t$, one obtains that Ψ is left $Q^1(\mathbf{Q})$-invariant on $\mathrm{Sp}_{2n}(\mathbf{A}_f) \times (Q^1(\mathbf{R})K_\infty)$.

But Ψ corresponds to a holomorphic function on $\mathcal{H}_n$, so by the identity theorem for holomorphic functions, one can check that Ψ is left $Q^1(\mathbf{Q})$-invariant on all of $\mathrm{Sp}_{2n}(\mathbf{A})$. As Ψ is both left invariant for $P(\mathbf{Q})$ and $Q^1(\mathbf{Q})$, it is left $\mathrm{Sp}_{2n}(\mathbf{Q})$-invariant. □

The aim of the rest of note is prove that assumption that the a_T grow polynomially with T is unnecessary.

Theorem 3.2 (Automatic convergence for Siegel modular forms)**.** *Suppose the $\{a_T\}_T$ are a collection of Siegel Whittaker functions that satisfy the P and Q symmeties, and are uniformly smooth. Then the a_T grow polynomially with T. Consequently, every such collection is the set of Fourier coefficients of an honest cuspidal Siegel modular form of weight ℓ.*

The proof of Theorem 3.2 combines some reduction theory with a "quantitative Sturm bound".

4. Reduction theory and the Quantitative Sturm bound

In this section, we review some reduction theory and prove the quantitative Sturm bound.

4.1. **Reduction theory.** There are two results we will need from reduction theory. The first involves Minkowski reduction theory for GL_n. It is standard. See, e.g., [And09, equation (1.22)].

Theorem 4.1. *There is a positive constant C_n with the following property: Suppose $T \in S_n(\mathbf{R})$ is positive-definite. Then there is $\gamma \in \mathrm{GL}_n(\mathbf{Z})$ so that $T \cdot \gamma$ has (11) entry at most $C_n \det(T)^{1/n}$.*

The second involves a Siegel set for the action of $\mathrm{Sp}_{2n}(\mathbf{Z})$ on the Siegel upper half-plane. One can see, e.g., [And09, Theorem 1.16].

Theorem 4.2. *There is a positive constant ϵ_n with the following property: Suppose $Z \in \mathcal{H}_n$. Then there is $\gamma \in \mathrm{Sp}_{2n}(\mathbf{Z})$ so that $Im(\gamma \cdot Z) > \epsilon_n 1_n$.*

Set $\mathcal{S}(\epsilon_n) = \{g \in \mathrm{Sp}_{2n}(\mathbf{R}) : Im(g \cdot i) > \epsilon_n 1_n\}$, and let $\widetilde{\mathcal{S}}(\epsilon_n)$ denote the inverse image of $\mathcal{S}(\epsilon_n)$ in $\widetilde{\mathrm{Sp}}_{2n}(\mathbf{R})$. Let $K_f = \prod_{p<\infty} \mathrm{Sp}_{2n}(\mathbf{Z}_p)$. Let $\widetilde{K}_f$ be the inverse image of K_f in $\widetilde{Sp}_{2n}(\mathbf{A}_f)$.

Corollary 4.3. *Suppose $g \in \mathrm{Sp}_{2n}(\mathbf{A})$. Then there is $\gamma \in \mathrm{Sp}_{2n}(\mathbf{Q})$ so that $\gamma g \in \mathcal{S}(\epsilon_n)K_f$. Likewise, if $g \in \widetilde{\mathrm{Sp}}_{2n}(\mathbf{A})$, then there is $\gamma \in \mathrm{Sp}_{2n}(\mathbf{Q})$ so that $\gamma g \in \widetilde{\mathcal{S}}(\epsilon_n)\widetilde{K}_f$.*

Proof. By approximation, one has $\mathrm{Sp}_{2n}(\mathbf{A}_f) = \mathrm{Sp}_{2n}(\mathbf{Q})K_f$. Thus we can write $g = \gamma_1 g_1$ with $g_{1,f} \in K_f$. By Theorem 4.2, there is $\gamma_2 \in \mathrm{Sp}_{2n}(\mathbf{Z})$ so that $\gamma_{2,\infty} g_{1,\infty} \in \mathcal{S}(\epsilon_n)$. The corollary follows. □

4.2. **Quantitative Sturm bound.** Recall that the classical Sturm bound result says that if the first several Fourier coefficients of a holomorphic modular form of some weight ℓ are 0, then the modular form is identitcally 0. By a quantitative Sturm bound we mean a result of the form "if the first several Fourier coefficients of a holomorphic modular form of some weight ℓ are at most $\epsilon \geq 0$, then the automorphic function φ is bounded by a constant times ϵ". In particular, all the Fourier coefficients of φ are bounded by a constant times ϵ.

We explain the proof of such a result on Sp_{2n} here. We begin with a few simple lemmas.

Lemma 4.4. *Suppose $X \in \mathbf{R}_{>0}$.*

(1) *The number of $T \in S_n(\mathbf{Z})^\vee$ positive-definite with $\mathrm{tr}(T) \leq X$ is bounded by $2^{n(n-1)/2}X^{n(n+1)/2}$.*
(2) *If $M \in \mathbf{Z}_{\geq 1}$, the number of $T \in M^{-1}S_n(\mathbf{Z})^\vee$ positive-definite with $\mathrm{tr}(T) \leq X$ is bounded by $2^{n(n-1)/2}M^{n(n+1)/2}X^{n(n+1)/2}$.*

Proof. We prove the first statement of the lemma. The second follows from the first. Thus suppose $T \in S_n(\mathbf{Z})^\vee$ is positive-definite and $\mathrm{tr}(T) \leq X$. Then each diagonal entry T_{ij} is an integer between 1 and X. For the off diagonal entries, we have $T_{ii}T_{jj} - \frac{T_{ij}^2}{4} > 0$. Thus

$$|T_{ij}| < 2\sqrt{T_{ii}T_{jj}} \leq T_{ii} + T_{jj}.$$

So, $|T_{ij}| \leq T_{ii} + T_{jj} - 1 \leq X - 1$. Thus there at most $2X$ possibilities for T_{ij}. □

Suppose $r > 0, N > 0$. Finding the critical point of the function $f(v) = v^N e^{-rv}$ for $v > 0$ gives:

Lemma 4.5. *One has $v^N e^{-rv} \leq (N/r)^N e^{-N}$ for $v \geq 0$.*

We now bound some infinite sums.

Lemma 4.6. *Suppose $R > 0$ and $M \in \mathbf{Z}_{\geq 1}$. For $y \in S_n(\mathbf{R})$ positive-definite and $\ell \geq 0$, define*

$$S_\ell(M,y) = \sum_{T \in M^{-1}S_n(\mathbf{Z})^\vee} \det(T)^{\ell/2}\det(y)^{\ell/2}e^{-2\pi(T,y)}$$

and

$$T_\ell(R,M,y) = \sum_{T \in M^{-1}S_n(\mathbf{Z})^\vee, \det(T) \geq R} \det(T)^{\ell/2}\det(y)^{\ell/2}e^{-2\pi(T,y)}.$$

If $1 > \mu > 0$ and $y \geq \mu 1_n$, then there are positive constants $D_{\ell,n,\mu}$ and α so that

$$S_\ell(M,y) \leq D_{\ell,n,\mu}M^\alpha$$

for all $M \geq 1$. Likewise,

$$T_\ell(R,M,y) \leq D_{\ell,n,\mu}M^\alpha e^{-\pi\mu n R^{1/n}/2}.$$

Proof. First note that, by the AM-GM inequality,

$$\det(Ty) = \det(y^{1/2}Ty^{1/2}) \leq \left(\frac{1}{n}\mathrm{tr}(y^{1/2}Ty^{1/2})\right)^n = n^{-n}(T,Y)^n. \quad (2)$$

Thus $\det(TY)^{\ell/2}e^{-\pi(T,Y)} \leq n^{-n\ell/2}(T,Y)^{\ell n/2}e^{-\pi(T,Y)}$, which by Lemma 4.5 is bounded by

$$C_{\ell,n} := \left(\frac{\ell n}{2\pi}\right)^{\ell n/2} n^{-n\ell/2}e^{-\ell n/2}.$$

Thus

$$S_\ell(M,y) \leq C_{\ell,n}\sum_{T \in M^{-1}S_n(\mathbf{Z})^\vee} e^{-\pi(T,y)} \leq C_{\ell,n}\sum_{T \in M^{-1}S_n(\mathbf{Z})^\vee} e^{-\pi\mu\,\mathrm{tr}(T)}.$$

Applying Lemma 4.4, one obtains

$$S_\ell(M,y) \leq C_{\ell,n}2^{n(n-1)/2}\sum_{k \geq 1} k^{n(n+1)/2}e^{-\pi\mu k/M}.$$

Applying Lemma 4.5 again, summing a geometric series, and applying the inequality $\frac{e^{-r}}{1-e^{-r}} \leq r^{-1}$ for $r > 0$ gives the bound on $S_\ell(M, y)$.

To bound $T_\ell(R, M, y)$, first note that if $\det(T) \geq R$ and $y > \mu 1_n$, then using (2), one sees that (T, y) is bounded below by $nR^{1/n}\mu$. Thus, to bound $T_{\ell,M,y}$ it suffices to bound

$$\sum_{k \geq R^{1/n} nM} k^{n(n+1)/2} e^{-\pi\mu k/M}.$$

Arguing as above, one gets the claim. □

If φ corresponds to a Siegel modular form of weight ℓ with Fourier coefficients $a_T(g_f)$, let $\beta_T(g_f) = \det(T)^{-\ell/2} a_T(g_f)$ be the normalized Fourier coefficients.

Lemma 4.7. *Suppose $|\varphi(g)| \leq L$ for all $g \in \mathrm{Sp}_{2n}(\mathbf{A})$. Then $|\beta_T(g_f)| \leq e^{2\pi n} L$ for all $T > 0$ and all $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$.*

Proof. This is standard. From $|\varphi(g)| \leq L$ for all g, one obtains by integration that

$$|\beta_T(g_f)| \det(T)^{\ell/2} |\mathcal{W}_{\ell,T}(g_\infty)| \leq L$$

for all g_f, all g_∞. Letting $g_\infty = m_n(T^{-1/2})$ gives the lemma. □

Here is the quantitative Sturm bound.

Theorem 4.8 (Quantitative Sturm bound)**.** *Suppose $\varphi(g)$ is a cuspidal automorphic form on $\mathrm{Sp}_{2n}(\mathbf{A})$ or its double cover corresponding to a holomorphic modular form of weight $\ell \geq 0$. Let $M \geq 1$ be a positive integer so that $\beta_T(k) \neq 0$ implies $T \in M^{-1} S_n(\mathbf{Z})^\vee$ for all T and all $k \in K_f = \mathrm{Sp}_{2n}(\widehat{\mathbf{Z}})$ or $\widetilde{K}_f$. Suppose $\epsilon \geq 0$ is such that that the normalized Fourier coefficients $\beta_T(g_f)$ satisfy $|\beta_T(k)| \leq \epsilon$ for all $k \in K_f$ and all T with $\det(T) \leq R$, where*

$$R^{1/n} = \frac{2}{\pi n \epsilon_n} \log(2e^{2\pi n} D_{\ell,n,\epsilon_n} M^\alpha).$$

Then

$$|\varphi(g)| \leq \epsilon \left(2e^{2\pi n} D_{\ell,n,\epsilon_n} M^\alpha\right).$$

In particular,

$$|\beta_T(g_f)| \leq \epsilon \cdot \left(2e^{4\pi n} D_{\ell,n,\epsilon_n} M^\alpha\right)$$

for all T and all $g_f \in \mathrm{Sp}_{2n}(\mathbf{A}_f)$.

Proof. We write out the proof in the case of the linear group Sp_{2n}. The proof in the case of the double cover is identical.

Let $g_* \in \mathrm{Sp}_{2n}(\mathbf{A})$ be such that $|\varphi(g)|$ attains its maximum at g_*. Let $L = |\varphi(g_*)|$. By Corollary 4.3, we can assume $g_* = g_\infty k \in \mathcal{S}(\epsilon_n) K_f$. Write $y = Im(g_\infty \cdot i)$. By Lemma 4.7, we obtain

$$\begin{aligned} L = |\varphi(g_*)| &\leq \sum_{T \in M^{-1} S_n(\mathbf{Z})^\vee, T > 0} |\beta_T(k)| \det(Ty)^{\ell/2} e^{-2\pi(T,y)} \\ &\leq e^{2\pi n}(\epsilon S_\ell(M, y) + L T_\ell(R, M, y)). \end{aligned}$$

Applying Lemma 4.6 gives

$$L \leq e^{2\pi n} D_{\ell,n,\epsilon_n} M^\alpha (\epsilon + L e^{-\pi \epsilon_n n R^{1/n}/2}).$$

The constant R is chosen so that

$$e^{2\pi n} D_{\ell,n,\epsilon_n} M^\alpha e^{-\pi\epsilon_n n R^{1/n}/2} = \frac{1}{2}.$$

Rearranging the inequality gives the theorem. □

5. Automatic convergence

In this section, we prove Theorem 3.2, the automatic convergence theorem. We will need some prepatory lemmas.

5.1. **Prepatory lemmas.**

Lemma 5.1. *Let $U_{r,g} \subseteq X(\mathbf{A}_f)$ be an open compact subset such that $a_{\mathrm{diag}(t,T')}(xrg) \neq 0$ implies $x \in U_{r,g}$. Let B be a positive real number so that $|a_{\mathrm{diag}(t,T')}(xrg)| \leq B$ for all $x \in X(\mathbf{A}_f)$. Then $|a_{t,T'}(r,g,\phi)| \leq ||\phi||_{L^2} \cdot B \cdot \mathrm{vol}(U_{r,g})^{1/2}$.*

Proof. The proof is identical to [Pol24, Lemma 13.4]. □

Conversely, we can bound the $a_{\mathrm{diag}(t,T')}(xg)$ in terms of the $a_{t,T'}(r,g,\phi)$.

Lemma 5.2. *Suppose $B'_{t,T',g} > 0$ is a constant so that $|a_{t,T'}(1,g,\phi)| \leq B'_{t,T',g} \cdot ||\phi||_{L^2}$ for all $\phi \in S(X(\mathbf{A}_f))$. Suppose $V_{t,T',g} \subseteq X(\mathbf{A}_f)$ is a compact open subgroup with the property that $a_{\mathrm{diag}(t,T')}(xvg) = a_{\mathrm{diag}(t,T')}(xg)$ if $v \in V_{t,T',g}$ and $x \in X(\mathbf{A}_f)$. Then $|a_{\mathrm{diag}(t,T')}(xg)| \leq B'_{t,T',g} \cdot \mathrm{vol}(V_{t,T',g})^{-1/2}$.*

Proof. The proof is identical to [Pol24, Lemma 13.5]. □

5.2. **Proof of automatic convergence.** To prove that the $a_T(g_f)$ grow polynomially for all g_f, it suffices to check it for $g_f \in K_f$. Indeed, this follows from the P symmetries and from the decomposition $\mathrm{Sp}_{2n}(\mathbf{A}_f) = N_P(\mathbf{A}_f)M_P(\mathbf{Q})K_f$, where $P = M_P N_P$ is the Siegel parabolic subgroup.

To prove the polynomial growth, we will proceed by induction. To set up the induction, let $\delta > 1$ be a positive real number, to be determined, and let $D_0 > 0$ be a sufficiently large real number. If $T \in S_n(\mathbf{Q})$ is positive-definite, then $\det(T) \leq D_0^{\delta^n}$ for some $n = 0, 1, 2, \ldots$. We will show that, there are positive constants $Q, E > 0$ so that if $D_0^{\delta^{n-1}} \leq \det(T) \leq D_0^{\delta^n}$, then

$$|\beta_T(k)| \leq Q(1 \cdot D_0 \cdot D_0^{\delta} \cdots D_0^{\delta^{n-1}})^E = QD_0^{E \cdot \frac{\delta^n - 1}{\delta - 1}} \leq Q\det(T)^{\delta E/(\delta - 1)}.$$

We will prove the first inequality by induction on n. The second inequality shows that the growth is polynomial.

To begin, first note that there is a lattice $\Lambda \subseteq S_n(\mathbf{Q})$ so that $a_T(k) \neq 0$ and $k \in K_f$ implies $T \in \Lambda$. Indeed, this follows from the right U-invariance of the a_T's and the fact that they are Siegel Whittaker functions. See, e.g., [Pol24, Lemma 9.1]. Without loss of generality, we can assume that U is normal in K_f and Λ is $\Gamma_U := \mathrm{SL}_n(\mathbf{Q}) \cap U$ invariant. There are only finitely many Γ_U orbits on the elements $T \in \Lambda$ with $\det(T)$ bounded. See, e.g., [Pol24, Lemma 13.2]. Moreover, $|a_{T\cdot\gamma}(k)| = |a_T(k)|$ if $\gamma \in \Gamma_U$. Thus, the base case of our induction can be satisfied for any D_0, for some Q depending on D_0.

We now proceed with the induction step. For ease of notation, if $n \geq 1$ and $D_0^{\delta^{n-1}} \leq D < D_0^{\delta^n}$, we set

$$f(D) = Q(1 \cdot D_0 \cdot D_0^{\delta} \cdots D_0^{\delta^{n-1}})^E = QD_0^{E \cdot \frac{\delta^n - 1}{\delta - 1}}.$$

We suppose that $|\beta_T(k)| \leq Qf(\det(T))$ for all T with $\det(T) < D_0^{\delta^n}$.

Claim 5.3. *Let $D_1 = D_0^{\delta^n}$. Suppose $\epsilon > 0$ is small, D_0 is sufficiently large, and $t < D_1^{1-\epsilon}$. Suppose $x \in X(\mathbf{A}_f)$. Then there are positive constants C' and α so that $|a_{\mathrm{diag}(t,T')}(xk)| \leq QC' \det(T')^{\ell/2} t^{\alpha'} f(D_1)$ for all $k \in K_f$.*

Proof. We apply the quantitative Sturm bound for cusp forms of weight $\ell - \frac{1}{2}$ on $\widetilde{\mathrm{Sp}}_{2n-2}(\mathbf{A})$. Fix $\phi \in S(X(\mathbf{A}_f))$. The Fourier coefficients of our cusp form are

$$a_{t,T'}(k_1, k, \phi) = \int_{X(\mathbf{A}_f)} \omega(k_1)\phi(x) a_{\mathrm{diag}(t,T')}(xk_1k)\, dx.$$

The constant M of the quantitative Sturm bound can be taken to be proportional to t. Applying Lemma 5.1, we see that the normalized Fourier coefficients are bounded by a constant times $||\phi||^2 t^{\ell/2} \det(T')^{1/2} f(D_1)$ if $t\det(T') < D_1$. As $t < D_1^{1-\epsilon}$, we see that if $\det(T') < (a_1 + b_1 \log(t))^{n-1}$ then $t\det(T') < D_1$. Here a_1, b_1 are fixed postive constants. We can thus apply the quantitative Sturm bound. Applying it, we find

$$|a_{t,T'}(r,k,\phi)| \leq QC||\phi||^2 \det(T')^{\ell/2} t^{\alpha'} f(D_1)$$

for some positive constants C and α'. We now apply Lemma 5.2 to finish the proof. □

Suppose now $T \in S_n(\mathbf{Q})$ and $D_1 = D_0^{\delta^n} \leq \det(T) \leq D_1^{\delta}$. We wish to bound $|\beta_T(k)|$ for $k \in K_f$. By Theorem 4.1, we can assume $t = T_{11} \leq C_n D_1^{\delta/n}$. Thus, if $1 < \delta < n$, $t < D_1^{1-\epsilon}$, so we can apply Claim 5.3. We obtain $|\beta_T(k)| \leq QD_1^E f(D_1)$ for some positive number $E > 0$. This completes the induction, and with it, the automatic convegence theorem.

References

[And09] Anatoli Andrianov, *Introduction to Siegel modular forms and Dirichlet series*, Universitext, Springer, New York, 2009. MR 2468862

[BR24] Jan Hendrik Bruinier and Martin Raum, *Formal siegel modular forms for arithmetic subgroups*, Preprint (2024).

[Bru15] Jan Hendrik Bruinier, *Vector valued formal Fourier-Jacobi series*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 505–512. MR 3283640

[BWR15] Jan Hendrik Bruinier and Martin Westerholt-Raum, *Kudla's modularity conjecture and formal Fourier-Jacobi series*, Forum Math. Pi **3** (2015), e7, 30. MR 3406827

[IPY13] Tomoyoshi Ibukiyama, Cris Poor, and David S. Yuen, *Jacobi forms that characterize paramodular forms*, Abh. Math. Semin. Univ. Hambg. **83** (2013), no. 1, 111–128. MR 3055825

[Pol24] Aaron Pollack, *Automatic convergence and arithmeticity of modular forms on exceptional groups*, Preprint (2024).

[WR15] Martin Westerholt-Raum, *Formal Fourier Jacobi expansions and special cycles of codimension two*, Compos. Math. **151** (2015), no. 12, 2187–2211. MR 3433884

[Xia22] Jiacheng Xia, *Some cases of Kudla's modularity conjecture for unitary Shimura varieties*, Forum Math. Sigma **10** (2022), Paper No. e37, 31. MR 4436595

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