ON THE RESIDUE METHOD FOR PERIOD INTEGRALS

AARON POLLACK, CHEN WAN, AND MICHAŁ ZYDOR

Abstract. By applying the residue method for period integrals, we prove identities between period integrals on cuspidal automorphic representations and period integrals on residual representations for seven spherical varieties. Combining with Langlands-Shahidi's theory for residues of Eisenstein series, for each case, we prove some relations between the period integrals and certain automorphic L-functions. In some cases, we also study the local multiplicity of the spherical varieties.

1. Introduction and main results

Let k be a number field and $\mathbb{A}$ its ring of adeles. Let G be a reductive group defined over k, and H a closed subgroup of G. Assume that $X = H\backslash G$ is a spherical G-variety (i.e., a Borel subgroup of G has a dense orbit in X). Let A_G be the maximal split torus of the center of G and let $A_{G,H} = A_G \cap H$. For an automorphic form ϕ on $G(\mathbb{A})$ whose central character is trivial on $A_{G,H}(\mathbb{A})$, we define the period integral $\mathcal{P}_H(\phi)$ to be

$$\mathcal{P}_H(\phi) := \int_{H(k)A_{G,H}(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)dh.$$

The integral is not absolutely convergent in general (unless ϕ is a cusp form, see Proposition 1 of [AGR93] and Proposition A.1.1 of [B16]) but can be regularized (see Section 2.5 for more details).

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ whose central character is trivial on $A_{G,H}(\mathbb{A})$. One of the most fundamental problems in the relative Langlands program is to find the relation between $\mathcal{P}_H|_\pi$-the period integral restricted to the space of π-and some automorphic L-functions of π. This point of view was most systematically put forward by Sakellaridis [S12], and Sakellaridis-Venkatesh [SV17].

In this paper, we prove such relations for seven spherical varieties by establishing identities between $\mathcal{P}_H(\phi)$ and another period $\mathcal{P}_{\underline{H}}(Res_{s=s_0}(E(\phi,s)))$ where $(\underline{G},\underline{H})$ is another (spherical) pair, $\phi \in \pi$ is a cusp form, and $Res_{s=s_0}(E(\phi,s))$ is the residue of the Eisenstein series on $\underline{G}(\mathbb{A})$ induced from π. The connection with special values of L-functions is then derived through the Langlands-Shahidi theory. In some cases, we also study the local multiplicity of the spherical varieties.

1.1. Main global results.

Let $\underline{G}$ be a reductive group over k and let $\underline{H}$ be a reductive subgroup of $\underline{G}$. Suppose G is a maximal, self-dual, Levi subgroup of $\underline{G}$ and let $H = \underline{H} \cap G$. The cases of interest are listed in the table below (the Levi G is listed up to GL_1 factors):

№	$\underline{G}$	$\underline{H}$	G	$H = \underline{H} \cap G$	s_0
1	SO_{2n+3}	$\mathrm{SO}_{n+3} \times \mathrm{SO}_n$	SO_{2n+1}	$\mathrm{SO}_{n+1} \times \mathrm{SO}_n$	$1/2$
2	SO_{2n+2}	$\mathrm{SO}_{n+3} \times \mathrm{SO}_{n-1}$	SO_{2n}	$\mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1}$	1
3	U_{2n+2}	$U_{n+2} \times U_n$	U_{2n}	$U_n \times U_n$	$1/2$
4	Sp_{4n}	$Res_{k'/k}\mathrm{Sp}_{2n}$	GL_{2n}	$Res_{k'/k}\mathrm{GL}_n$	$1/2$
5	E_7	$A_1 \times D_6$	GE_6	$A_1 \times A_6$	1
6	D_5	B_3	$\mathrm{GL}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	$1/2$
7	Sp_6	$\mathrm{Sp}_4 \times \mathrm{GL}_1$	Sp_4	$\mathrm{Sp}_2 \times \mathrm{GL}_1$	1

Table 1. Spherical pairs considered in the global setting

Here k'/k denotes a quadratic extension, which is also used to define the unitary groups in case 3. In cases 5 and 6 we only provide the Cartan type of the Lie algebra of the underlying group, the details for these exceptional cases will be provided in Sections 8 and 9. In case 6, the embedding of $\mathrm{GL}_2 \times \mathrm{GL}_2$ into $\mathrm{GL}_4 \times \mathrm{GL}_2$ is given by $(a,b) \mapsto (\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a)$.

Let $\underline{P} = \underline{M}\underline{N}$ be a maximal Levi subgroup of $\underline{G}$ with $G \subset \underline{M}$ (for all of our cases, we either have $G = \underline{M}$ or $\underline{M} = G \times \mathrm{GL}_1$). Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. We extend π to $\underline{M}$ by making it trivial on the GL_1-factor. Let $\mathcal{A}_\pi$ be the space of automorphic forms ϕ on $\underline{N}(\mathbb{A})\underline{M}(F)\backslash\underline{G}(\mathbb{A})$ such that the function $m \in \underline{M}(\mathbb{A}) \mapsto \phi(mg)$ belongs to the space of π for all $g \in \underline{G}(\mathbb{A})$ (see Section 2.4). For $\phi \in \mathcal{A}_\pi$ let $E(x, \phi, s)$ be the associated Eisenstein series, which is an automorphic form on $\underline{G}$. The main global result of this paper is, in essence, the following equality

$$\mathcal{P}_H(\phi) = \mathcal{P}_{\underline{H}}(\mathrm{Res}_{s=s_0}(E(\phi, s_0)).$$

More precisely, we prove

Theorem 1.1. *Let π be an irreducible, cuspidal automorphic representation of $G(\mathbb{A})$ which is generic at one non-archimedean place of k. Then, for all cases listed in the table above, we have*

$$\int_{K_{\underline{H}}} \mathcal{P}_H(\phi_k)\, dk = \mathcal{P}_{\underline{H},reg}(\mathrm{Res}_{s=s_0}(E(\phi, s_0))$$

where $K_{\underline{H}}$ is a good maximal compact subgroup of $\underline{H}(\mathbb{A})$ (we refer the reader to Section 2.2 for the choice of the measure dk, the integral over $K_{\underline{H}}$ is identified with the integral over $P_{\underline{H}}(\mathbb{A})\backslash\underline{H}(\mathbb{A})$ where $P_{\underline{H}} = \underline{P} \cap \underline{H}$), $\phi_k(x) = \phi(xk)$ and $\mathcal{P}_{\underline{H},reg}$ denotes the possibly regularized period integral depending on the particular inner forms of the groups considered (see Sections 2.5 and 3.2).

As an immediate application of Theorem 1.1 we can relate the non-vanishing of the period integral $\mathcal{P}_H$ on the space of π to the non-vanishing of the residue $\mathrm{Res}_{s=s_0}(E(\phi, s_0))$ (see Proposition 3.7 for more details). Note that by a similar argument as in Proposition 2 of [JR92], the integral over $K_{\underline{H}}$ does not kill the period integral.

Corollary 1.2. *With the same assumptions as in Theorem 1.1, if the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then the cuspidal Eisenstein series $E(\phi, s)$ has a pole at $s = s_0$.*

Invoking the Langlands-Shahidi theory, we get the following corollary which gives a relation between the period integrals and some automorphic L-functions.

Corollary 1.3. *Assume that G is quasi-split over k. Let π be a cuspidal generic automorphic representation of $G(\mathbb{A})$. In Case (5) (resp. Case (6)), we assume that the standard L-function $L(s,\pi)$ (resp. the L-function $L(s,\pi,\wedge^2 \otimes Std)$) is nonzero at $s = 2$ (resp. $s = 3/2$) - this is always the case if π is tempered. If the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, then, for each respective case listed in Table 1, the following holds.*

1. *The standard L-function $L(s,\pi)$ is nonzero at $s = 1/2$.*
2. *The standard L-function $L(s,\pi)$ has a pole at $s = 1$.*
3. *The standard L-function $L(s,\pi)$ is nonzero at $s = 1/2$. Moreover, if π is a discrete series representation at a local split place (with respect to the quadratic extension k'/k) then the exterior square L-function $L(s,\pi,\wedge^2)$ has a pole at $s = 1$.*
4. *The standard L-function $L(s,\pi)$ is nonzero at $s = 1/2$ and the exterior square L-function $L(s,\pi,\wedge^2)$ has a pole at $s = 1$.*
5. *The standard L-function $L(s,\pi)$ has a pole at $s = 1$.*
6. *The L-function $L(s,\pi,\wedge^2 \otimes Std)$ is nonzero at $s = 1/2$.*
7. *The standard L-function $L(s,\pi)$ has a pole at $s = 1$.*

Throughout this paper, all the L-functions are the completed Langlands-Shahidi L-functions; in particular, the global L-functions include local factors from the archimedean places.

Locally, we say a representation of a quasi-split group $G(F)$ is generic if it admits a Whittaker model with respect to a generic character of the maximal unipotent subgroup of $G(F)$. Globally, we say a cuspidal automorphic representation is generic if it has nonzero Whittaker-Fourier coefficient (in particular, it is generic at all the local places).

1.2. **The method of proof.** In this subsection, we briefly explain the proof of Theorem 1.1. More details will be provided in Section 3.2 after we introduce some notation in Section 2. The starting point is to consider the integral

$$\int_{\underline{H}(k)A_{\underline{G},\underline{H}}(\mathbb{A})\backslash\underline{H}(\mathbb{A})} \Lambda^{T,\underline{H}}E(h,\phi,s)dh = \mathcal{P}_{\underline{H}}(\Lambda^{T,\underline{H}}E(\phi,s))$$

where $T \in \mathbb{R}$ is the truncation parameter and $\Lambda^{T,\underline{H}}$ is the truncation operator developed in [Zyd19] (based on the classical truncation operator developed by Arthur [A]). The truncation makes the integral absolutely convergent, while the integral of the Eisenstein series alone diverges in this case. As follows from the analysis in [Zyd19], the truncated period can be explicitly expressed as

$$\mathcal{P}_{\underline{H}}(\Lambda^{T,\underline{H}}E(\phi,s)) = \mathcal{P}_{\underline{H},reg}(E(\phi,s)) + \frac{e^{T(s-s_0)}}{s-s_0}\int_{K_{\underline{H}}} \mathcal{P}_H(\phi_k)\,dk + \Psi(T,s,\phi). \tag{1.1}$$

Here $\mathcal{P}_{\underline{H},reg}$, the so called regularized period, is a functional on the space of almost all automorphic forms (i.e. automorphic forms with regular parameters) on $\underline{G}$ that extends in an $\underline{H}(\mathbb{A})$-invariant way the period integral $\mathcal{P}_{\underline{H}}$ and $\Psi(T,s,\phi)$ is an explicit term that we will ignore in the introduction (we refer the readers to Section 3.2 for details). We point out that the identity (1.1) is not merely a technical result as the singularity at $s = s_0$ hints at the existence of a pole of the Eisenstein series at s_0 and its relation to the period $\mathcal{P}_H(\phi)$, both entities being completely independent of the regularization method.

Since the regularized period is $\underline{H}(\mathbb{A})$-invariant, by applying a local result of Prasad and Sakellaridis in [Pra18], together with our assumption that π is generic at one non-archimedean place of k, we know that $\mathcal{P}_{\underline{H},reg}(E(\phi,s))$ must be zero. Then the equation (1.1) simplifies to

$$\mathcal{P}_{\underline{H}}(\Lambda^{T,\underline{H}}E(\phi,s)) = \frac{e^{T(s-s_0)}}{s-s_0}\int_{K_{\underline{H}}} \mathcal{P}_H(\phi_k)\,dk + \Psi(T,s,\phi).$$

Taking the residue at $s = s_0$, which can be taken inside the integrals, and then letting T go to infinity, one obtains the desired identity of Theorem 1.1. Examination of $\Psi(T,s,\phi)$ specifies conditions on the exponents of $\mathrm{Res}_{s=s_0}(E(\phi,s_0))$ which determine whether the residue has a convergent period over $\underline{H}$ or if the regularization is necessary.

1.3. **Comparison with other methods and results.** The idea of relating period integrals of cuspidal representations to period integrals of certain residual representations is not new and goes back to Jacquet-Rallis [JR92]. We refer to it as the residue method. It has been applied by Jiang [Jia98], Ginzburg-Rallis-Soudry [GRS99], Ginzburg-Jiang-Rallis [GJR04a],[GJR05],[GJR09], Ginzburg-Jiang-Soudry [GJS10], Pollack-Wan-Zydor [PWZ], where the classical truncation operator of Arthur [A] was used. A different approach is to use pseudo-Eisenstein series to handle the divergence issues as introduced by Lapid-Rogawski [LR03] and used in this "residual" context by Ginzburg-Lapid [GL07]. Both methods have the disadvantage, in our view, that one needs to study the orbit set $\underline{H}(k)\backslash\underline{G}(k)/\underline{P}(k)$ which can be complicated and/or even infinite (for example, the orbit set is infinite for the models (1)-(3) in Table 1).

The truncation operator defined in [Zyd19] circumvents this need and allows for a uniform treatment of many different groups. A truncation operator adapted to a particular period was first developed by Jacquet-Lapid-Rogawski [JLR99] in the case of Galois periods as well as Ichino-Yamana [IY] also in the "residual" context. It is often referred to as the mixed truncation operator. Using the general theory of such operators [Zyd19], that allows to define $\mathcal{P}_{H,reg}$ whenever H is reductive, we push the residue method to its limit by considering essentially all the remaining untreated cases in a uniform way (see Section 3.1 for more details).

The residue method can also be applied to cases when H is not reductive, as done in [GL07, PWZ]. However, at this moment, the orbit space analysis can not be avoided for the non-reductive case because the truncation operator defined in [Zyd19] is only available for the reductive case. Some other applications of the residue method can be found in [Lap04] and [LR03].

Some of the models we consider in this paper have already appeared in previous literature. For example, when G is an orthogonal group, the connection between periods of cusp forms, L-values, and the theta correspondence has been studied by other authors. In case $G = \mathrm{SO}(V)$, Theorem 1.1 and Corollary 1.3 can be interpreted in this context. In particular, we point the interested reader to Ginzburg-Jiang-Soudry [GJSI] (see also [GJSII].) It seems likely that the combination of various results on the theta correspondence, Rallis inner product formula, and Siegel-Weil formula (e.g., [KR94], [M97a], [M97b], [GJSI], [GQT14]) could imply Theorem 1.1 and Corollary 1.3 when $G = \mathrm{SO}(V)$. One feature of our method is that we can obtain results relating non-vanishing of L-values and periods on orthogonal groups simply and directly, and without the Weil representation as an intermediary.

We also mention that the model $(U_{2n}, U_n \times U_n)$ relates to the principle put forward by Getz and Wambach in [GW14]. They conjectured that for any reductive group H and any involution σ of H, the non-vanishing of the period integrals of the model (H, H^σ) (H^σ being the group of fixed points of σ) for a cuspidal automorphic representation π of $H(\mathbb{A})$ should be (roughly) equivalent to the non-vanishing of the period integrals of the model (G, G^σ) for the base change of π to $G(\mathbb{A})$. Our result in Corollary 1.3(3) confirms one direction of a special case of their conjecture. Also in a special case when $n = 2$, the model $(\mathrm{U}_4, \mathrm{U}_2 \times \mathrm{U}_2)$ and its twists also appear in the work of Ichino-Prasanna [IP18] in the context of algebraic cycles on Shimura varieties.

On the other hand, the model $(\mathrm{GL}_{2n}, Res_{k'/k}GL_n)$ is the Jacquet-Guo model, first studied in [Guo96]. Our complete result in this case is Corollary 1.3(4). In [FMW18], under some local assumptions on π (i.e. π is supercuspidal at some split place and H-elliptic at another place), the authors proved the same result by the relative trace formula method.

1.4. **Local result.** In this subsection, we state our local result for some of the models in Table 1. Throughout this paper, all local fields have characteristic 0 and all the representations are smooth admissible complex representations. Let F be a p-adic field and π be an irreducible representation of $G(F)$. We define the multiplicity

$$m(\pi) = \dim(\mathrm{Hom}_{H(F)}(\pi, 1)).$$

We say π is H-distinguished if $m(\pi) \neq 0$.

We first consider the following four cases:

$\underline{G}$	$\underline{H}$	G	$H = \underline{H} \cap G$	s_0	ρ
SO_{2n+2}	$\mathrm{SO}_{n+3} \times \mathrm{SO}_{n-1}$	SO_{2n}	$\mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1}$	1	std
Sp_{4n}	$Res_{k'/k}\mathrm{Sp}_{2n}$	GL_{2n}	$Res_{k'/k}\mathrm{GL}_n$	1/2	$\wedge^2$
E_7	$A_1 \times D_6$	GE_6	$A_1 \times A_6$	1	std
Sp_6	$\mathrm{Sp}_4 \times \mathrm{GL}_1$	Sp_4	$\mathrm{Sp}_2 \times \mathrm{GL}_1$	1	std

TABLE 2. Spherical pairs considered in the local setting

Here ρ is a representation of the dual group $\hat{G}$ of G.

Theorem 1.4. *Let (G,H) be one of the models above and π be an irreducible tempered generic representation of $G(F)$. If π is H-distinguished, then the local L-function $L(s,\pi,\rho)$ has a pole at $s=0$.*

Now we briefly explain the proof of Theorem 1.4. Recall that $\underline{P}=\underline{MN}$ is a maximal parabolic subgroup of $\underline{G}$ with $G\subset\underline{M}$. For an irreducible representation π of $G(F)$, we extend it to $\underline{M}(F)$ by making it trivial on the GL_1-factor. For $s\in\mathbb{C}$, let $\pi_s=\pi\otimes\varpi^s$ where ϖ is a character of $\underline{M}(F)$ induced by the fundamental weight associated to $\underline{P}$. Then we let $\Pi_s=I_{\underline{P}}^{\underline{G}}(\pi_s)$ where $I_{\underline{P}}^{\underline{G}}$ is the normalized parabolic induction.

By using Frobenius reciprocity, one can show that for all the models in the table above, if π is H-distinguished, then Π_{s_0} is $\underline{H}$-distinguished. Combining with the result of Prasad and Sakellaridis in [Pra18], we know that Π_{s_0} is not generic. Now if we assume π is tempered and generic as in Theorem 1.4, by the results in [GI16] and [HO13], Π_{s_0} being not generic would imply that some local gamma function has a pole at $s=s_0$. This will then imply Theorem 1.4. Our method can be viewed as a local analogue of the residue method. It has been used by Jiang-Nien-Qin ([JNY], [JQ]) for the Shalika model case.

Another local result is for the model $(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2)$. In Section 9.5, we will show that the summation of the multiplicities of this model is always equal to 1 over every tempered local Vogan L-packet.

1.5. **Relation to conjectures of Sakellaridis-Venkatesh.** In [SV17], Sakellaridis and Venkatesh formulated some global and local conjectures for general spherical varieties. In their global conjecture (Section 17 of [SV17]), they conjectured a decomposition of period integrals into an Euler product of local relative characters (this is the so-called "Ichino-Ikeda" type formula). Since the local relative characters are closely related to some local L-functions, the "Ichino-Ikeda" type formula will imply a relation between the period integrals and some automorphic L-functions. The relation will look like the ones in Corollary 1.3 except that it will have two directions (i.e. the period integrals are nonzero if and only if some automorphic L-functions satisfy certain conditions). Locally, they conjectured (Section 16 of [SV17]) a Plancherel decomposition for $L^2(H\backslash G)$ which relates the condition of H-distinguish to Langlands functoriality. One important point in their local and global conjectures is that instead of considering one spherical pair (G,H) and one representation π, we need to consider all the pure inner forms of (G,H) and all the representations in a local/global L-packet.

Although the local and global conjectures of Sakellaridis and Venkatesh are formulated for the general case, they have two requirements for the spherical variety: the spherical variety needs to be wavefront and does not have Type N spherical root (see Section 2 of [SV17] for details). The most famous example for non-wavefront spherical variety (resp. spherical variety with Type N spherical root) is $(\mathrm{SO}_{2n+1},\mathrm{GL}_n)$ (resp. $(\mathrm{GL}_n,\mathrm{SO}_n)$). For those cases, one does not expect the period integrals to have an Euler product decomposition into local relative characters.

For the seven models we considered in this paper, there are two models ($(\mathrm{SO}_{2n+1},\mathrm{SO}_{n+1}\times\mathrm{SO}_n)$ and $(\mathrm{SO}_{2n},\mathrm{SO}_{n+1}\times\mathrm{SO}_{n-1})$) that have Type N spherical root, and one model ($(\mathrm{Sp}_4,\mathrm{Sp}_2\times\mathrm{GL}_1)$, which is just $(\mathrm{SO}_5,\mathrm{GL}_2)$) that is not wavefront. Our results for these three models show that for spherical varieties with Type N root or spherical varieties that are not wavefront, although the period integrals may not have an Euler product decomposition, they can still be related to some special values of L-functions.

Globally, for the four models in Table 1 that are wavefront and do not have Type N spherical root, guided by the global conjecture of Sakellaridis and Venkatesh, we expect the opposite direction of Corollary 1.3 to be true once we include all the pure inner forms of the model and all the cuspidal automorphic representations in a single global L-packet.

Locally, for the two models in Table 2 that are wavefront and do not have Type N spherical root, guided by the local conjecture of Sakellaridis and Venkatesh, if we consider all the pure inner forms of the model and all the representations in a local Vogan L-packet, we expect that a local tempered Vogan L-packet of G contains an H-distinguished element if and only if its Arthur parameter factors through the dual group of the spherical variety $X = H\backslash G$ (see Section 3.1 for the dual groups of these spherical varieties). The condition on Arthur parameter implies the condition of the local L-functions in Theorem 1.4. Moreover, if the L-packet is discrete, then the condition on Arthur parameter is equivalent to the condition of local L-function (assuming that the L-function defined by the Langlands-Shahidi method is the same as the L-function defined by the local Langlands correspondence).

1.6. **Organization of the paper.** The paper is organized as follows. In Section 2, we set up notation relating to Eisenstein series and truncation operators. Then in Section 3, we explain the strategy of the proofs of the main theorems (i.e. the residue method). In Sections 4–10, we prove the main theorems for all seven spherical varieties.

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2. Eisenstein series and the truncation operators

2.1. **General notation.** Let k be a number field and $|k|$ be the set of places of k. Let G be a connected reductive algebraic group over k. We fix a maximal k-split torus A_0 of G. Let P_0 be a minimal parabolic subgroup of G defined over k containing A_0, M_0 be the Levi part of P_0 containing A_0 and N_0 be the unipotent radical of P_0. Let $\mathcal{F}(P_0) = \mathcal{F}^G(P_0)$ be the set of parabolic subgroups of G containing P_0. Elements in $\mathcal{F}(P_0)$ are called standard parabolic subgroups of G. We also use $\mathcal{F}(M_0) = \mathcal{F}(A_0)$ (resp. $\mathcal{L}(M_0)$) to denote the set of parabolic subgroups (resp. Levi subgroups) of G containing A_0; these are the semi-standard parabolic subgroups (resp. Levi subgroups). In particular, they are finite sets.

For $P \in \mathcal{F}(M_0)$, we have the Levi decomposition $P = MN$ with N be the unipotent radical of P and M be the Levi subgroup containing A_0. We use $A_P \subset A_0$ to denote the maximal k-split torus of the center of M. Put

$$\mathfrak{a}_0^* = X(A_0) \otimes_{\mathbb{Z}} \mathbb{R} = X(M_0) \otimes_{\mathbb{Z}} \mathbb{R}$$

and let $\mathfrak{a}_0$ be its dual vector space. Here $X(H)$, for any k-group H, denotes the group of rational characters of H. Similarly we can also define $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$. The inclusions $A_P \subset A_0$ and $M_0 \subset M$ identify $\mathfrak{a}_P$ as a direct factor of $\mathfrak{a}_0$, we use $\mathfrak{a}_0^P$ to denote its complement. Similarly, $\mathfrak{a}_P^* = X(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$ is a direct factor of $\mathfrak{a}_0^*$ and we use $\mathfrak{a}_0^{P,*}$ to denote its complement. Let $\Delta_P \subset \mathfrak{a}_P^*$ be the set of simple roots for the action of A_P on N and we use Δ_0 to denote Δ_{P_0}. Similarly, for $P \subset Q$, we can also define the subset $\Delta_P^Q \subset \Delta_P$. We also define the positive chamber

$$\mathfrak{a}_P^+ = \{H \in \mathfrak{a}_P |\ \langle H, \alpha \rangle\ > 0,\ \forall \alpha \in \Delta_P\}.$$

Let $\widehat{\Delta}_0 \subset \mathfrak{a}_0^{G,*}$ be the set of fundamental weights. By the theory of root systems, we get a natural bijection between Δ_0 and $\widehat{\Delta}_0$ which we denote by $\alpha \mapsto \varpi_\alpha$. Let $\widehat{\Delta}_P \subset \widehat{\Delta}_0$ be the set corresponding to $\Delta_0 \smallsetminus \Delta_0^P$.

For any subgroup $H \subset G$, let $H(\mathbb{A})^1$ denote the common kernel of all characters on $H(\mathbb{A})$ of the form $|\chi(\cdot)|_{\mathbb{A}}$ where $\chi \in X(H)$ and $|\cdot|_{\mathbb{A}}$ is the absolute value on the ideles of $\mathbb{A}$. Fix K a maximal compact subgroup of $G(\mathbb{A})$ adapted to M_0. We define the Harish-Chandra map $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ via the relation

$$\langle \chi, H_P(x)\rangle = |\chi(x)|_{\mathbb{A}}, \quad \forall \chi \in X(P) = \mathrm{Hom}(P, \mathbb{G}_m)$$

where $x = pk$ is the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$. Let A_P^∞ be the connected component of the identity of $\mathrm{Res}_{k/\mathbb{Q}} A_P(\mathbb{R})$. Then $M(\mathbb{A})^1$ is the kernel of H_P restricted to $M(\mathbb{A})$ and we have the direct product decomposition of commuting subgroups $M(\mathbb{A}) = A_P^\infty M(\mathbb{A})^1$.

For any group H we use $[H]$ to denote $H(k)\backslash H(\mathbb{A})$ and $[H]^1$ to denote $H(k)\backslash H(\mathbb{A})^1$.

2.2. **Haar measures.** We fix compatible Haar measures on $G(\mathbb{A})$, $G(\mathbb{A})^1$ and A_G^∞. For all unipotent subgroups N of G, we fix a Haar measure on $N(\mathbb{A})$ so that $[N]$ is of volume one. On K we also fix a Haar measure of volume 1. For any $P = MN \in \mathcal{F}(A_0)$, let $\rho_P \in \mathfrak{a}_P^*$ be the half sum of the weights of the action of A_P on N. We choose compatible Haar measures on A_P^∞ and $M_P(\mathbb{A})^1$ such that

$$\int_{P(k)\backslash G(\mathbb{A})} f(h)\, dh = \int_K \int_{[M]^1} \int_{A_P^\infty} \int_{[N]} e^{\langle -2\rho_P, H_P(a)\rangle} f(uamk)\, du\, da\, dm\, dk$$

for $f \in C_c^\infty(P(k)\backslash G(\mathbb{A}))$.

2.3. **The computation of ρ_P when P is maximal.** Let $P \in \mathcal{F}(P_0)$ be the maximal parabolic subgroup that corresponds to the simple root α, i.e. $\{\alpha\} = \Delta_0 \smallsetminus \Delta_0^P$. Let ϖ_α be the corresponding fundamental weight. We have $\rho_P \in \mathfrak{a}_P^{G,*}$. Since P is maximal, $\mathfrak{a}_P^{G,*}$ is one dimensional. Hence there exists a constant $c \in \mathbb{R}$ such that $\rho_P = c\varpi_\alpha$. In the following proposition, we write down the constant c in six cases. It will be used in later sections. The computation is easy and standard, and hence we will skip it.

Proposition 2.1. *(1) If $G = \mathrm{SO}_n$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{SO}_{n-2} \times \mathrm{GL}_1$, then $c = \frac{n-2}{2}$.*

(2) If $G = \mathrm{Sp}_{2n}$ and P is the Siegel parabolic subgroup, then $c = \frac{n+1}{2}$.

(3) If $G = \mathrm{Sp}_6$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{Sp}_4 \times \mathrm{GL}_1$, then $c = 3$.

(4) If $G = \mathrm{U}_n$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{U}_{n-2} \times \mathrm{GL}_1$, then $c = \frac{n-1}{2}$.

(5) If $G = \mathrm{SO}_{10}$ and P is the parabolic subgroup whose Levi part is isomorphic to $\mathrm{SO}_6 \times \mathrm{GL}_2$, then $c = \frac{7}{2}$.

(6) If G is the simply-connected group of type E_7 and P is the parabolic subgroup whose Levi part is of type E_6, then $c = 9$.

2.4. **Eisenstein series.** Let $P = MN$ be a standard parabolic subgroup of G. Given a cuspidal automorphic representation π of $M(\mathbb{A})$, let $\mathcal{A}_\pi = \mathcal{A}_\pi(P)$ be the space of automorphic forms ϕ on $N(\mathbb{A})M(k)\backslash G(\mathbb{A})$ such that $M(\mathbb{A})^1 \ni m \mapsto \phi(mg) \in L^2_\pi([M]^1)$ for any $g \in G(\mathbb{A})$, where $L^2_\pi([M]^1)$ is the π-isotypic part of the L^2-space $L^2([M]^1)$, and such that

$$\phi(ag) = e^{\langle \rho_P, H_P(a)\rangle}\phi(g), \quad \forall g \in G(\mathbb{A}),\ a \in A_P^\infty.$$

Suppose that P is a maximal parabolic subgroup. Let $\varpi \in \widehat{\Delta}_P$ be the corresponding weight. We then define

$$E(g, \phi, s) = \sum_{\delta \in P(k)\backslash G(k)} \phi(\delta g) e^{\langle s\varpi, H_P(\delta g)\rangle}, \quad s \in \mathbb{C},\ g \in G(\mathbb{A}).$$

The series converges absolutely for $s \gg 0$ and admits a meromorphic continuation to all $s \in \mathbb{C}$ ([L], [MW95], [BL]).

Suppose moreover that M is stable for the conjugation by the longest element, denoted here w, in the Weyl group of G. Such M is called self-dual. We have in this case the intertwining operator $M(s) : \mathcal{A}_\pi \to \mathcal{A}_{\pi^w}$, where π^w is the twist of π by w, that satisfies $E(M(s)\phi, -s) = E(\phi, s)$ and

$$E(g, \phi, s)_P = \phi(g)e^{\langle s\varpi, H_P(g)\rangle} + e^{\langle -s\varpi, H_P(g)\rangle} M(s)\phi(g), \quad g \in G(\mathbb{A})$$

where $E(\cdot, \phi, s)_P$ is the constant term of $E(\cdot, \phi, s)$ along P

$$E(g, \phi, s)_P := \int_{[N]} E(ug, \phi, s)\, du.$$

When the Eisenstein series $E(g, \phi, s)$ has a pole at $s = s_0$, the intertwining operator also has a pole at $s = s_0$, we use $Res_{s=s_0} E(g, \phi, s)$ (resp. $Res_{s=s_0} M(s)$) to denote the residue of the Eisenstein series (resp. intertwining operator). By IV.1.11 of [MW95], the poles of the Eisenstein series $E(g, \phi, s)$ are simple when the real part $\mathrm{Re}(s)$ of s is positive and their residues are square integrable automorphic forms. Also the Eisenstein series, their derivatives and residues are of moderate growth (cf. [MW95]).

2.5. **The relative truncation operator and the regularized period integral.** For later applications, we recall the relative truncation operator which was recently introduced by the third author in [Zyd19]. Let $H \subset G$ be a closed connected reductive subgroup, and let $P = MN \subset G$ still be a maximal parabolic subgroup. With the same notation as in Section 2.3, let ϖ_P be the corresponding weight and $c \in \mathbb{R}$ be the constant such that $\rho_P = c\varpi_P$. Fix a maximal split torus $A_{0,H}$ (resp. A_0) of H (resp. G) such that $A_{0,H} \subset A_0$. Then $\mathfrak{a}_H := \mathfrak{a}_{0,H}$ is a subspace of $\mathfrak{a}_0$. For simplicity, we assume that G has trivial split center (i.e. $A_G = \{1\}$).

Remark 2.2. *In* [Zyd19]*, the author defined the relative truncation operator for general automorphic functions and also for a general pair (G, H) with H reductive (H does not need to be a spherical subgroup). But for our applications in this paper, we only consider the case when the automorphic function is a cuspidal Eisenstein series induced from a maximal parabolic subgroup.*

We fix a minimal subgroup $P_{0,H}$ of H with $A_{0,H} \subset P_{0,H}$. This allows us to define the set of standard (resp. semi-standard) parabolic subgroups of H. We will use $\mathcal{F}_H(P_{0,H})$ (resp. $\mathcal{F}_H(A_{0,H})$) to denote this set. We can also define the chamber $\mathfrak{a}_H^+ = \mathfrak{a}_{P_{0,H}}^+$ of $\mathfrak{a}_H$. Let $\bar{\mathfrak{a}}_H^+$ be the closure of $\mathfrak{a}_H^+$.

Definition 2.3. *We use $\mathcal{F}^G(P_{0,H}, P)$ to denote the set of semi-standard parabolic subgroups $Q \in \mathcal{F}(A_0)$ of G that satisfy the following two conditions.*

(1) Q is a conjugate of P.
(2) $\mathfrak{a}_Q^+ \cap \bar{\mathfrak{a}}_H^+ \neq \emptyset$.

For any $Q \in \mathcal{F}(A_0)$, we denote $Q = LU$ its Levi decomposition with L the Levi component containing M_0. The next proposition was proved in Proposition 3.1 of [Zyd19].

Proposition 2.4. *Let Q be a semi-standard parabolic subgroup of G that is conjugate to P. Then the following statements hold.*

(1) If $Q \in \mathcal{F}^G(P_{0,H}, P)$, then $Q_H = Q \cap H$ is a standard parabolic subgroup of H with the Levi decomposition $Q_H = L_H U_H$ such that $L_H = L \cap H$ and $U_H = U \cap H$.
(2) Let A_L be the maximal split torus of the center of L. If $A_L \subset A_{0,H}$ (this is always the case when $A_0 = A_{0,H}$), then the converse of (1) holds. In other words, if $Q_H = Q \cap H$ is a standard parabolic subgroup of H with the Levi decomposition $Q_H = L_H U_H$ such that $L_H = L \cap H$ and $U_H = U \cap H$, then $Q \in \mathcal{F}^G(P_{0,H}, P)$.

For $Q \in \mathcal{F}^G(P_{0,H}, P)$, we let $Q_H = L_H U_H = Q \cap H$ as in Proposition 2.4 above. We also let $A_L \subset A_0$ to be the maximal split torus of the center of L, and let ϖ_Q be the weight corresponding to Q. By the definition of the constant c, we have

$$\rho_Q = c\varpi_Q.$$

Since $\mathfrak{a}_Q^+ \cap \bar{\mathfrak{a}}_H^+ \neq \emptyset$ and $\mathfrak{a}_Q$ is one-dimensional, we have $\mathfrak{a}_Q \subset \mathfrak{a}_H$. We can therefore restrict ρ_{Q_H} to $\mathfrak{a}_Q$ and we define the real number c_Q^H to satisfy

$$\rho_{Q_H}|_{\mathfrak{a}_Q} = c_Q^H \rho_Q.$$

Remark 2.5. *In the case when U is abelian, let $n_Q = \dim(U)$ and $n_{Q,H} = \dim(U_H)$. Then*

$$c_Q^H = \frac{n_{Q,H}}{n_Q}.$$

We fix a cuspidal automorphic representation π of $M(\mathbb{A})$ and let $E(g, \phi, s)$ be the Eisenstein series defined in the previous section. We assume that M is self-dual. For $Q \in \mathcal{F}^G(P_{0,H}, P)$, let $W(P, Q)$ be the two element set of isometries between $\mathfrak{a}_P$ and $\mathfrak{a}_Q$. For $w \in W(P, Q)$ let $sgn(w) \in \{-1, 1\}$ be such that $w\varpi_P = sgn(w)\varpi_Q$. We have then

$$E(g, \phi, s)_Q = \sum_{w \in W(P,Q)} M(w, s)\phi(g)e^{\langle sgn(w)s\varpi_Q, H_Q(g)\rangle}$$

for some explicit intertwining operators $M(w, s)$. When $sgn(w) = 1$, $M(w, s)$ is just the isomorphism given by the w-conjugation.

In [Zyd19], the author defined a relative truncation operator, denoted by $\Lambda^{T,H}$, on the space $\mathcal{A}(G)$ of autormorphic forms on G, where $T \in \mathfrak{a}_H$. It depends on a choice of a good maximal compact K_H of $H(\mathbb{A})$ with respect to the maximal split torus $A_{0,H}$ of H. We fix such K_H. For all $\varphi \in \mathcal{A}(G)$, the truncation $\Lambda^{T,H}\varphi$ is a rapidly decreasing function on $[H]$. The following is a consequence of Theorem 4.1 of [Zyd19] and the discussion in Paragraph 4.7 of loc. cit.

Theorem 2.6. *(1) For all $\phi \in \mathcal{A}_\pi$ and $T \in \mathfrak{a}_H^+$ sufficiently regular, the integral*

$$\int_{[H]} \Lambda^{T,H} E(h, \phi, s)dh$$

is absolutely convergent for all $s \in \mathbb{C}$ in the domain of holomorphy of the Eisenstein series $E(\phi, s)$. Moreover, it defines a meromorphic function on $\mathbb{C}$.

(2) Define the regularized period for $E(\phi, s)$ to be

$$\mathcal{P}_{H,reg}(E(\phi, s)) := \int_{[H]} \Lambda^{T,H} E(h, \phi, s)dh - \sum_{Q \in \mathcal{F}^G(P_{0,H},P)} \sum_{w \in W(P,Q)} \frac{e^{\langle (sgn(w)s + c(1-2c_Q^H))\varpi_Q, T\rangle}}{sgn(w)s + c(1 - 2c_Q^H)} \int_{K_H} \int_{[L_H]^1} M(w, s)\phi(mk)dm\, dk.$$

The integrals defining $\mathcal{P}_{H,reg}(E(\phi, s))$ are absolutely convergent. The functional $\mathcal{P}_{H,reg}(\cdot)$ is independent of the choice of parameter T and it is $H(\mathbb{A})$-invariant.

Let us comment on the second part of Theorem 2.6 above. For $Q \in \mathcal{F}^G(P_{0,H}, P)$ and $w \in W(P, Q)$, the corresponding term in the definition of $\mathcal{P}_{H,reg}(E(\phi, s))$ can be written as the integral (or meromorphic continuation of it):

$$\int_{(Q\cap H)(k)\backslash H(\mathbb{A})} [H_Q(h) > T]e^{\langle sw\varpi_Q, H_P(h)\rangle} M(w, s)\phi(h)\, dh$$

where $[H_Q(h) > T]$ equals 1 if the condition $H_{Q\cap H}(h) > T$ is met and zero otherwise. Using the Iwasawa decomposition in the integral above and performing the integral over the torus $A_{Q\cap H}^\infty$ gives the term in the Theorem. This in particular explains the appearance of the fraction

$$\frac{e^{\langle (sgn(w)s + c(1-2c_Q^H))\varpi_Q, T\rangle}}{sgn(w)s + c(1 - 2c_Q^H)}.$$

This fraction is important in our work as the denominator of the fraction will be used to infer the poles of the Eisenstein series $E(\phi, s)$. Finally, the H-invariance of $\mathcal{P}_{H,reg}(E(\phi, s))$ follows from the first part of the theorem and the same argument as in Theorem 9 of [JLR99].

2.6. Some nonvanishing results on automorphic L-functions.

Proposition 2.7. *(1) Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Then the standard L-function $L(s, \pi)$ is holomorphic nonzero when $Re(s) > 1$.*

(2) Let π be a generic cuspidal automorphic representation of $\mathrm{SO}_n(\mathbb{A})$, $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\mathrm{U}_n(\mathbb{A})$. Then the standard L-function $L(s, \pi)$ is holomorphic nonzero when $Re(s) > 1$.

(3) Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Then the exterior square L-function $L(s, \pi, \wedge^2)$ is holomorphic nonzero when $Re(s) > 1$.

Proof. By Theorem 5.3 of [JS81], the partial L-function $L^S(s, \tau \times \sigma)$ is holomorphic nonzero for $Re(s) > 1$ for any unitary cuspidal automorphic representation τ (resp. σ) of $\mathrm{GL}_{m_1}(\mathbb{A})$ (resp. $\mathrm{GL}_{m_2}(\mathbb{A})$). By the results in [JPSS] (p-adic case) and [Jac] (archimedean case), we know that the local tensor product L-function of generic unitary representations is holomorphic nonzero for $Re(s) > 1$. Hence the complete L-function $L(s, \tau \times \sigma)$ is holomorphic nonzero for $Re(s) > 1$ for any unitary cuspidal automorphic representation τ (resp. σ) of $\mathrm{GL}_{m_1}(\mathbb{A})$ (resp. $\mathrm{GL}_{m_2}(\mathbb{A})$). This proves (1) by taking $m_1 = n, m_2 = 1, \tau = \pi$ and $\sigma = 1$. (2) is a direct consequence of (1) together with the results in [CKPSS04] (for orthogonal and symplectic group) and [KK05] (for unitary group).

For (3), by taking $\sigma = \tau = \pi$, we know that the L-function $L(s, \pi \times \pi) = L(s, \pi, Sym^2)L(s, \pi, \wedge^2)$ is holomorphic nonzero for $Re(s) > 1$. Hence it is enough to show that the L-functions $L(s, \pi, Sym^2)$ and $L(s, \pi, \wedge^2)$ are holomorphic when $Re(s) > 1$. This follows from Corollary 5.8 of [Tak14] (see also [BG]), Theorem 3.1(3) of [Kim00], and Lemma 5.2 of [FL17]. □

2.7. A criterion on closed orbit.

Let F be a local field, G be a linear algebraic group defined over F and H, P be two closed subgroups of G.

Proposition 2.8. *Suppose that the immersion*

$$H/(H \cap P) \hookrightarrow G/P$$

is closed. Then $H(F)P(F)$ is closed in $G(F)$.

Proof. Let $P_H = H \cap P$, and $H \times^{P_H} P$ be the quotient on the right of $H \times P$ by the diagonally embedded subgroup P_H. Let $p : H \times^{P_H} P \to G$ be the morphism induced by the map $H \times P \to G$ given by $(h, p) \mapsto hp^{-1}$. We have the following cartesian diagram

$$\begin{array}{ccc} H \times^{P_H} P & \xrightarrow{p} & G \\ \downarrow & & \downarrow \\ H/P_H & \longrightarrow & G/P \end{array}$$

which shows that $H \times^{P_H} P$ is isomorphic to the fiber product $(H/P_H) \times_{G/P} G$. By assumption, $H/P_H \hookrightarrow G/P$ is a closed immersion. By the property of pullbacks, $p : H \times^{P_H} P \to G$ is also a closed immersion. Hence $(H \times^{P_H} P)(F)$ is closed in $G(F)$.

The set $H(F)P(F)$ is identified with the subset $(H(F) \times P(F))/P_H(F)$ of $(H \times^{P_H} P)(F)$. The inclusion $(H(F) \times P(F))/P_H(F) \hookrightarrow (H \times^{P_H} P)(F)$ is closed by Corollary A.1.6 of [AG09]. This proves the proposition. □

Corollary 2.9. *Suppose that $H \cap P$ is a parabolic subgroup of H. Then $H(F)P(F)$ is closed in $G(F)$.*

Proof. The variety $H/(H \cap P)$ is projective hence the assumptions of Proposition 2.8 are satisfied. □

2.8. **The set $\mathcal{F}^G(P_{0,H}, P)$.** In this subsection we prove a few simple facts about the set $\mathcal{F}^G(P_{0,H}, P)$. This will simplify our computations in the next sections.

Recall the notation: $H \subset G$, $P_{0,H}$ is a minimal parabolic subgroup of H, and $P \subset G$ is a parabolic subgroup. We have $A_{0,H} \subseteq M_{0,H} \subseteq P_{0,H}$ denotes a maximal split torus inside a Levi subgroup of $P_{0,H}$. $A_{0,G}$ is a maximal split torus of G with $A_{0,H} \subseteq A_{0,G}$.

Moreover, $\mathcal{F}^G(P_{0,H}, P)$ denotes the set of semistandard parabolic subgroups Q of G, conjugate to P, that satisfy the condition $\mathfrak{a}_Q^+ \cap \mathfrak{a}_H^+ \neq \emptyset$ of Definition 2.3. Now we assume that the group G has trivial connected center and P is a maximal parabolic subgroup of G. We use $\mathcal{F}^G(P_{0,H}, P)'$ to denote the set of semistandard (i.e., containing $A_{0,G}$) parabolic subgroups Q of G, conjugate to P, for which $Q \cap H$ contains $P_{0,H}$. By Proposition 2.4, we know that $\mathcal{F}^G(P_{0,H}, P) \subset \mathcal{F}^G(P_{0,H}, P)'$.

We introduce an auxiliary set $\mathcal{F}^G_{geom}(P_{0,H}, P)$, similar to the set $\mathcal{F}^G(P_{0,H}, P)'$ already defined. Specifically, we denote by $\mathcal{F}^G_{geom}(P_{0,H}, P)$ the parabolic subgroups Q of G, conjugate to P, whose intersection $Q \cap H$ with H contains $P_{0,H}$. Note that in defining $\mathcal{F}^G_{geom}(P_{0,H}, P)$ we have dropped the condition that Q be semistandard. The next lemma follows from the definitions of $\mathcal{F}^G(P_{0,H}, P)'$ and $\mathcal{F}^G_{geom}(P_{0,H}, P)$.

Lemma 2.10. *The identity map $Q \mapsto Q$ defines an injection $\mathcal{F}^G(P_{0,H}, P)' \hookrightarrow \mathcal{F}^G_{geom}(P_{0,H}, P)$. The image is identified with those parabolics $Q \in \mathcal{F}^G_{geom}(P_{0,H}, P)$ that contain (equivalently, are normalized by) the maximal split torus $A_{0,G}$ of G. If G and H have the same split rank (i.e. $A_{0,H} = A_{0,G}$), this injection is a bijection.*

We will relate the sets $\mathcal{F}^G(P_{0,H}, P)'$ and $\mathcal{F}^G_{geom}(P_{0,H}, P)$ to H-orbits on the flag variety $P\backslash G$. This is done in the following lemma. If Q is a parabolic subgroup of G conjugate to P, then we write $[Q]$ for the corresponding element of $P\backslash G$. We write $[P\backslash G/H]^{cl}$ for the closed H-orbits on this flag variety $P\backslash G$. Finally, the maximal split torus $A_{0,G}$ acts on $P\backslash G$ on the right, and we set $[P\backslash G/H]^{cl,A}$ for the subset of $[P\backslash G/H]^{cl}$ consisting of orbits that contain a fixed point for $A_{0,G}$.

Lemma 2.11. *Let the notation be as above. The map*

$$Q \in \mathcal{F}^G_{geom}(P_{0,H}, P) \mapsto [Q]H \subseteq P\backslash G$$

induces a bijection between $\mathcal{F}^G_{geom}(P_{0,H}, P)$ and $[P\backslash G/H]^{cl}$. It also induces a bijection between $\mathcal{F}^G(P_{0,H}, P)'$ and $[P\backslash G/H]^{cl,A}$.

Proof. First, if $Q \in \mathcal{F}^G_{geom}(P_{0,H}, P)$, then $Q \cap H \supseteq P_{0,H}$ so $Q \cap H$ is a parabolic subgroup of H. Thus, $[Q]H = (Q \cap H)\backslash H \subseteq P\backslash G$ is projective and hence closed in $P\backslash G$. So the map does indeed take values in $[P\backslash G/H]^{cl}$.

For the injectivity, suppose Q_1, Q_2 are in $\mathcal{F}^G_{geom}(P_{0,H}, P)$ and define the same H-orbit, i.e., $hQ_1h^{-1} = Q_2$ for some $h \in H$. Then, because both Q_1 and Q_2 contain $P_{0,H}$ (by assumption), $Q_1 \cap H$ and $Q_2 \cap H$ are standard parabolic subgroups of H. Because $hQ_1h^{-1} = Q_2$, these parabolics are conjugate and thus equal. Therefore, h normalizes $Q_1 \cap H$, so $h \in Q_1 \cap H \subseteq Q_1$. Consequently, $Q_1 = Q_2$ as desired.

For the surjectivity, suppose that $[Q]H \subseteq P\backslash G$ is closed. As $P\backslash G$ is projective, $[Q]H = (Q \cap H)\backslash H$ is projective, so $Q \cap H$ is a parabolic subgroup of H. Consequently, there is $h \in H$ so that $Q' := hQh^{-1} \supseteq P_{0,H}$. Then $Q' \in \mathcal{F}^G_{geom}(P_{0,H}, P)$ and $[Q']H = [Q]H$, giving the first part of the lemma.

For the second part of the lemma, note that the action of $A_{0,G}$ on $P\backslash G$ is conjugation of a parabolic Q by $A_{0,G}$. Thus $[Q]$ is a fixed point for this action precisely when $A_{0,G} \subseteq N_G(Q) = Q$, i.e., when Q is semistandard. □

In all the cases of periods in this paper, the group G comes together with a finite dimensional algebraic representation V. Moreover, with the exception of the Jacquet-Guo model (Section 7), the subgroup $H \subseteq G$ stabilizes a decomposition $V = V_0 \oplus V_1$ of V, and the following assumption holds:

Assumption A: The set of isomorphism classes of irreducible representations of $M_{0,H}$ appearing in its action on V_0 are disjoint from those appearing in its action on V_1.

We will usually apply this assumption in the case that $M_{0,H} = A_{0,H}$, so that the irreducible representations of $M_{0,H}$ are just characters of $A_{0,H}$. However, for working with nonsplit groups, it is useful to write things in terms of the larger group $M_{0,H}$. Applying the above assumption gives the following lemma.

Lemma 2.12. *Suppose that $Q \in \mathcal{F}^G_{geom}(P_{0,H}, P)$, and Q stabilizes a subspace V_Q of V. Then $V_Q = (V_Q \cap V_0) \oplus (V_Q \cap V_1)$.*

Proof. Indeed, because $Q \in \mathcal{F}^G_{geom}(P_{0,H}, P)$, $M_{0,H} \subseteq Q$ so $M_{0,H}$ stabilizes V_Q. The lemma follows. □

For a parabolic subgroup Q of G conjugate to P, denote by V_Q the subspace of V on which the unipotent radical of Q acts as the identity. Combining the above two lemmas, we arrive at the following proposition.

Proposition 2.13. *Let the notation be as above. The set $\mathcal{F}^G_{geom}(P_{0,H}, P)$ is in bijection with the H-orbits $V_Q H$ that are closed. Moreover, every such orbit has a representative V_Q^A satisfying $V_Q^A = (V_Q^A \cap V_0) \oplus (V_Q^A \cap V_1)$. In particular, if the subspaces V_Q are one-dimensional, then every closed H-orbit $V_Q H$ has a representative V_Q^A with either $V_Q^A \subseteq V_0$ or $V_Q^A \subseteq V_1$.*

3. The strategy of the proof

3.1. **The genesis of the cases considered.** Let (G, H) be reductive groups with $H \subset G$ as before. Assume $X = H\backslash G$ is spherical. Suppose there exists another such pair $(\underline{G}, \underline{H})$ with $\underline{X} = \underline{H}\backslash\underline{G}$ spherical satisfying the following conditions

1. G is isomorphic to the Levi component $\underline{M}$ of a maximal parabolic subgroup $\underline{P} = \underline{MU}$ of $\underline{G}$ (up to the center and some finite isogeny).
2. $\underline{H} \cap \underline{P} = (\underline{H} \cap \underline{M}) \ltimes (\underline{H} \cap \underline{U})$ is a maximal parabolic subgroup of $\underline{H}$ such that $\underline{H} \cap \underline{M}$ is isomorphic to the group H (up to the center).

Such a spherical variety does not exist in general, but if it exists, we can try to use it to prove a relation between the period integral $\mathcal{P}_H$ and $\mathcal{P}_{\underline{H}}$ as in Theorem 1.1. The cases listed in Table 1 satisfy these conditions. We can assume $\underline{G}$ has trivial split center as this is the only case we consider.

In [KS17] (also see [GN10]), Knop and Schalke have defined the dual group $G_X^\vee$ for every affine spherical variety $X = H\backslash G$ together with a natural morphism $\iota_X : G_X^\vee \to G^\vee$ from the dual group of the spherical variety to the dual group of G. Let $Cent_{G^\vee}(Im(\iota_X))$ be the centralizer of the image of the map ι_X in $G^\vee$. Following the notation in [KS17], we use $\mathfrak{l}_X^\wedge$ to denote the Lie algebra of $Cent_{G^\vee}(Im(\iota_X))$. We say the spherical variety X is tempered if $\mathfrak{l}_X^\wedge = 0$ (this is equivalent to say that the associated parabolic subgroup $P(X)$, which is the stabilizer in G of the open Borel orbit of X is the Borel subgroup).

For all the data (G, H) and $(\underline{G}, \underline{H})$ of Table 1 (as well as all the other presently known cases), the dual groups of the spherical varieties $X = H\backslash G$ and $\underline{X} = \underline{H}\backslash\underline{G}$ satisfy the following two conditions.

3. $G_X^\vee = G_{\underline{X}}^\vee$.
4. $\mathfrak{l}_X^\wedge = 0$, $\mathfrak{l}_{\underline{X}}^\wedge = \mathfrak{sl}_2$.

In general, we believe that for a given spherical pair (G,H), if we can find another spherical pair $(\underline{G},\underline{H})$ satisfying conditions (1),(3) and (4), then it should also satisfy the Conditions (2) (up to conjugating the parabolic subgroup $\underline{P}$).

Definition 3.1. *We say that spherical pairs (G,H) and $(\underline{G},\underline{H})$ are related if they satisfy the conditions (1)-(4) above.*

In this paper, we have considered all related spherical pairs (G,H) and $(\underline{G},\underline{H})$, with H and $\underline{H}$ reductive and $\underline{G}$ simple (except those pairs that have already been studied by other authors). To be specific, by Table 3 of [KS17], the following table contains all the cases with $\underline{G}$ simple.

	G	H	$\underline{G}$	$\underline{H}$	$G_X^\vee = G_{\underline{X}}^\vee$
1	GL_{2n}	$\mathrm{GL}_n \times \mathrm{GL}_n$	GL_{2n+2}	$\mathrm{GL}_{n+2} \times \mathrm{GL}_n$	$\mathrm{Sp}_{2n}(\mathbb{C})$
2	GL_{2n}	$\mathrm{GL}_n \times \mathrm{GL}_n$	Sp_{4n}	$\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$	$\mathrm{Sp}_{2n}(\mathbb{C})$
3	GL_{2n}	$\mathrm{GL}_n \times \mathrm{GL}_n$	SO_{4n}	GL_{2n}	$\mathrm{Sp}_{2n}(\mathbb{C})$
4	$\mathrm{GL}_n \times \mathrm{GL}_n$	GL_n	GL_{2n}	Sp_{2n}	$\mathrm{GL}_n(\mathbb{C})$
5	GL_{2n+1}	$\mathrm{GL}_{n+1} \times \mathrm{GL}_n$	SO_{4n+2}	GL_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
6	GL_2	$\mathrm{GL}_1 \times \mathrm{GL}_1$	G_2	SL_3	$\mathrm{GL}_2(\mathbb{C})$
7	$\mathrm{GL}_2 \times \mathrm{GL}_2$	GL_2	GSO_7	G_2	$\mathrm{GL}_2(\mathbb{C})$
8	$\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$	GL_2	GSO_8	G_2	$\mathrm{GL}_2(\mathbb{C})^3$
9	$\mathrm{GL}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	GSO_{10}	$\mathrm{GSpin}_7 \times \mathrm{GL}_1$	$\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$
10	SO_{2n+1}	$\mathrm{SO}_{n+1} \times \mathrm{SO}_n$	SO_{2n+3}	$\mathrm{SO}_{n+3} \times \mathrm{SO}_n$	$\mathrm{Sp}_{2n}(\mathbb{C})$
11	SO_{2n}	$\mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1}$	SO_{2n+2}	$\mathrm{SO}_{n+3} \times \mathrm{SO}_{n-1}$	$\mathrm{SO}_{2n-1}(\mathbb{C})$
12	Sp_4	$\mathrm{Sp}_2 \times \mathrm{GL}_1$	Sp_6	$\mathrm{Sp}_4 \times \mathrm{GL}_1$	$\mathrm{SO}_4(\mathbb{C})$
13	$\mathrm{SO}_5 \times \mathrm{SL}_2$	$\mathrm{SL}_2 \times \mathrm{SL}_2$	SO_9	Spin_7	$\mathrm{SL}_2(\mathbb{C})^2$
14	GE_6	$\mathrm{GL}_6 \times \mathrm{SL}_2$	E_7	$\mathrm{SO}_{12} \times \mathrm{SL}_2$	$F_4(\mathbb{C})$

The split version of model (1)-(3) is the linear model, which has been studied by Friedberg-Jacquet [FJ93] via different method. In this paper, we studied the quasi-split versions of this model (i.e. the models $(U_{2n}, U_n \times U_n)$ and $(\mathrm{GL}_{2n}, Res_{k'/k}\mathrm{GL}_n)$). The model (4) was studied by Jacquet-Rallis [JR92]. The model (5) is not interesting because an easy unfolding argument shows that the period integral is zero for all the cusp forms of GL_{2n+1}. The models (6)-(8) were studied by Jiang [Jia98]. The models (9)-(12) and (14) are studied in this paper. The model (13) was studied by Piatetski-Shapiro and Soudry [PSS] via different method.

3.2. **A local result of Prasad and Sakellaridis.** In this subsection, we recall a recent result of Prasad and Sakellaridis for the distinguished representations of spherical varieties. Let F be a p-adic field of characteristic 0. Let G be a quasi-split reductive group defined over F, and H be a closed reductive spherical subgroup of G defined over F. Let $H_1 = [H,H]$ be the derived group of H. The following result was proved by Prasad in [Pra18] for symmetric pairs (i.e. H is the group of fixed points of an involutaion of G). In the appendix of the same paper, Sakellaridis extended the result to all spherical varieties.

Theorem 3.2 (Prasad, Sakellaridis)**.** *If (G,H) is not tempered, then $\mathrm{Hom}_{H_1(F)}(\pi,1) = \{0\}$ for all irreducible generic representations π of $G(F)$. In other words, there is no $H_1(F)$-distinguished irreducible generic representation of $G(F)$.*

Corollary 3.3. *Suppose (G,H) is not tempered, and $P = MN$ is a maximal parabolic subgroup of G. Let π be an irreducible, generic representation of $M(F)$ and let $\Pi_s = I_{\underline{P}}^{\underline{G}}\pi_s$. Here $\pi_s = \pi \otimes \varpi_{\underline{P}}^s$, and $I_{\underline{P}}^{\underline{G}}(\cdot)$ is the normalized parabolic induction. Then*

$$\mathrm{Hom}_{\underline{H}(F)}(\Pi_s, 1) = \{0\}$$

for $Re(s) \gg 0$. In other words, Π_s is not $\underline{H}(F)$-distinguished.

Proof. This follows from the fact that Π_s is irreducible and generic for $Re(s) \gg 0$. □

By our definition of tempered spherical variety and Table 3 of [KS17], we have the following corollary.

Corollary 3.4. *The following spherical pairs are not tempered. In particular, there is no $H_1(F)$-distinguished irreducible generic representation of $G(F)$.*

(1) G is a connected reductive group that is not abelian, and $H = G$.
(2) $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_{n+1} \times \mathrm{GL}_{n-1}$.
(3) $G = \mathrm{SO}_{2n+3}$ the split odd orthogonal group, and $H = \mathrm{SO}_{n+k} \times \mathrm{SO}_{n+3-k}$ with $k \geq 3$.
(4) $G = \mathrm{SO}_{2n+2}$ a quasi-split even orthogonal group, and $H = \mathrm{SO}_{n+k} \times \mathrm{SO}_{n+2-k}$ with $k \geq 3$.
(5) $G = \mathrm{SO}_{2n}$ the split even orthogonal group ($n \geq 1$), and $H = \mathrm{GL}_n$ be the Levi subgroup of the Siegel parabolic subgroup of G.
(6) $G = \mathrm{GE}_6$ the similitude group of the split exceptional group of type E_6, and H be symmetric subgroup of G of type D_5.

3.3. **The proof of Theorem 1.1.** Let us go back to the global setting. Let (G, H) and $(\underline{G}, \underline{H})$ be a related pair. We will use results of Section 2 for $\underline{G}$ and $\underline{H}$ as well as for G and H. Let $\underline{P} = \underline{M}\underline{N}$ be a parabolic subgroup of $\underline{G}$ with $\underline{M}$ equal to G up to center and isogeny. Let π be a cuspidal automorphic representation of $\underline{M}$. Our starting point is the (regularized) period of Eisenstein series induced from π. It turns out that it is always zero.

Proposition 3.5. *Suppose (G, H) and $(\underline{G}, \underline{H})$ are related. Assume as well that π is generic at a finite place of k. Then, the regularized period $\mathcal{P}_{\underline{H},reg}(E(\phi, s))$ is identically zero for all $\phi \in \mathcal{A}_\pi(\underline{P})$.*

Proof. The Proposition follows from Corollary 3.3. Indeed, let v be the place of k at which π is generic and let π_v be the local component of π at v. In particular, we must have that $\underline{M}$ and $\underline{G}$ are quasi-split over k_v. By Theorem 2.6 and using the notation of Corollary 3.3 the period $\mathcal{P}_{\underline{H},reg}(E(\phi, s))$ defines an element of $\mathrm{Hom}_{\underline{H}(F)}(\Pi_{v,s}, 1)$. By Corollary 3.3, the latter space is zero and the result follows. □

In spite of the period $\mathcal{P}_{\underline{H},reg}(E(\phi, s))$ being trivially zero, part (2) of Theorem 2.6 gives the following non-trivial identity

$$\int_{[\underline{H}]} \Lambda^{T,\underline{H}} E(h, \phi, s) dh = \sum_{\underline{Q} \in \mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})} \sum_{w \in W(\underline{P}, \underline{Q})} \frac{e^{\langle (sgn(w)s + c(1 - 2c^{\underline{H}}_{\underline{Q}}))\varpi_{\underline{Q}}, T\rangle}}{sgn(w)s + c(1 - 2c^{\underline{H}}_{\underline{Q}})} \int_{K_{\underline{H}}} \int_{[\underline{L}_{\underline{H}}]^1} M(w, s)\phi(mk) dm\, dk. \tag{3.1}$$

We have the following Claim, for which we do not provide a theoretical explanation, rather we will verify it explicitly in each case of Table 1.

Claim 3.6. *For all $\underline{Q} = \underline{L}\underline{U} \in \mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ with $\underline{Q} \neq \underline{P}$ the spherical variety $\underline{H} \cap \underline{L} \backslash \underline{L}$ is not tempered.*

Assume Claim 3.6 holds. Under assumptions of Proposition 3.5, the equation (3.1) yields

$$\int_{[\underline{H}]} \Lambda^{T,\underline{H}} E(h, \phi, s) dh = \frac{e^{\langle (s + c(1 - 2c^{\underline{H}}_{\underline{P}}))\varpi_{\underline{P}}, T\rangle}}{s + c(1 - 2c^{\underline{H}}_{\underline{P}})} \int_{K_{\underline{H}}} \int_{[\underline{M}_{\underline{H}}]^1} \phi(hk) dh\, dk + \frac{e^{\langle (-s + c(1 - 2c^{\underline{H}}_{\underline{P}}))\varpi_{\underline{P}}, T\rangle}}{-s + c(1 - 2c^{\underline{H}}_{\underline{P}})} \int_{K_{\underline{H}}} \int_{[\underline{M}_{\underline{H}}]^1} M(s)\phi(hk) dh\, dk. \tag{3.2}$$

Indeed, all the terms with $Q \neq P$ are zero by Theorem 3.2. Let $s_0 = -c(1 - 2c^{H}_{\underline{P}})$ (in all the case we consider $s_0 > 0$). Taking the residue at $s = s_0$ in the above equation, we get

(3.3)
$$\int_{[\underline{H}]} \Lambda^{T,\underline{H}} Res_{s=s_0} E(h,\phi,s)dh = \int_{K_{\underline{H}}} \int_{[\underline{M}_{\underline{H}}]^1} \phi(hk)dh\, dk + \frac{e^{\langle -2s_0\varpi_{\underline{P}},T\rangle}}{-2s_0} \int_{K_{\underline{H}}} \int_{[\underline{M}_{\underline{H}}]^1} Res_{s=s_0} M(s)\phi(hk)dh\, dk.$$

By Theorem 4.1 of [Zyd19], the regularized period $\mathcal{P}_{\underline{H},reg}(Res_{s=s_0}E(h,\phi,s))$ is defined as the constant term in T of the polynomial exponential given by the right hand side of the above equation (3.3). This constant term is simply $\int_{K_{\underline{H}}} \int_{[\underline{M}_{\underline{H}}]^1} \phi(hk)dhdk$. Combining with Condition (2) of the model $(\underline{G}, \underline{H})$, we proved Theorem 1.1.

3.4. **Application to special values of L-functions.** We continue with the notation above. In particular, π is an automorphic representation of $\underline{M}$ which is generic at a finite place of k.

Proposition 3.7. *Assume Claim 3.6 holds. If the period integral $\mathcal{P}_{\underline{M}_{\underline{H}}}$ is nonzero on π, then the cuspidal Eisenstein series $E(\phi,s)$ and the intertwining operator $M(s)$ have a pole at $s = s_0$.*

Proof. If the period integral $\mathcal{P}_{\underline{M}_{\underline{H}}}$ is nonzero on π, by a similar argument as in Proposition 2 of [JR92], there exists a $\phi \in \mathcal{A}_\pi(\underline{P})$ such that
$$\int_{K_{\underline{H}}} \int_{[\underline{M}_{\underline{H}}]^1} \phi(hk)dh\, dk \neq 0.$$
By (3.3), we know that $Res_{s=s_0}E(h,\phi,s) \neq 0$ or $Res_{s=s_0}M(s) \neq 0$. Now the proposition follows from the fact that $Res_{s=s_0}E(h,\phi,s) \neq 0$ if and only if $Res_{s=s_0}M(s) \neq 0$. □

Remark 3.8. *When π is generic, by the Langlands-Shahidi method, the above proposition implies that if the period integral $\mathcal{P}_H(\phi)$ is nonzero for some $\phi \in \pi$, some Langlands-Shahidi L-function has a pole at $s = s_0$. This would imply a relation between the period integral $\mathcal{P}_H(\phi)$ and some automorphic L-functions.*

Remark 3.9. *A similar version of this method has been used by Ichino-Yamana* [IY] *in the unitary Gan-Gross-Prasad model case.*

3.5. **Convergence of the period of the residue.** We will address the question of whether $Res_{s=s_0}E(h,\phi,s)$ is integrable over $\underline{H}$ or not. Theorem 4.1 of [Zyd19] provides an explicit condition for convergence of periods of automorphic forms which is a generalization of the square-integrability criterion of [MW95], Lemma I.4.11. The condition reflects the basic fact that the growth of an automorphic form is dictated by its cuspidal exponents. In this particular case, and since $s_0 > 0$, it translates to the following simple criterion

$$s_0 > c(1 - 2c^{H}_{\underline{Q}}),\ \forall \underline{Q} \in \mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}). \tag{3.4}$$

In fact, one simply requires that when taking residue at $s = s_0$ in the equation (3.1), all the exponents $sgn(w)s + c(1 - 2c^{H}_{\underline{Q}}))$ become negative when $s = s_0$. We state this as a proposition.

Proposition 3.10. *The period $\mathcal{P}_{\underline{H}}(Res_{s=s_0}E(h,\phi,s))$ is absolutely convergent and equal to the regularized one $\mathcal{P}_{\underline{H},reg}(Res_{s=s_0}E(h,\phi,s))$ if and only if the condition* (3.4) *holds.*

We point out that the argument laid out in Lemma 9 of [L11] also proves the if part of the above Proposition.

In the concrete examples of Table 1, some cases satisfy (3.4) and some do not. Note that for $\underline{Q} = \underline{P}$ condition (3.4) is trivial since $s_0 = -c(1 - 2c^{H}_{\underline{P}})$ is positive (in fact, always equal to 1/2 or 1). In particular, if the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ only contains one element $\underline{P}$, (3.4) holds. It turns out that this condition is also a necessary condition in all cases we consider. The following claim will be verified in a case by case manner.

Claim 3.11. *The condition (3.4) fails for all $\underline{Q} \in \mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ different from $\underline{P}$. Moreover, the following are all cases when $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$ and $\mathcal{P}_{\underline{H}}(Res_{s=s_0}E(h, \phi, s))$ is absolutely convergent:*

- $\underline{G} = \mathrm{SO}_{2n+3}$, $\underline{H} = \mathrm{SO}_{n+3} \times \mathrm{SO}_n$ *and the* SO_n*-part of* $\underline{H}$ *is anisotropic, discussed in Section 4.*
- $\underline{G} = \mathrm{SO}_{2n+2}$, $\underline{H} = \mathrm{SO}_{n+3} \times \mathrm{SO}_{n-1}$ *and the* SO_{n-1}*-part of* $\underline{H}$ *is anisotropic, discussed in Section 5.*
- $\underline{G} = \mathrm{U}_{2n+2}$, $\underline{H} = \mathrm{U}_{n+2} \times \mathrm{U}_n$ *and* U_n*-part of* $\underline{H}$ *is anisotropic, discussed in Section 6.*
- $\underline{G} = \mathrm{Sp}_{4n}$ *and* $\underline{H} = Res_{k'/k}\mathrm{Sp}_{2n}$, *discussed in Section 7.*
- $\underline{G} = E_7^{sc}$ *the semisimple, simply-connected group of type* E_7, $\underline{H}$ *the symmetric subgroup of type* $D_6 \times A_1$, *and* $\underline{H}$ *not split. This will be discussed in Section 8.*
- $\underline{G} = \mathrm{GSO}_{10}$ *and* $\underline{H} = \mathrm{GSpin}_7 \times \mathrm{GL}_1$. *This will be discussed in Section 9.*

In particular, we provide many examples when regularization of periods of residual automorphic representations is necessary. This phenomenon (although not for residual representations associated with maximal parabolic subgroups) has already been observed for the model $(\mathrm{Sp}_{4n}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n})$ by Lapid and Offen in [LO18].

4. The model $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{n+1} \times \mathrm{SO}_n)$

4.1. **The result.** Let W_1 (resp. W_2) be a quadratic space defined over k of dimension $n+1$ (resp. n), and $W = W_1 \oplus W_2$. Let $G = \mathrm{SO}(W)$ and $H = \mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$. Let $D = Span\{v_{0,+}, v_{0,-}\}$ be a two-dimensional quadratic space with $< v_{0,+}, v_{0,+} >=< v_{0,-}, v_{0,-} >= 0$ and $< v_{0,+}, v_{0,-} >= 1$, $V_1 = W_1 \oplus D$, $V_2 = W_2$, $V = V_1 \oplus V_2$, $\underline{G} = \mathrm{SO}(V)$, and $\underline{H} = \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$.

Let $D_+ = Span\{v_{0,+}\}$ and $D_- = Span\{v_{0,-}\}$. Then $D = D_+ \oplus D_-$ as a vector space. Let $\underline{P} = \underline{MN}$ be the maximal parabolic subgroup of $\underline{G}$ that stabilizes the subspace D_+ where $\underline{M}$ is the Levi subgroup that stabilizes the subspaces D_+, W and D_-. Then $\underline{M} = \mathrm{SO}(W) \times \mathrm{GL}_1$.

Let $\pi = \otimes_{v \in |k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Then $\pi \otimes 1$ is a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$. To simplify the notation, we will still use π to denote this cuspidal automorphic representation. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. The goal of this section is to prove Theorem 1.1 for the current model.

Theorem 4.1. *Assume that there exists a local non-archimedean place $v \in |k|$ such that π_v is a generic representation of $G(k_v)$ (in particular, $G(k_v)$ is split). If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$.*

Theorem 4.1 will be proved in the last subsection of this section. The next proposition shows that Corollary 1.3(1) follows from Theorem 4.1.

Proposition 4.2. *Theorem 4.1 implies Corollary 1.3(1).*

Proof. We first recall the statement of Corollary 1.3(1). Let $G = \mathrm{SO}_{2n+1}$ be the split odd orthogonal group, $H = \mathrm{SO}_{n+1} \times \mathrm{SO}_n$ be a closed subgroup of G (not necessarily split), and π be a generic cuspidal automorphic representation of $G(\mathbb{A})$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, we need to show that the standard L-function $L(s, \pi)$ is nonzero at $s = 1/2$.

By Theorem 4.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$, and thus the intertwining operator $M(w, s)$, has a pole at $s = 1/2$. In this case, the intertwining operator is

$$M(w, s) = \frac{L(s, \pi)\zeta_k(2s)}{L(s+1, \pi)\zeta_k(2s+1)} N(w, s)$$

where $\zeta_k(s)$ is the Dedekind zeta function and $N(s, w)$ is the normalized intertwining operator. Note that the Dedekind zeta function shows up because we are inducing from the representation $\pi \otimes 1$

of $\underline{M}(\mathbb{A})$. By Theorem 4.7 of [KK11], the normalized intertwining operator $N(w,s)$ is holomorphic at $s=1/2$. By Proposition 2.7, $L(3/2,\pi)\neq 0$. It follows that the numerator $L(s,\pi)\zeta_k(2s)$ has a pole at $s=1/2$, which implies that $L(\frac{1}{2},\pi)\neq 0$ because $\zeta_k(s)$ has a simple pole at $s=1$. This proves Corollary 1.3(1). □

Remark 4.3. *For a generic representation π of SO_{2n+1}, the central value $L(1/2,\pi)$ is linked to the so-called Bessel periods by the Gan-Gross-Prasad conjecture* [GGP12] *and has been studied in* [GJR04a, GJR05]. *In* [JS07b, JS07a] *it is linked on the other hand to the first occurrence problem in theta correspondence.*

4.2. **The parabolic subgroups.** Let $P_{0,\underline{H}}=\underline{P}_{0,1}\times\underline{P}_{0,2}$ be a minimal parabolic subgroup of $\underline{H}=\mathrm{SO}(V_1)\times\mathrm{SO}(V_2)=\mathrm{SO}(V_1)\times\mathrm{SO}(W_2)$ with $P_{0,\underline{H}}\subset\underline{P}\cap\underline{H}$. We also fix minimal Levi subgroup $M_{0,\underline{H}}=\underline{M}_{0,1}\times\underline{M}_{0,2}$ with $M_{0,\underline{H}}\subset\underline{M}\cap\underline{H}$. The goal of this subsection is to study the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})$. When W_2 is not anisotropic, we let $D'_+\subset V_2$ be the unique isotropic line stabilized by $\underline{P}_{0,2}$. We also decompose V_2 as $D'_+\oplus D'_-\oplus V'_2$ accordingly so that D'_+,D'_-,V'_2 are fixed by $\underline{M}_{0,2}$.

Proposition 4.4. *If W_2 is anisotropic, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P}\}$. If W_2 is not anisotropic, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P},\underline{P}'\}$ where $\underline{P}'$ is the maximal standard parabolic subgroup of $\underline{G}$ that stabilizes the isotropic line D'_+ in W_2.*

Proof. First, it is clear from Definition 2.3 that $\underline{P},\underline{P}'\in\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})$. For the other direction, we apply Proposition 2.13 to the orthogonal decomposition $V=(W_1\oplus D)\oplus W_2$ and the spaces $V_{\underline{Q}}$ of that proposition being isotropic lines in V. For $\underline{Q}\in\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})$, let D_Q be the isotropic line defining Q. Then this isotropic line can be in either $W_1\oplus D$ or W_2, so long as W_2 has Witt rank at least one. But since $P_{0,\underline{H}}\subset\underline{Q}$, we must have $D_Q=D_+$ or D'_+. This proves the proposition. □

To end this subsection, we discuss the intersections $\underline{P}\cap\underline{H}$ and $\underline{P}'\cap\underline{H}$. Let $P_1=M_1N_1$ be the maximal parabolic subgroup of $\mathrm{SO}(V_1)$ that stabilizes the space D_+, and M_1 be the subgroup of $\mathrm{SO}(V_1)$ that stabilizes the subspaces D_+,D_- and W_1. Then $M_1\simeq\mathrm{SO}(W_1)\times\mathrm{GL}(D_+)$ and we have

$$\underline{P}\cap\underline{H}=P_1\times\mathrm{SO}(W_2),\ \underline{M}\cap\underline{H}=M_1\times\mathrm{SO}(W_2),\ \underline{N}\cap\underline{H}=N_1\times\{1\}.$$

For $\underline{P}'\cap\underline{H}$, assume that we are in the situation where the Witt rank of W_2 is at least one. Let $P_2=M_2N_2$ be the maximal parabolic subgroup of $\mathrm{SO}(V_2)=\mathrm{SO}(W_2)$ that stabilizes the isotropic line D'_+, and M_2 be the subgroup of $\mathrm{SO}(V_2)$ that stabilizes the subspaces D'_+,D'_- and W'_2. Then $M_2\simeq\mathrm{SO}(W'_2)\times\mathrm{GL}_1$ and we have

$$\underline{P}'\cap\underline{H}=\mathrm{SO}(V_1)\times P_2,\ \underline{M}'\cap\underline{H}=\mathrm{SO}(V_1)\times M_2,\ \underline{N}'\cap\underline{H}=\{1\}\times N_2.$$

4.3. **The proof of Theorem 4.1.** In this section, we will prove Theorem 4.1 by the method discussed in Section 3. We first compute the constant $s_0=-c(1-2c_{\underline{P}}^{\underline{H}})$ for the current case. By Remark 2.5, we have

$$c_{\underline{P}}^{\underline{H}}=\frac{\dim(N_1)}{\dim\underline{N}}=\frac{n+1}{2n+1}.$$

By Proposition 2.1, we have $c=\frac{2n+1}{2}$. This implies that

$$s_0=-c(1-2c_{\underline{P}}^{\underline{H}})=1/2.$$

It remains to show that Claim 3.6 holds. We only need to show that $(\underline{L},L_{\underline{H}})=(\underline{M}',\underline{M}'\cap\underline{H})$ is not tempered. By the discussion in the previous subsection, we have

$$(\underline{L},L_{\underline{H}})=(\underline{M}',\underline{M}'\cap\underline{H})=(\mathrm{SO}(2n+1)\times\mathrm{GL}_1,\mathrm{SO}_{n+3}\times\mathrm{SO}_{n-2}\times\mathrm{GL}_1).$$

By Corollary 3.4, $(\underline{L},L_{\underline{H}})$ is not tempered. Theorem 4.1 follows from Proposition 3.7.

Remark 4.5. *When $m_2 \neq 0$ (i.e. when W_2 is not anisotropic), according to our discussion above, the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ contains two elements $\underline{P}$ and $\underline{P}'$. We have $c(1-2c\frac{H}{\underline{P}'}) = \frac{2n+1}{2}(1-2\frac{n-2}{2n+1}) = \frac{5}{2} > s_0 = \frac{1}{2}$. This confirms the Claim 3.11.*

5. The model $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$

5.1. **The global result.** Let W_1 (resp. W_2) be a quadratic space defined over k of dimension $n+1$ (resp. $n-1$), and $W = W_1 \oplus W_2$. Let $G = \mathrm{SO}(W)$ and $H = \mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$. Let $D = Span\{v_{0,+}, v_{0,-}\}$ be a two-dimensional quadratic space with $< v_{0,+}, v_{0,+} >=< v_{0,-}, v_{0,-} >= 0$ and $< v_{0,+}, v_{0,-} >= 1$, $V_1 = W_1 \oplus D$, $V_2 = W_2$, $V = V_1 \oplus V_2$, $\underline{G} = \mathrm{SO}(V)$, and $\underline{H} = \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$.

Let $D_+ = Span\{v_{0,+}\}$ and $D_- = Span\{v_{0,-}\}$. Then $D = D_+ \oplus D_-$ as a vector space. Let $\underline{P} = \underline{M}\underline{N}$ be the maximal parabolic subgroup of G that stabilizes the subspace D_+ with $\underline{M}$ be the Levi subgroup that stabilizes the subspaces D_+, W and D_-. Then $\underline{M} = \mathrm{SO}(W) \times \mathrm{GL}_1$.

By a similar argument as in Section 4.2, we can prove the following proposition.

Proposition 5.1. *If W_2 is anisotropic, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$. If W_2 is not anisotropic, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}, \underline{P}' = \underline{M}'\underline{N}'\}$ where $\underline{M}' \cap \underline{H} = \mathrm{SO}(V_1) \times M_2$ with M_2 be a maximal Levi subgroup of $\mathrm{SO}(W_2)$ that is isomorphic to $\mathrm{SO}(n-3) \times \mathrm{GL}_1$.*

Let $\pi = \otimes_{v \in |k|} \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Like in the previous section, by abusing of language, we also use π to denote the cuspidal automorphic representation $\pi \otimes 1$ of $\underline{M}(\mathbb{A})$. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$.

Theorem 5.2. *Assume that there exists a local non-archimedean place $v \in |k|$ such that π_v is a generic representation of $G(k_v)$ (in particular, $G(k_v)$ is quasi-split). If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$.*

Proof. The proof is very similar to the proof of Theorem 4.1, we will skip it here. The only thing worth to point out is that in the case of Theorem 4.1, the constant $-c(1-2c\frac{H}{\underline{P}}) = -\frac{2n+1}{2}(1-\frac{2(n+1)}{2n+1})$ is equal to $1/2$ and this is why we can show that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$. For the current case, the constant $-c(1-2c\frac{H}{\underline{P}}) = -\frac{2n}{2}(1-\frac{2(n+1)}{2n})$ is equal to 1. This is why we can show that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$. □

Remark 5.3. *As in the previous case, when W_2 is not anisotropic, the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ will contain two elements $\underline{P}$ and $\underline{P}'$. We have $c(1-2c\frac{H}{\underline{P}'}) = \frac{2n}{2}(1-2\frac{n-3}{2n}) = 3 > s_0 = 1$, which confirms the Claim 3.11.*

Remark 5.4. *In* [GRS97] *the existence of pole at $s = 1$ of $L(s, \pi)$ for π a generic cuspidal representation of SO_{2n} is linked to a different (non-reductive) period and also to the so called first occurrence problem in theta correspondence.*

Proposition 5.5. *Theorem 5.2 implies Corollary 1.3(2).*

Proof. We first recall the statement of Corollary 1.3(2). Let $G = \mathrm{SO}_{2n}$ be a quasi-split even orthogonal group, $H = \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1}$ be a closed subgroup of G (not necessarily quasi-split), and π be a generic cuspidal automorphic representation of $G(\mathbb{A})$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, we need to show that the standard L-function $L(s, \pi)$ has a pole at $s = 1$.

By Theorem 5.2, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$, and thus the intertwining operator $M(w, s)$, has a pole at $s = 1$. In this case, the intertwining operator is

$$M(w,s) = \frac{L(s,\pi)}{L(s+1,\pi)} N(w,s)$$

where $N(s,w)$ is the normalized intertwining operator. By Theorem 4.7 of [KK11], the normalized intertwining operator $N(w,s)$ is holomorphic at $s=1$. By Proposition 2.7, $L(2,\pi)\neq 0$. It follows that the numerator $L(s,\pi)$ has a pole at $s=1$. This proves Corollary 1.3(2). □

5.2. **The local result.** Let F be a p-adic field, and $\underline{G},\underline{H},\underline{P}=\underline{M}\underline{N},G,H$ be the groups defined in the previous subsection with $G(F)$ quasi-split. Let π be an irreducible smooth representation of $G(F)$. By abusing of notation, we still use π to denote the representation $\pi\otimes 1$ of $\underline{M}(F)\simeq G(F)\times \mathrm{GL}_1(F)$. We also extend π to $\underline{P}(F)$ by making it trivial on $\underline{N}(F)$. For $s\in\mathbb{C}$, we use π_s to denote the representation $\pi\otimes\varpi^s$. Here $\varpi=\varpi_{\underline{P}}\in\mathfrak{a}_{\underline{M}}^*$ is the fundamental weight associated to $\underline{P}$, and ϖ^s is the character of $\underline{M}(F)$ defined by

$$\varpi^s(m)=e^{\langle s\varpi,H_{\underline{M}}(m)\rangle},\ m\in\underline{M}(F).$$

Let $I_{\underline{P}}^{\underline{G}}(\cdot)$ be the normalized parabolic induction:

$$I_{\underline{P}}^{\underline{G}}(\pi)=\{f:\underline{G}(F)\to\pi|\ f \text{ locally constant},\ f(nmg)=\delta_{\underline{P}}(m)^{1/2}\cdot\pi(m)f(g),\\ \forall m\in\underline{M}(F),n\in\underline{N}(F),\ g\in\underline{G}(F)\},$$

and the $\underline{G}(F)$-action is the right translation. The goal of this section is to prove the following theorem.

Theorem 5.6. *If π is an irreducible representation of $G(F)$ such that the Hom space*

$$\mathrm{Hom}_{H(F)}(\pi,1)$$

is nonzero, then the representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is $\underline{H}(F)$-distinguished, i.e. $\mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{\underline{G}}(\pi_1),1)\neq\{0\}$.

Before we prove the theorem, we first show that Theorem 5.6 implies Theorem 1.4 for the current model.

Proposition 5.7. *Theorem 5.6 implies Theorem 1.4.*

Proof. Assume that G is quasi-split. Let π be an irreducible generic tempered representation of $G(F)$ such that the Hom space

$$\mathrm{Hom}_{H(F)}(\pi,1)$$

is nonzero, we need to show that the local standard L-function $L(s,\pi)$ has a pole at $s=0$.

By Theorem 5.6, we know that the induced representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is $\underline{H}(F)$-distinguished. If $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is irreducible, then it is generic since π is generic. On the other hand, by Corollary 3.4, we know that there is no $\underline{H}(F)$-distinguished generic representation of $\underline{G}(F)$. This is a contradiction and hence we know that $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is reducible.

By Lemma B.2 of [GI16] and the Standard Module Conjecture proved in [HO13] and [M01], we have that $I_{\underline{P}}^{\underline{G}}(\pi_1)$ is reducible if and only if the local gamma factor $\gamma(s,\pi,\psi)=\epsilon(s,\pi,\psi)\frac{L(1-s,\pi)}{L(s,\pi)}$ has a pole at $s=1$ (ψ is an additive character). Here the gamma factor was defined by Shahidi in [S90]. Since the epsilon factor $\epsilon(s,\pi,\psi)$ and the inverse of the local L-function $\frac{1}{L(s,\pi)}$ are holomorphic functions, we know that the L-function $L(s,\pi,\rho_X)$ has a pole at $s=0$. This proves the proposition. □

For the rest of this section, we will prove Theorem 5.6. Let $P_{\underline{H}}=\underline{H}\cap\underline{P}$, $M_{\underline{H}}=\underline{H}\cap\underline{M}$, and $N_{\underline{H}}=\underline{H}\cap\underline{N}$. Then $P_{\underline{H}}=M_{\underline{H}}N_{\underline{H}}$ is a maximal parabolic subgroup of $\underline{H}$ with $M_{\underline{H}}\simeq H\times\mathrm{GL}_1$. We need a lemma.

Lemma 5.8. *We have the following equality of characters of* $M_{\underline{H}}(F)$.

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}}=\varpi.$$

Here $\delta_{\underline{P}}$ *and* ϖ *are characters of* $\underline{M}(F)$, *and we view them as characters of* $M_{\underline{H}}(F)$ *by restriction.*

Proof. By the definition of the constants c and $c_{\underline{P}}^{H}$, we have

$$\delta_{\underline{P}}=\varpi^{2c},\ \delta_{P_{\underline{H}}}=\varpi^{2cc_{\underline{P}}^{H}}.$$

This implies that

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}}=\varpi^{-c+2cc_{\underline{P}}^{H}}=\varpi^{(-n)+2n\cdot\frac{n+1}{2n}}=\varpi.$$

□

Remark 5.9. *The statement in the lemma above is equivalent to the equality*

$$s_0=-c(1-2c_{\underline{P}}^{H})=1.$$

Now we are ready to prove the theorem. Let $\underline{G}(F)_0=\underline{G}(F)-\underline{P}(F)\underline{H}(F)$. By Corollary 2.9, it is an open subset of $\underline{G}(F)$. We realize the representation $I_{\underline{P}}^{\underline{G}}(\pi_1)$ on the space

$$\begin{aligned} I_{\underline{P}}^{\underline{G}}(\pi_1) &= \{f:\underline{G}(F)\to\pi|\ f \text{ locally constant},\ f(nmg)=\delta_{\underline{P}}(m)^{1/2}\varpi(m)\cdot\pi(m)f(g),\\ &\quad \forall m\in\underline{M}(F), n\in\underline{N}(F),\ g\in\underline{G}(F)\}\end{aligned}$$

with the $G(F)$-action given by the right translation. Let V' be the subspace of $I_{\underline{P}}^{\underline{G}}(\pi_1)$ consisting of all the functions whose support is contained in $\underline{G}(F)_0$. Then we know that V' is $\underline{H}(F)$-invariant. Moreover, as a representation of $\underline{H}(F)$, the map $f\in I_{\underline{P}}^{\underline{G}}(\pi_1)\mapsto f|_{\underline{H}}$ induced an isomorphism

$$I_{\underline{P}}^{\underline{G}}(\pi_1)/V'\simeq ind_{P_{\underline{H}}}^{\underline{H}}(\delta_{\underline{P}}^{1/2}\varpi\pi)$$

where *ind* is the compact induction. By the Frobenius reciprocity law and Lemma 5.8, we have

$$\begin{aligned}\mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{\underline{G}}(\pi_1)/V',1)&\simeq \mathrm{Hom}_{P_{\underline{H}}(F)}(\delta_{\underline{P}}^{1/2}\varpi\pi,\delta_{P_{\underline{H}}})=\mathrm{Hom}_{P_{\underline{H}}(F)}(\pi,1)\\ &=\mathrm{Hom}_{M_{\underline{H}}(F)}(\pi,1)=\mathrm{Hom}_{H(F)}(\pi,1)\neq\{0\}.\end{aligned}$$

This implies that $\mathrm{Hom}_{\underline{H}(F)}(I_{\underline{P}}^{\underline{G}}(\pi_1),1)\neq\{0\}$ and finishes the proof of Theorem 5.6.

6. The model $(\mathrm{U}_{2n},\mathrm{U}_n\times\mathrm{U}_n)$

Let k'/k be a quadratic extension, W_1 and W_2 be two Hermitian spaces of dimension n, and $W=W_1\oplus W_2$. Let $G=\mathrm{U}(W)$ and $H=\mathrm{U}(W_1)\times\mathrm{U}(W_2)$. Let $D=Span\{v_{0,+},v_{0,-}\}$ be a two-dimensional Hermitian space with $<v_{0,+},v_{0,+}>=<v_{0,-},v_{0,-}>=0$ and $<v_{0,+},v_{0,-}>=1$, $V_1=W_1\oplus D$, $V_2=W_2$, $V=V_1\oplus V_2$, $\underline{G}=\mathrm{U}(V)$, and $\underline{H}=\mathrm{U}(V_1)\times\mathrm{U}(V_2)$.

Let $D_+=Span\{v_{0,+}\}$ and $D_-=Span\{v_{0,-}\}$. Then $D=D_+\oplus D_-$ as a vector space. Let $\underline{P}=\underline{M}\underline{N}$ be the maximal parabolic subgroup of $\underline{G}$ that stabilizes the subspace D_+ with $\underline{M}$ be the Levi subgroup that stabilizes the subspaces D_+, W and D_-. Then $\underline{M}=\mathrm{U}(W)\times Res_{k'/k}\mathrm{GL}_1$.

By a similar argument as in Section 4.2, we can prove the following proposition.

Proposition 6.1. *If* W_2 *is anisotropic, then* $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P}\}$. *If* W_2 *is not anisotropic, then* $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P},\underline{P}'=\underline{M}'\underline{N}'\}$ *where* $\underline{M}'\cap\underline{H}=\mathrm{U}(V_1)\times M_2$ *with* M_2 *be a maximal Levi subgroup of* $\mathrm{U}(W_2)$ *that is isomorphic to* $\mathrm{U}_{n-2}\times Res_{k'/k}\mathrm{GL}_1$.

Let $\pi=\otimes_{v\in|k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Like in the previous subsection, by abusing of language, we also use π to denote the cuspidal automorphic representation $\pi\otimes 1$ of $\underline{M}(\mathbb{A})$. For $\phi\in\mathcal{A}_\pi$ and $s\in\mathbb{C}$, let $E(\phi,s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. Let Π be the base change of π to $\mathrm{GL}_{2n}(\mathbb{A}_{k'})$.

Theorem 6.2. *Assume that there exists a local non-archimedean place $v \in |k|$ such that π_v is a generic representation of $G(k_v)$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1/2$.*

Proof. The proof is very similar to the orthogonal group case (Theorem 4.1), we will skip it here. The only thing worth to mention is the computation of the constant $s_0 = -c(1 - 2c_{\underline{P}}^{\underline{H}})$. By Proposition 2.1(3), we have $c = \frac{2n+1}{2}$. On the other hand, although the unipotent radical $\underline{N}$ is not abelian in this case, it is easy to see that $c_{\underline{P}}^{\underline{H}} = \frac{n+1}{2n+1}$. As a result, we have

$$s_0 = -c(1 - 2c_{\underline{P}}^{\underline{H}}) = -\frac{2n+1}{2}(1 - \frac{2(n+1)}{2n+1}) = 1/2.$$

□

Remark 6.3. *As in the previous cases, when W_2 is not anisotropic, the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ will contain two elements $\underline{P}$ and $\underline{P}'$. We have $c(1 - 2c_{\underline{P'}}^{\underline{H}}) = \frac{2n+1}{2}(1 - 2\frac{n-1}{2n+1}) = \frac{3}{2} > s_0 = \frac{1}{2}$, which confirms the Claim 3.11.*

The next proposition proves the first part of Corollary 1.3(3). Its proof is very similar to the proof of Proposition 4.2 and we will skip it here.

Proposition 6.4. *Assume that G is quasi-split and π is generic. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then the standard L-function $L(s, \pi)$ is nonzero at $s = 1/2$.*

Now it remains to prove the second part of Corollary 1.3(3). We first recall the statement. Assume that Π is cuspidal. Also assume that there exists a local place $v_0 \in |k|$ such that k'/k splits at v_0 and π_{v_0} is a discrete series of $G(k_{v_0}) = \mathrm{GL}_{2n}(k_{v_0})$. We need to show that if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then the exterior square L-function $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$ (i.e. Π is of symplectic type). Recall that globally we say a cuspidal automorphic representation of GL_{2n} is of symplectic type (resp. orthogonal type) if the exterior square (resp. symmetric square) automorphic L-function has a pole at $s = 1$. Locally we say an irreducible smooth representation of GL_{2n} is of symplectic type (resp. orthogonal type) if its Langlands parameter factors through $\mathrm{Sp}_{2n}(\mathbb{C})$ (resp. $\mathrm{SO}_{2n}(\mathbb{C})$). Moreover, if a cuspidal automorphic representation of GL_{2n} is of symplectic (resp. orthogonal) type, then all its local components are of symplectic (resp. orthogonal) type (this follows from Theorem 7.1 and 7.2 of [CKPSS04]).

We first show that Π is self-dual. Let $\pi = \otimes_{v \in |k|} \pi_v$. By the automorphic Chebotarev density theorem proved in [Ram15], in order to show that Π is self-dual, it is enough to show that π_v is self-dual for all the non-archimedean places $v \in |k|$ such that the quadratic extension k'/k splits at v. We fix such a local place v. Then π_v is an irreducible smooth representation of $G(k_v) = \mathrm{GL}_{2n}(k_v)$. Since the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, we know that locally the Hom space

$$\mathrm{Hom}_{H(k_v)}(\pi_v, 1)$$

is nonzero. By Theorem 1.1 of [JR96], we know that π_v is self-dual. This proves that Π is self-dual. Since Π is cuspidal, this implies that Π is either of symplectic type or of orthogonal type.

Now in order to show that Π is of symplectic type, it is enough to show that at the split place $v_0 \in |k|$, π_{v_0} is not of orthogonal type. By our assumption, π_{v_0} is a discrete series of $G(k_{v_0}) = \mathrm{GL}_{2n}(k_{v_0})$, hence it is enough to show that π_{v_0} is of symplectic type (see the lemma below).

Lemma 6.5. *Let F be a p-adic field and τ be an irreducible discrete series of* $\mathrm{GL}_{2n}(F)$. *Then τ can not be of symplectic type and orthogonal type at the same time.*

Proof. If τ is of symplectic type and of orthogonal type, both the local exterior square L-function $L(s, \tau, \wedge^2)$ and the local symmetric square L-function $L(s, \tau, Sym^2)$ have a pole at $s = 0$. By Corollary 8.2 of [S92], $ord_{s=0} L(s, \tau \times \tau) = ord_{s=0} L(s, \tau, Sym^2) + ord_{s=0} L(s, \tau, \wedge^2) \leq -2$.

On the other hand, since τ is a discrete series, by Proposition 8.1 and Theorem 8.2 of [JPSS], we have $ord_{s=0}L(s,\tau\times\tau)=0$ or -1. This is a contradiction. □

By the discussion above, we know that the Hom space $\mathrm{Hom}_{H(k_{v_0})}(\pi_{v_0},1)$ is nonzero. In other words, π_{v_0} is distinguished by the linear model. By Theorem 5.1 of [Mat14], we know that π_{v_0} is distinguished by the Shalika model. Then by Proposition 3.4 of [LM17], we know that π_{v_0} is of symplectic type. This finishes the proof of Theorem 1.1(3).

7. The Jacquet-Guo model

7.1. **The global result.** Let k' be a quadratic extension of k. Let W be a k'-vector space of dimension $2n$. Fix a basis $\{w_1,\cdots,w_{2n}\}$ of W. We define a nondegenerate skew-symmetric k'-bilinear form B on W to be

$$B(w_j,w_k)=\delta_{j+k-1,2n},\ B(w_l,w_k)=-\delta_{l+k-1,2n},\ 1\leq j\leq n, n+1\leq l\leq 2n, 1\leq k\leq 2n.$$

In other words, in terms of the basis $\{w_1,\cdots,w_{2n}\}$, B is defined by the skew-symmetric matrix

$$J_{2n}=\begin{pmatrix}0 & w_n\\ -w_n & 0\end{pmatrix}$$

where w_n is the longest Weyl element in GL_n. Then we define the symplectic group $\underline{H}=\mathrm{Sp}(W,B)$. In other words, $\underline{H}=Res_{k'/k}\mathrm{Sp}_{2n}$.

Now we define the group $\underline{G}$. View W as a k-vector space of dimension $4n$. We define a nondegenerate skew-symmetric k-bilinear form B_k on W to be

$$B_k(v_1,v_2)=\mathrm{tr}_{k'/k}(B(v_1,v_2)),\ v_1,v_2\in W.$$

Then we define $\underline{G}=\mathrm{Sp}(W,B_F)$ (i.e. $\underline{G}=\mathrm{Sp}_{4n}$). We have $\underline{H}\subset\underline{G}$. For $1\leq j\leq 2n$, let W_j be the k'-subspace of W spanned by w_j.

Let $\underline{P}=\underline{M}\underline{N}$ be the Siegel parabolic subgroup of $\underline{G}$ that stabilizes the k-subspace $Span_{k'}\{w_1,\cdots,w_n\}$ and $\underline{M}$ be the Levi subgroup that stabilizes both $Span_{k'}\{w_1,\cdots,w_n\}$ and $Span_{k'}\{w_{n+1},\cdots,w_{2n}\}$. Similarly, let $P_{\underline{H}}=M_{\underline{H}}N_{\underline{H}}$ be the Siegel parabolic subgroup of $\underline{H}$ that stabilizes the k'-subspace $Span_{k'}\{w_1,\cdots,w_n\}$, and $\underline{M}_{\underline{H}}$ be the Levi subgroup that stabilizes the k'-subspaces $Span_{k'}\{w_1,\cdots,w_n\}$ and $Span_{k'}\{w_{n+1},\cdots,w_{2n}\}$. Then immediately one has $\underline{P}\cap\underline{H}=P_{\underline{H}}$, $\underline{M}\cap\underline{H}=M_{\underline{H}}$ and $\underline{N}\cap\underline{H}=N_{\underline{H}}$. We let $G=\underline{M}=\mathrm{GL}_{2n}$ and $H=M_{\underline{H}}=Res_{k'/k}\mathrm{GL}_n$.

Let $\pi=\otimes_{v\in|k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})=\underline{M}(\mathbb{A})$ with trivial central character. For $\phi\in\mathcal{A}_\pi$ and $s\in\mathbb{C}$, let $E(\phi,s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. The goal of this section is to prove the following theorem.

Theorem 7.1. *If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi\in\mathcal{A}_\pi$ such that the Eisenstein series $E(\phi,s)$ has a pole at $s=1/2$.*

Remark 7.2. *Since $G=\mathrm{GL}_{2n}$, all the cuspidal automorphic representations of $G(\mathbb{A})$ are generic.*

Theorem 7.1 will be proved in Section 7.3. The next proposition shows that Corollary 1.1(4) follows from Theorem 7.1.

Proposition 7.3. *Theorem 7.1 implies Corollary 1.1(4).*

Proof. We need to show that if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then the standard L-function $L(s,\pi)$ is nonzero at $s=1/2$ and the exterior square L-function $L(s,\pi,\wedge^2)$ has a pole at $s=1$.

By Theorem 7.1, if the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, there exists $\phi\in\mathcal{A}_\pi$ such that the Eisenstein series $E(\phi,s)$, and thus the intertwining operator $M(w,s)$, has a pole at $s=1/2$. In this case, the intertwining operator is

$$M(w,s)=\frac{L(s,\pi)L(2s,\pi,\wedge^2)}{L(s+1,\pi)L(2s+1,\pi,\wedge^2)}N(w,s).$$

where $N(w,s)$ is the normalized intertwining operator. By Theorem 4.7 of [KK11], the normalized intertwining operator $N(w,s)$ is holomorphic at $s=1/2$. By Proposition 2.7, the product $L(3/2,\pi)L(2,\pi,\wedge^2)\neq 0$. It follows that the numerator $L(s,\pi)L(2s,\pi,\wedge^2)$ has a pole at $s=1/2$. Since the standard L-function $L(s,\pi)$ is an entire function and the exterior square L-function $L(2s,\pi,\wedge^2)$ can only have a simple pole at $s=1/2$, we know that the standard L-function $L(s,\pi)$ is nonzero at $s=1/2$ and the exterior square L-function $L(s,\pi,\wedge^2)$ has a pole at $s=1$. □

7.2. **The parabolic subgroups.** Denote by $P_{0,\underline{H}}=\underline{T}_H\underline{N}_{0,H}$ be the minimal parabolic subgroup of $\underline{H}$ that stabilizes the filtration

$$Span_{k'}\{w_1\}\subset Span_{k'}\{w_1,w_2\}\subset\cdots\subset Span_{k'}\{w_1,\cdots,w_n\},$$

and $\underline{T}_H$ be the maximal torus of $\underline{G}$ that stabilizes the subspaces W_j for $1\leq j\leq 2n$.

Theorem 7.4. *The set* $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P}\}$.

Proof. To prove the theorem, it suffices to check that $\mathcal{F}^{\underline{G}}_{geom}(P_{0,\underline{H}},\underline{P})=\{\underline{P}\}$. Thus suppose $\underline{Q}$ is in $\mathcal{F}^{\underline{G}}_{geom}(P_{0,\underline{H}},\underline{P})$, so that in particular $\underline{Q}$ is a Siegel parabolic subgroup of $\underline{G}$. That is, $\underline{Q}$ stabilizes an isotropic $2n$-dimensional k-subspace $L_{\underline{Q}}$ of W. By assumption, $\underline{T}_H\subseteq\underline{Q}$, so $L_{\underline{Q}}$ is stabilized by $\underline{T}_H$ and thus is a direct sum of the spaces W_j for $j\in J_Q$ where J_Q is a subset of $\{1,2,\cdots,2n\}$ that contains n elements. As it is clear that there is only one $\underline{H}$-orbit of such spaces, the theorem follows from Lemma 2.11. □

7.3. **The proof of Theorem 7.1.** In this section, we will prove Theorem 7.1 by the method discussed in Section 3. We first compute the constant $s_0=-c(1-2c^{\underline{H}}_{\underline{P}})$ for the current case. By Remark 2.5, we have

$$c^{\underline{H}}_{\underline{P}}=\frac{\dim(N_{\underline{H}})}{\dim\underline{N}}=\frac{n+1}{2n+1}.$$

By Proposition 2.1, we have $c=\frac{2n+1}{2}$. This implies that

$$s_0=-c(1-2c^{\underline{H}}_{\underline{P}})=1/2.$$

It remains to show that Claim 3.6 holds. Since $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})=\{\underline{P}\}$, this is trivial. Theorem 7.1 follows from Proposition 3.7.

7.4. **The local result.** Let F be a p-adic field, and E/F be a quadratic extension. As in the previous subsections, we can define the groups $\underline{G},\underline{H},\underline{P}=\underline{M}\underline{N},G,H$ over F. Let π be an irreducible smooth representation of $G(F)=\underline{M}(F)=\mathrm{GL}_{2n}(F)$. We extend π to $\underline{P}(F)$ by making it trivial on $\underline{N}(F)$. As in Section 5.2, for $s\in\mathbb{C}$, we use π_s to denote the representation $\pi\otimes\varpi^s$ and use $I^{\underline{G}}_{\underline{P}}(\pi_s)$ to denote the normalized parabolic induction.

Theorem 7.5. *If π is an irreducible representation of $G(F)$ such that the Hom space*

$$\mathrm{Hom}_{H(F)}(\pi,1)$$

is nonzero, then the representation $I^{\underline{G}}_{\underline{P}}(\pi_{\frac{1}{2}})$ is $\underline{H}(F)$-distinguished, i.e. $\mathrm{Hom}_{\underline{H}(F)}(I^{\underline{G}}_{\underline{P}}(\pi_{\frac{1}{2}}),1)\neq\{0\}$.

Theorem 7.5 will follows from the exact same argument as the proof of Theorem 5.6 once we have proved the following lemma which is the analogue of Lemma 5.8.

Lemma 7.6. *We have the following equality of characters of $M_{\underline{H}}(F)$.*

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}}=\varpi^{1/2}.$$

Here $\delta_{\underline{P}}$ and ϖ are characters of $\underline{M}(F)$, and we view them as characters of $M_{\underline{H}}(F)$ by restriction.

Proof. By the same argument as in the proof of Lemma 5.8, we have

$$\delta_{\underline{P}}^{-1/2}\delta_{P_{\underline{H}}} = \varpi^{-c+2cc_{\underline{P}}^{H}} = \varpi^{\frac{-2n-1}{2}+\frac{2(2n+1)}{2}\cdot\frac{n+1}{2n+1}} = \varpi^{1/2}.$$

This proves the lemma. □

Now we are ready to prove Theorem 1.4 for the current model. Let π be a tempered representation of $G(F)$ with trivial central character (in particular, π is generic since $G = \mathrm{GL}_{2n}$). Assume that the Hom space $\mathrm{Hom}_{H(F)}(\pi, 1)$ is nonzero, we need to show that the local exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$. By the same argument as in Proposition 5.7, we know that the induced representation $I_{\underline{P}}^{G}(\pi_{\frac{1}{2}})$ is reducible. Again by applying Lemma B.2 of [GI16] and the Standard Module Conjecture proved in [HO13] (for the current case, the Standard Module Conjecture was proved much earlier in [JPSS] and [S92]), we have that $I_{\underline{P}}^{G}(\pi_{\frac{1}{2}})$ is reducible if and only if the local gamma factor

$$\gamma(s, \pi, \psi)\gamma(2s, \pi, \wedge^2, \psi) = \epsilon(s, \pi, \psi)\epsilon(2s, \pi, \wedge, \psi)\frac{L(1-s, \pi)L(1-2s, \pi, \wedge^2)}{L(s, \pi)L(2s, \pi, \wedge^2)}$$

has a pole at $s = \frac{1}{2}$ where ψ is an additive character. Since π is tempered, $L(s, \pi)$ and $L(s, \pi, \wedge^2)$ are holomorphic and nonzero when $Re(s) > 0$ ([JPSS], [S92]). Moreover, the epsilon factors are holomorphic functions. Hence the L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$. This finishes the proof of Theorem 1.4.

8. The model $(\mathrm{GE}_6, A_1 \times A_5)$

8.1. **The result.** Fix a quaternion algebra B over the number field k. Denote by $J_B = H_3(B)$ the Hermitian 3×3 matrices over B. Let $\Theta = B \oplus B$ be an octonion algebra over k defined via the Cayley-Dickson construction and denote by $J_\Theta = H_3(\Theta)$ the Hermitian 3×3 matrices over Θ. Then $\dim_k J_B = 15$ and $\dim_k J_\Theta = 27$; J_Θ is the exceptional cubic norm structure. The Cayley-Dickson construction induces an identification $J_\Theta = J_B \oplus B^3$. See for example [SV00] for the octonions, the Cayley-Dickson construction, and cubic norm structures. See also [Pol17, Appendix A], or [Pol18, section 4 and section 8.1] for all of these notions. In particular, see [Pol18, section 8.1], J_Θ comes equipped with a symmetric non-degenerate bilinear form, and the decomposition $J_\Theta = J_B \oplus B^3$ is an orthogonal decomposition with respect to this form.

Let $G = GE_6$ be the group preserving the cubic norm on J_Θ up to similitude. Let

$$H = (\mathrm{GL}_1(B) \times \mathrm{GL}_3(B))^0 = \{(x, g) \in \mathrm{GL}_1(B) \times \mathrm{GL}_3(B) : n_B(x) = N_6(g)\}$$

where n_B, resp. N_6, denotes the reduced norm on B (of degree two), resp. on $M_3(B)$ (of degree six). In this section, we will consider H-periods of cusp forms on G.

Denote by $\underline{G}$ the semisimple, simply-connected group of type E_7 defined in terms of J_Θ. Precisely, see for example [Pol18, section 4], $\underline{G}$ is the group preserving Freudenthal's symplectic and quartic form on $W_\Theta = k \oplus J_\Theta \oplus J_\Theta^\vee \oplus k$. Denote by $W_B := k \oplus J_B \oplus J_B^\vee \oplus k$ the Freudenthal space associated to the cubic norm structure J_B. We write elements of W_B as ordered four-tuples (a, b, c, d), so that $a, d \in k$, $b \in J_B$ and $c \in J_B^\vee$, and similarly for W_Θ. The 32-dimensional space W_B affords one of the half-spin representations of a group of type D_6. The decomposition $J_\Theta = J_B \oplus B^3$ induces a decomposition

$$W_\Theta = W_B \oplus B^6 \tag{8.1}$$

of symplectic vector spaces [Pol18, section 8.1], [Pol17, Appendix A]. Note that the trace form on B induces an identification between $B^\vee$ and B. We will use the decomposition (8.1) to define $\underline{H}$ and the map $\underline{H} \to \underline{G}$.

In more detail, let $\underline{H}'$ be the subgroup of elements with similitude equal to 1 of the group denoted $\widetilde{G}$ in [Pol17, Appendix A]. Recall that $\underline{H}'$ is defined as follows. Denote by D_6^+ the semisimple half-spin group of type D_6 whose defining representation is W_B and denote by $U_6(B)$ the subgroup of $\mathrm{GL}_6(B)$ satisfying $g\begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix}g^* = \begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix}$ where g^* is the transpose conjugate of g. The group $\underline{H}'$ is defined to be the set of pairs $(g,h) \in D_6^+ \times U_6(B)$ whose action on $W_B \oplus B^6 = W_\Theta$ preserves Freudenthal's symplectic and quartic form on W_Θ. Consequently, $\underline{H}' \subseteq \underline{G}$, and $\underline{H}'$ preserves the decomposition 8.1. The group $\underline{H}'$ is semisimple and simply-connected of type D_6, and is split precisely when the quaternion algebra B is split.

Denote by B^1 the subgroup of $\mathrm{GL}_1(B)$ with reduced norm equal to 1. Let B^1 act on W_Θ by $x(w,v) = (w, xv)$ where $x \in B^1$, $w \in W_B$ and $v \in B^6$. This action commutes with the action of $\underline{H}'$ on W_Θ because $\underline{H}'$ acts on the right of B^6. We set $\underline{H} = B^1 \times \underline{H}'$. The action of $\underline{H}$ on W_Θ defines a map $\underline{H} \to \underline{G}$. This map has a diagonal μ_2-kernel. We will consider an $\underline{H}$-period of automorphic forms on $\underline{G}$, not periods of the (different) algebraic group $\underline{H}/\mu_2$.

Denote by $\underline{P}$ the subgroup of $\underline{G}$ that stabilizes the line $k(0,0,0,1)$ in W_Θ. The group $\underline{P}$ is a parabolic subgroup of $\underline{G}$ with reductive quotient of type E_6. Denote by $\underline{M}$ the subgroup of $\underline{P}$ that also fixes the line $k(1,0,0,0)$. Then $\underline{M}$ is a Levi subgroup of $\underline{P}$, and one has $\underline{P} = \underline{M}\underline{N}$ with the unipotent radical $\underline{N}$ of $\underline{P}$ abelian; in fact, $\underline{N} \simeq J_\Theta$. The group $\underline{M}$ is isomorphic to the subgroup $\mathrm{GL}_1 \times GE_6$ consisting of pairs (δ, m) with $\delta^2 = \nu(m)$, where $\nu : GE_6 \to \mathrm{GL}_1$ is the similitude. The map $\underline{M} \to \mathrm{GL}_1$ defined as $(\delta, m) \mapsto \delta$ is the fundamental weight of $\underline{M}$. One has that

$$\underline{M} \cap \underline{H} = \{(x, \delta, g) \in B^1 \times \mathrm{GL}_1 \times \mathrm{GL}_3(B) : \delta^2 = N_6(g)\}.$$

Note that the image of $\underline{M} \cap \underline{H}$ in PGE_6 is contained in the image of H in PGE_6.

Let $\pi = \otimes_{v \in |k|}\pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. It can also be viewed as a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$ with trivial central character. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$. The goal of this section is to prove the following theorem.

Theorem 8.1. *Assume that there exists a local non-archimedean place $v \in |k|$ such that $G(k_v)$ is split and π_v is a generic representation of $G(k_v)$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$.*

Proposition 8.2. *Theorem 8.1 implies Corollary 1.3(5).*

Proof. The proof is very similar to the proof of Proposition 5.5, we will skip it here. □

8.2. **The parabolic subgroups.** The purpose of this subsection is to prove the following theorem.

Proposition 8.3. *If the quaternion algebra B is not split, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})' = \{\underline{P}\}$. If the quaternion algebra B is split, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})' = \{\underline{P}, \underline{P}'\}$, where $\underline{P}' \cap \underline{H}'$ is a maximal parabolic of subgroup of $\underline{H}'$ with reductive quotient of type D_5.*

Proof. Since the maximal split torus of $\underline{H}$ is also the maximal split torus of $\underline{G}$, we have $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})' = \mathcal{F}^{\underline{G}}_{geom}(P_{0,\underline{H}}, \underline{P})$. We are in the situation of Proposition 2.13, with the spaces $V_{\underline{Q}}$ of that proposition the rank one lines of W_Θ (see, e.g., [Pol17, Definition 2.4]), $V_0 = W_B$, and $V_1 = B^6$. Consequently, to compute $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})'$, we must find the $\underline{H}$-orbits of rank one lines contained in either W_B or B^6. Note that there is exactly one $\underline{H}$-orbit of rank one lines contained in W_B, so we must analyze the $\underline{H}$-orbits of rank one lines contained in $V_1 = B^6$.

When B is not split, we claim that there are no rank one lines of W_Θ contained in $V_1 = B^6$. To see this, suppose that $(0, v, w, 0) \in W_\Theta$ spans such a line, with $v, w \in B^3$, considered as row vectors. Then from [Pol18, Lemma 4.3.4 and section 8.1.1], $v^*v = w^*w = 0$. But as B is anisotropic, this implies $v = w = 0$, as desired. Consequently, when B is not split, $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})' = \{\underline{P}\}$, as claimed.

When B is split, $V_1 = B^6$ is the tensor product $V_2 \boxtimes V_{12}$ of the two-dimensional representation V_2 of $B^1 = \mathrm{SL}_2$ and the twelve-dimensional representation V_{12} of $\underline{H}'$. We claim that in this case there is one $\underline{H}$-orbit of rank one lines in $V_2 \boxtimes V_{12}$. Suppose that v spans such a rank one line. Because the representation W_Θ of $\underline{G}$ is minuscule, and because $A_{0,\underline{H}}$ is a maximal split torus of $\underline{G}$, it is easy to see that if v is an eigenvector for $A_{0,\underline{H}}$ then $v = v' \otimes v''$ is a pure tensor in $V_2 \boxtimes V_{12}$, with v'' isotropic. As SL_2 acts transitively on the lines in V_2, and $\underline{H}'$ acts transitively on the isotropic lines in V_{12}, there is one $\underline{H}$-orbit of such $v' \otimes v''$. Moreover, it is clear that the subgroup of $\underline{H}'$ stabilizing such a v'' is a maximal parabolic of type D_5. This proves the theorem, with the exception of the fact that we have not yet checked that the $v' \otimes v''$ with v'' isotropic actually span rank one lines. That is, we have not yet checked that $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ strictly contains $\underline{P}$.

Thus, to complete the proof, suppose that $v = v' \otimes v'' \in V_2 \boxtimes V_{12} \subseteq W_\Theta$, with v'' isotropic. Because v'' is isotropic, by an $\underline{H}'$ translation, we can assume that v'' is contained inside any fixed Lagrangian subspace of V_{12}. In coordinates, this means that we can assume $v = (0, b, 0, 0) \in W_\Theta$. That v is rank one now follows easily. □

Remark 8.4. *By the proof of the proposition, we know that the parabolic subgroup $\underline{P}'$ satisfies the conditions in Proposition 2.4(2). Hence we have $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$ if the quaternion algebra B is not split and $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}, \underline{P}'\}$ if B is split.*

8.3. **Proof of Theorem 8.1.** In this section, we will prove Theorem 8.1 by the method discussed in Section 3. We first compute the constant $s_0 = -c(1 - 2c_{\underline{P}}^{\underline{H}})$ for the current case. We have $c = 9$, and, because the unipotent radicals are abelian, $c_{\underline{P},\underline{H}} = \frac{\dim_k J_B}{\dim_k J_\Theta} = \frac{15}{27}$. Thus $s_0 = 1$.

It remains to show that Claim 3.6 holds. If B is not split, this is trivial since $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$. If B is split, then $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}, \underline{P}'\}$. The pair $(\underline{L}, L_{\underline{H}}) = (\underline{M}', \underline{M}' \cap \underline{H})$ is not tempered by Proposition 8.3 and Corollary 3.4. Theorem 8.1 follows from Proposition 3.7.

Remark 8.5. *As in the previous cases, if B is not split, the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$ will contain two elements $\underline{P}$ and $\underline{P}'$. We have $c(1 - 2c_{\underline{P}'}^{\underline{H}}) = 9(1 - 2\frac{10}{27}) = \frac{5}{3} > s_0 = 1$, which confirms the Claim 3.11.*

8.4. **The local result.** The proof of Theorem 1.4 for current model is the same as the proof of the model $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$ in Section 5.2. We will skip it here.

9. The model $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$

The purpose of this section is to prove the local and global results for the pair $(\mathrm{GL}_4 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2)$. The first four subsections are concerned with the global results, while the final subsection concerns the local results.

9.1. **Overview of argument.** The purpose of this and the following three subsections is to prove the global results. In this subsection, we give an overview of the proof.

Denote by $E = k \times k$ the split quadratic étale extension of k. In fact, almost all of this section is unchanged if E is replaced by a general quadratic étale extension of k, so we frequently write E instead of $k \times k$. Let Θ be a split octonion algebra over k. Define the quadratic space $V = \Theta \oplus E$ with quadratic form $q(x, \lambda) = n_\Theta(x) - n_E(\lambda)$ where $x \in \Theta$, $\lambda \in E$, and n_Θ resp. n_E denote the quadratic norms on Θ resp. E. We define $\underline{G} = \mathrm{GSO}(V)$, which by definition is the subgroup of $\mathrm{GO}(V)$ consisting of those g with $\det(g) = \nu(g)^{\dim(V)/2}$, where $\nu : \mathrm{GO}(V) \to \mathrm{GL}_1$ is the similitude factor.

In the next subsection, we specify a group $\underline{H}_7$, isomorphic to $\mathrm{GSpin}(7)$, together with its 8-dimensional spin representation on Θ. Set

$$\begin{aligned}\underline{H} = \underline{H}_7 \boxtimes \mathrm{Res}_{E/k}(\mathrm{GL}_1) :&= \{(h, \lambda) \in \underline{H}_7 \times \mathrm{Res}_{E/k}(\mathrm{GL}_1) : \nu(h) = n_E(\lambda)\} \\ &= \{(h, \lambda_1, \lambda_2) \in \underline{H}_7 \times \mathrm{GL}_1 \times \mathrm{GL}_1 : \nu(h) = \lambda_1\lambda_2\}.\end{aligned}$$

Via the representation $t_1 : \mathrm{GSpin}(7) \to \mathrm{GSO}(\Theta)$ specified below, we obtain an inclusion $\underline{H} \to \underline{G}$.

Denote by $\underline{P} = \underline{MN}$ the Heisenberg parabolic of $\underline{G}$, so that the Levi subgroup $\underline{M}$ of $\underline{P}$ is of type $A_1 \times D_3 = A_1 \times A_3$. Suppose that $\pi = \otimes_{v\in|k|}\pi_v$ is a cuspidal automorphic representation of $\underline{M}$ or $G = \mathrm{GL}_2 \times \mathrm{GL}_4$ with trivial central character[1]. Suppose that $\phi \in \mathcal{A}_\pi$, $s \in \mathbb{C}$ and $E(\phi, s)$ denotes the associated Eisenstein series.

Let

$$\begin{aligned} H &= (\mathrm{GL}_2 \times \mathrm{GL}_2) \boxtimes \mathrm{Res}_{E/k}(\mathrm{GL}_1) \\ &= \{(g, h, \lambda) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{Res}_{E/k}(\mathrm{GL}_1) : \det(g)\det(h) = N_{E/k}(\lambda)\}. \end{aligned}$$

Denote by $Z \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ the subgroup of $(\mathrm{GL}_2 \times \mathrm{GL}_2) \boxtimes \mathrm{Res}_{E/F}(\mathrm{GL}_1)$ consisting of the elements $(\mathrm{diag}(z, z), \mathrm{diag}(w, w), (zw, zw))$. From Lemma 9.4 below we obtain an embedding $H \to \underline{M}$ so that $Z = H \cap Z_{\underline{M}}$ where $Z_{\underline{M}}$ is the center of $\underline{M}$. Note that the map $H \to \mathrm{GL}_2 \times \mathrm{GL}_2$ given by $(g, h, (\lambda_1, \lambda_2)) \mapsto (g, \lambda_1^{-1}\det(g)h)$ induces an isomorphism $H/Z \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)/\Delta(\mathrm{GL}_1)$, with $\Delta(\mathrm{GL}_1)$ the diagonally embedded central GL_1. For a cuspidal automorphic form φ of $\underline{M}$ with trivial central character, denote by $\mathcal{P}_H$ the period

$$\mathcal{P}_H(\varphi) = \int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})} \varphi(h)\, dh.$$

Theorem 9.1. *Suppose that the period $\mathcal{P}_H(\cdot)$ is nonvanishing on the space of π. Then there exists $\phi \in \mathcal{A}_\pi$ such that $E(\phi, s)$ has a pole at $s = 1/2$.*

Note that even though G, H and $\underline{G}$ are classical groups, our proof of Theorem 9.1 proceeds through the non-classical group $\underline{H}_7 \simeq \mathrm{GSpin}(7)$, which appears via its *spin* representation. This is in complete analogy with the triple product period integral considered by Jiang in [Jia98], later generalized by Ginzburg-Jiang-Rallis in [GJR04b], where one considers G_2-periods of certain residual representations on groups of type D.

From Theorem 9.1 we obtain Corollary 1.3(6) of the introduction.

Proposition 9.2. *Theorem 9.1 implies Corollary 1.3(6).*

Proof. We first recall the statement of Corollary 1.3(6). Let π be a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ with trivial central character. We embed $\mathrm{GL}_2 \times \mathrm{GL}_2$ into $\mathrm{GL}_4 \times \mathrm{GL}_2$ as $a \times b \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times a$. Assume that the $\mathrm{GL}_2 \times \mathrm{GL}_2$-period integral is nonzero on the space of π. Moreover assume that the L-function $L(s, \pi, \rho_X)$ is nonzero at $s = 3/2$ where $\rho_X = \wedge^2 \otimes std$ is a 12-dimensional representation of ${}^LG$. Then we need to show that the L-function $L(s, \pi, \rho_X)$ is nonzero at $s = 1/2$.

As we explained in the previous page, we can view π as a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$ with trivial central character. Moreover, by the discussion in the end of Section 9.4, we know that the $\mathrm{GL}_2 \times \mathrm{GL}_2$-period integral on π (viewed as a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$) is just the H-period integral on π (viewed as a cuspidal automorphic representation of $\underline{M}(\mathbb{A})$). As a result, we know that the period $\mathcal{P}_H(\cdot)$ is nonvanishing on the space of π. By Theorem 9.1, there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$, and thus the intertwining operator $M(w, s)$, has a pole at $s = 1/2$.

In this case, the intertwining operator is

$$M(w, s) = \frac{L(s, \pi, \rho_X)\zeta_k(2s)}{L(s + 1, \pi, \rho_X)\zeta_k(2s + 1)} N(w, s)$$

[1]Note that the exterior square representation of GL_4 induces a map of algebraic groups $\mathrm{GL}_1 \times \mathrm{GL}_4 \to \mathrm{GSO}(6)$ which is surjective on k-points and has central kernel. Thus this map induces an isomorphism $\mathrm{PGL}_4 \simeq \mathrm{PGSO}_6$, so that a cuspidal automorphic representation on GL_4 with trivial central character may be considered as an automorhpic representation of $\mathrm{GSO}(6)$.

where $\zeta_k(s)$ is the Dedekind zeta function and $N(w,s)$ is the normalized intertwining operator. By Theorem 4.7 of [KK11], the normalized intertwining operator $N(w,s)$ is holomorphic at $s=1/2$. By Proposition 2.7, $L(3/2,\pi,\rho_X)\neq 0$. It follows that the numerator $L(s,\pi,\rho_X)\zeta_k(2s)$ has a pole at $s=1/2$, which implies that $L(\frac{1}{2},\pi,\rho_X)\neq 0$ because $\zeta_k(s)$ has a simple pole at $s=1$. This proves Theorem 1.1(6). □

In the next subsection, we define the group $\underline{H}_7$ precisely, its representation $t_1:\underline{H}_7\to \mathrm{GSO}(\Theta)$, and some special subgroups of it. In subsection 9.3 we compute $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}},\underline{P})$. In subsection 9.4 we deduce Theorem 9.1.

9.2. **Non-classical groups.** In this subsection we define the group $\underline{H}_7$, specify its Lie algebra concretely, and define certain subgroups of it. Note that we go to the trouble of defining $\underline{H}_7$ concretely–as opposed to making definitions in terms of root data–because it makes the necessary orbit computations more transparent.

First, recall from (say) [SV00] or [PWZ, section 1.1.1] the Zorn model of the octonions Θ. We will use the notation

$$\epsilon_1, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*, \epsilon_2 \tag{9.1}$$

of [PWZ, section 1.1.1] to denote a particular basis of Θ. We write $u_0=\epsilon_1-\epsilon_2$. By definition, the group G_2 is the automorphisms of Θ, and so fixes $1=\epsilon_1+\epsilon_2$. One can choose a split maximal torus of G_2 so that the basis (9.1) are eigenvectors for this torus. Moreover, if $\mathrm{SL}_3\subseteq G_2$ is generated by the long root subgroups of G_2, then this SL_3 fixes ϵ_1, ϵ_2, and stabilizes $\mathrm{Span}\{e_1,e_2,e_3\}$ and $\mathrm{Span}\{e_1^*,e_2^*,e_3^*\}$.

Define

$$GT(\Theta)=\{(g_1,g_2,g_3)\in \mathrm{GSO}(\Theta)\times\mathrm{GSO}(\Theta)\times\mathrm{SO}(\Theta):(g_1x,g_2y,g_3z)=(x,y,z)\text{ for all }x,y,z\in\Theta\}.$$

Here $\mathrm{SO}(\Theta)$ is defined via the norm $n_\Theta:\Theta\to F$ and the trilinear form is $(x,y,z):=\mathrm{tr}_\Theta(xyz)$. Denote by $t_1: GT(\Theta)\to\mathrm{GSO}(\Theta)$ the first projection, and $\nu: GT(\Theta)\to\mathrm{GL}_1$ the map that is t_1 composed with the similitude factor on $\mathrm{GSO}(\Theta)$. The subgroup of $GT(\Theta)$ with $\nu=1$ is the group $\mathrm{Spin}(8)=\mathrm{Spin}(\Theta)$. Define $\underline{H}_7$ to be the subgroup of $GT(\Theta)$ consisting of triples (g_1,g_2,g_3) so that $g_3\cdot 1=1$. One can check that the similitude $\nu:\underline{H}_7\to\mathrm{GL}_1$ is not the trivial map on k-points. We slightly abuse notation and let $t_1:\underline{H}_7\to\mathrm{GSO}(\Theta)$ denote the restriction of t_1 from $GT(\Theta)$ to $\underline{H}_7$.

We record facts about the group $\underline{H}_7$ that we will need later. Denote by σ the map $x\mapsto x^*$ on Θ. We begin with a simple lemma, whose proof is an exercise.

Lemma 9.3. *Suppose* $(g_1,g_2,g_3)\in\mathrm{Spin}(\Theta)$.

(1) Then

$$g_1(x)g_2(y)=(\sigma g_3\sigma)(xy)$$

for all $x,y\in\Theta$.

(2) Suppose $g=(g_1,g_2,g_3)\in GT(\Theta)$*, and define* $\nu=\nu(g_1)$*. If* $g_3(1)=1$*, then* $g_2=\nu^{-1}\sigma g_1\sigma$*. Consequently, if* $(g_1,g_2,g_3)\in GT(\Theta)$ *and* $g_3(1)=1$*, then*

$$g_1(x)g_1(y)^*=\nu(\sigma g_3\sigma)(xy^*)$$

for all $x,y\in\Theta$.

(3) Conversely, suppose $(g_1,g_2,g_3)\in\mathrm{Spin}(\Theta)$*, and* $g_2=\sigma g_1\sigma$*. Then* g_3 *stabilizes* 1.

Recall that we define the parabolic subgroup $\underline{P}$ of $\underline{G}$ to be the Heisenberg parabolic. This means that $\underline{P}$ stabilizes an isotropic two-dimensional subspace of V and that the flag variety $\underline{P}(k)\backslash\underline{G}(k)$ is the set of isotropic two-dimensional subspaces of V. In order to compute the double coset space $\underline{P}(k)\backslash\underline{G}(k)/\underline{H}(k)$, we will need to use $\underline{H}_7$ to move around various isotropic subspaces of V. To do this, it is helpful to have handy large concrete subgroups of $\underline{H}_7$. We specify such subgroups now.

The subgroups of $\underline{H}_7$ we will use are G_2, $\mathrm{GSpin}(6)$, and $\mathrm{GL}_2 \times \mathrm{GL}_2$. It is clear that $G_2 \subseteq \underline{H}_7$. For $\mathrm{GL}_2 \times \mathrm{GL}_2$, we compute inside the Cayley-Dickson construction (see, e.g., [PWZ, section 1.1]), so that the multiplication is $(x_1, y_1)(x_2, y_2) = (x_1x_2 + \gamma y_2' y_1, y_2x_1 + y_1x_2')$ and the conjugation is $(x, y)^* = (x', -y)$. Here for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(k)$, $h' = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ so that $hh' = \det(h)I_2$. Now, suppose $g \in \mathrm{GL}_2$ and $h \in \mathrm{GL}_2$. Define $(g, h) \cdot (x, y) = (gxh', \mu_h y g')$, where $\mu_h = \mathrm{diag}(\det(h), 1)$.

Lemma 9.4. *This action of* $\mathrm{GL}_2 \times \mathrm{GL}_2$ *on* Θ *defines a map* $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \underline{H}_7$.

Proof. Indeed, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then one computes

$$\begin{aligned} ((g,h)z_1) \cdot ((g,h)z_2)^* &= (gx_1h', \mu_h y_1 g')(gx_2h', \mu_h y_2 g')^* \\ &= \det(g)\det(h)(g(x_1x_2' - \gamma y_2' y_1)g^{-1}, \mu_h(-y_2x_1 + y_1x_2)h^{-1}). \end{aligned} \tag{9.2}$$

From this, the lemma is clear. □

We now describe the flag variety of Heisenberg parabolic subgroups in $\underline{H}_7$. For a two-dimensional isotropic subspace W of Θ, define $\kappa'(W) = \{xy^* : x, y \in W\}$. Then, because W is two-dimensional and isotropic, $\kappa'(W)$ is contained in $V_7 = \Theta^{\mathrm{tr}=0}$, and is either 0 or a line. If $\kappa'(W) = 0$, we say that W is *null*; otherwise, we say that W is not null. By Lemma 9.3, whether or not W is null is an $\underline{H}_7$-invariant. Moreover, it is clear that being isotropic and null is a closed condition on the Grassmanian of two-spaces in Θ, and thus the set of null isotropic two-spaces is a projective variety.

Lemma 9.5. *One has the following:*

(1) *The group* $\underline{H}_7$ *acts transitively on the set of null-isotropic two-spaces* W *of* Θ*, and thus the stabilizer* P_W *of any such* W *is a parabolic subgroup of* $\underline{H}_7$*; these are the Heisenberg parabolics.*
(2) *Denote by* W *the null isotropic two-dimensional subspace of* Θ *that consists of the elements* $(0, \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right))$*. The map* $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \underline{H}_7$ *of Lemma 9.4 identifies* $\mathrm{GL}_2 \times \mathrm{GL}_2$ *with a Levi subgroup of* P_W*.*

Proof. The first statement is easily checked, and in any case, is surely well-known. For the second statement, it is easy to see that this $\mathrm{GL}_2 \times \mathrm{GL}_2$ embeds into $\underline{H}_7$, and that the image preserves W. As the reductive quotient of the parabolic subgroup P_W is exactly $\mathrm{GL}_2 \times \mathrm{GSpin}(3) = \mathrm{GL}_2 \times \mathrm{GL}_2$, the lemma follows. □

We next describe the subgroup $\mathrm{GSpin}(6)$ of $\underline{H}_7$ and how it acts on Θ. Recall the elements $\epsilon_1, \epsilon_2 \in \Theta$ with $1 = \epsilon_1 + \epsilon_2$. Define H_6 to be the subgroup of triples (g_1, g_2, g_3) in $GT(\Theta)$ for which $g_3(\epsilon_j) = \epsilon_j$ for $j = 1, 2$.

Lemma 9.6. *The group* H_6 *fixes the four-dimensional subspaces* $U_+ = \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$ *and* $U_- = \left(\begin{smallmatrix} 0 & 0 \\ * & * \end{smallmatrix}\right)$ *of* Θ *under the* t_1*-representation. Moreover, the image of the map* $H_6 \to \mathrm{GL}(U_+)$ *includes* $\mathrm{SL}(U_+)$*.*

Proof. We have the relation $g_3(x)g_1(y) = (\sigma g_2 \sigma)(xy)$ for general triples $(g_1, g_2, g_3) \in \mathrm{Spin}(\Theta)$. Now, U_+ is the subset of $y \in \Theta$ with $\epsilon_2 y = 0$ and $U_- = \{y \in \Theta : \epsilon_1 y = 0\}$. The first part of the lemma follows immediately from this.

For the second part, it is clear that the image contains the $\mathrm{SL}_3 \subseteq G_2$ that stabilizes ϵ_1 and ϵ_2. From (9.2), the subgroup $1 \times \mathrm{SL}_2 \subseteq \mathrm{GL}_2 \times \mathrm{GL}_2$ is in H_6. Under the identification of the Cayley-Dickson and the Zorn model of the octonions, the subspace spanned by e_1, e_2, e_3 in the Zorn model becomes the subspace of elements $(\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 \\ * & 0 \end{smallmatrix}\right))$ in the Cayley-Dickson model. Thus, the subgroup $1 \times \mathrm{SL}_2$ of H_6 is the SL_2 that acts on the span of ϵ_1 and (say) e_1. Because the image of H_6 in $\mathrm{GL}(U_+)$ contains these two subgroups of $\mathrm{SL}(U_+)$, the image must contain all of $\mathrm{SL}(U_+)$, giving the lemma. □

The following lemma will be used in the next subsection.

Lemma 9.7. *Suppose $v, w \in \Theta$ are nonzero. If $n_\Theta(v) = n_\Theta(w)$, whether 0 or not, then there exists $g \in \underline{H}'_7$ with $gv = w$.*

Proof. Suppose $v \in \Theta$ as above. We claim that we can use the H'_6 inside $\underline{H}'_7$ to move v to $V_7 = \Theta^{\mathrm{tr}=0}$. From this, the lemma follows from the corresponding fact for G_2, by moving both v and w into V_7. To move v into V_7, write $v = v_1 + v_2$, with $v_1 \in U_+$ and $v_2 \in U_-$ in the notation of Lemma 9.6. Thinking about the action of SL_4 on its defining representation and its dual, it is clear that we can simultaneously move v_1 into the three-dimensional subspace $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ of Θ and v_2 into the three-dimensional subspace $\left(\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right)$. This proves the lemma. □

9.3. **Parabolic subgroups.** The purpose of this subsection is to compute the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$. Specifically, we prove the following theorem.

Theorem 9.8. *The set $\mathcal{F}^{\underline{G}}_{geom}(P_{0,\underline{H}}, \underline{P})$ has three elements, $\{\underline{P}, \underline{P}'_L, \underline{P}'_R\}$. Moreover, $\underline{P}'_L \cap \underline{H}_7 = \underline{P}'_R \cap \underline{H}_7$ is a maximal parabolic of $\underline{H}_7$ of type A_2.*

The following proposition will be used in the proof of Theorem 9.8.

Proposition 9.9. *There are two $\underline{H}'_7$ orbits on isotropic two-dimensional subspaces of Θ, consisting of the orbit of null isotropic spaces and of non-null spaces. Moreover, the stabilizer of a non-null such space is not a parabolic subgroup of $\underline{H}'_7$.*

Proof. To see that there are at least two orbits, note that $\mathrm{Span}\{\epsilon_1, e_3\}$ is not null, whereas the span $\mathrm{Span}\{e_1, e_3^*\}$ is null.

To see that there are exactly two orbits, one argues as follows. First, suppose W is two-dimensional isotropic. We claim that there is $g \in \underline{H}'_7$ so that $gW \subseteq V_7$. From this claim, the lemma follows from Lemma 2.5 of [PWZ], by using the action of $G_2 \subseteq H'_7$.

To see that there is $g \in \underline{H}'_7$ with $gW \subseteq V_7$, write x, y for a basis of W. Applying Lemma 9.7, we can assume $x = \epsilon_1$. Because $(x, y) = 0$, we get that $y \in \left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)$, and thus may assume $y \in \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right) \subseteq V_7$. Acting by H'_6, it is then clear that we can move all of W into V_7, as claimed. This proves the first part of the proposition.

The second part of the proposition follows from the following lemma, which is easily proved once stated. □

Lemma 9.10. *Suppose that $W = \mathrm{Span}\{x, y\}$ is a two-dimensional isotropic but non-null subspace of Θ. Set $b = xy^*$ and $U(b) = \{z \in \Theta : bz = 0\}$.*

1. *The space $U(b)$ is a four-dimensional isotropic subspace of Θ, that comes equipped with the symplectic form $\langle z_1, z_2 \rangle$ defined by $z_1 z_2^* = \langle z_1, z_2 \rangle b$.*
2. *Denote by $Q_{U(b)}$ the subgroup of $\underline{H}_7$ that stabilizes $U(b)$. Then $Q_{U(b)} = L_{U(b)} V_{U(b)}$ is a parabolic subgroup of $\underline{H}_7$ with Levi subgroup $L_{U(b)} = \mathrm{GL}_1 \times \mathrm{GSpin}(5) = \mathrm{GL}_1 \times \mathrm{GSp}_4$. The unipotent radical $V_{U(b)}$ is abelian of dimension 5 and the map $Q_{U(b)} \to \mathrm{GSp}_4$ is induced by the action of $Q_{U(b)}$ on $U(b)$.*
3. *The stabilizer of W inside $\underline{H}_7$ is $L' V_{U(b)}$ with $L' = \mathrm{GL}_1 \times (\mathrm{GL}_2 \times \mathrm{GL}_2)^0 \subseteq \mathrm{GL}_1 \times \mathrm{GSp}_4$. Here $(\mathrm{GL}_2 \times \mathrm{GL}_2)^0$ is the subgroup of pairs $(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2$ with $\det(g_1) = \det(g_2)$.*

We are now in a position to prove Theorem 9.8.

Proof of Theorem 9.8. To compute $\mathcal{F}^{\underline{G}}_{geom}(P_{0,\underline{H}}, \underline{P})$ we apply Proposition 2.13, with the spaces $V_{\underline{Q}}$ of that proposition being the isotropic two-dimensional subspaces of $V = \Theta \oplus E$, $V_0 = \Theta$ and $V_1 = E$. Therefore, if $V^A_{\underline{Q}}$ is one of the subspaces of V in the conclusion of Proposition 2.13, then either $V^A_{\underline{Q}} \subseteq \Theta$ or $V^A_{\underline{Q}}$ is a direct sum of a one dimensional isotropic subspace of Θ and a one-dimensional isotropic subspace of E. Note that we cannot have $V^A_{\underline{Q}} \subseteq V_1 = E$, because both are two-dimensional and E is not isotropic as a subspace of V.

Suppose first that $V^A_{\underline{Q}} \subseteq V_0 = \Theta$. By Proposition 9.9, there are at most two $\underline{H}'_7$-orbits of such spaces, corresponding to whether $V^A_{\underline{Q}}$ is null or non-null. However, again by Proposition 9.9, because the stabilizer of a non-null subspace is not a parabolic subgroup of $\underline{H}'_7$, we must have that such a $V^A_{\underline{Q}}$ is null. By Lemma 9.5, this $\underline{H}'_7$-orbit corresponds to the element $\underline{P}$ of $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$.

Next suppose that $V^A_{\underline{Q}}$ is a direct sum of a one-dimensional isotropic subspace of Θ and a one-dimensional isotropic subspace of E. In this case, the one-dimensional isotropic subspace of E is either $(k, 0)$ or $(0, k)$, and note that both are stabilized by $\underline{H}$. Moreover, by Lemma 9.7, there is one $\underline{H}'_7$-orbit of isotropic lines of $V_0 = \Theta$, so the stabilizer in $\underline{H}'_7$ of such a line is a parabolic subgroup; it is easily seen to be a parabolic of type A_2. Consequently, we obtain the elements $\underline{P}'_L$ and $\underline{P}'_R$ of $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$, corresponding to the two different isotropic lines $(k, 0)$ and $(0, k)$ of E. This completes the calculation of $\mathcal{F}^{\underline{G}}_{geom}(P_{0,\underline{H}}, \underline{P})$. This finishes the proof of the theorem. □

Corollary 9.11. $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$.

Proof. By Theorem 9.8, it is enough to show that $\underline{P}'_L$ and $\underline{P}'_R$ do not belongs to the set $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P})$. Applying Theorem 9.8 again, we know that the unipotent radical of $\underline{P}'_L \cap \underline{H}$ (resp. $\underline{P}'_R \cap \underline{H}$) is not a subgroup of the unipotent radical of $\underline{P}'_L$ (resp. $\underline{P}'_R$). Then the corollary follows from Proposition 2.4. □

9.4. **The proof of Theorem 9.1.** In this subsection, we prove Theorem 9.1. First we compute the constant $s_0 = -c(1 - 2c^{\underline{H}}_{\underline{P}})$. In this case, although the unipotent group $\underline{N}$ is not abelian, it is easy to see that $c^{\underline{H}}_{\underline{P}} = \frac{8}{14}$. On the other hand, by Proposition 2.1(4), we have $c = \frac{7}{2}$. This implies that $s_0 = \frac{1}{2}$.

It remains to show that Claim 3.6 holds. But this is trivial since $\mathcal{F}^{\underline{G}}(P_{0,\underline{H}}, \underline{P}) = \{\underline{P}\}$. Theorem 9.1 follows from Proposition 3.7

To conclude this subsection, we make explicit the period integral $\mathcal{P}_H$ in terms of the isomorphism $\mathrm{PGL}_4 \simeq \mathrm{PGSO}_6$. More precisely, the map $H \to \underline{M} \simeq \mathrm{GL}_2 \times \mathrm{GSO}(6)$ induces

$$H/Z \to \mathrm{PGL}_2 \times \mathrm{PGSO}(6) \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_4.$$

We have already noted that $H/Z \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)/\Delta(\mathrm{GL}_1)$. Here, $\Delta(\mathrm{GL}_1)$ is the subgroup $(z1_2, z1_2)$ of $\mathrm{GL}_2 \times \mathrm{GL}_2$, for $z \in \mathrm{GL}_1$. In the following lemma, we make explicit the induced map

$$(\mathrm{GL}_2 \times \mathrm{GL}_2)/\Delta(\mathrm{GL}_1) \to \mathrm{PGL}_2 \times \mathrm{PGL}_4. \tag{9.3}$$

Lemma 9.12. *The map* (9.3) *is induced by the map* $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_2 \times \mathrm{GL}_4$ *given by* $(a, b) \mapsto (a, (\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)))$ *in* 2×2 *block form.*

Proof. From Lemma 9.4 and Lemma 9.5, the map $H \to \underline{M} = \mathrm{GL}_2 \times \mathrm{GSO}(6)$ is given by

$$(g, h, (\lambda_1, \lambda_2)) \mapsto (g, j(g, h, \lambda))$$

where $j(g, h, \lambda)$ acts on $V_6 = M_2(k) \oplus E$ as $(m, \mu) \mapsto (gmh', \lambda\mu)$. Up to the action of $Z = \mathrm{GL}_1 \times \mathrm{GL}_1$ which sits inside H as triples $(z, w, (zw, zw))$, we can assume that $(g, h, \lambda) = (g, h, (\det(g), \det(h)))$. Denote by $V_4 = V_2 \oplus V_2$ the decomposition of the defining representation of GL_4 into two GL_2 representations. Recall that our map $\mathrm{GL}_4 \to \mathrm{GSO}(6)$ is induced by the exterior square representation. The element $\left(\begin{smallmatrix} g & \\ & h \end{smallmatrix}\right) \in \mathrm{GL}_4$ acts on $V_6 = \wedge^2(V_4)$ by $(m, \mu) \mapsto (gmh', (\det(g), \det(h))\mu)$ for an appropriate choice of basis. The lemma follows. □

9.5. **The local result.** Let F be a local field (archimedean or p-adic), and D/F be the unique quaternion algebra if $F \neq \mathbb{C}$. Let

$$G(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F),\ H(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times \begin{pmatrix} a \end{pmatrix} \mid a, b \in \mathrm{GL}_2(F) \right\}$$

as in the previous subsections. Let π be an irreducible admissible smooth representation of $G(F)$ with trivial central character (we can also consider the nontrivial central character case, but we assume it is trivial here for simplicity), define the multiplicity

$$m(\pi)=\dim(\mathrm{Hom}_{H(F)}(\pi,1)).$$

If F is archimedean, $\mathrm{Hom}_{H(F)}(\pi,1)$ is the space of continuous homomorphisms. Similarly, if $F\neq\mathbb{C}$, we can define the quaternion version of the model (G_D,H_D) with $G_D(F)=\mathrm{GL}_2(D)\times\mathrm{GL}_1(D)$ and $H_D(F)\simeq\mathrm{GL}_1(D)\times\mathrm{GL}_1(D)$. We can also define the multiplicity $m(\pi_D)$ for all irreducible smooth representations of $G_D(F)$ with trivial central character.

Assume that F is p-adic. Let $\pi=\pi_1\otimes\pi_2$ (resp. $\pi_D=\pi_{1,D}\otimes\pi_{2,D}$) be an irreducible tempered representation of $G(F)$ (resp. $G_D(F)$) with trivial central character. We define the geometric multiplicity

$$m_{geom}(\pi)=c_{\pi_1}(1)c_{\pi_2}(1)+\Sigma_{T\in\mathcal{T}_{ell}(\mathrm{GL}_2)}|W(\mathrm{GL}_2,T)|^{-1}\int_{T(F)/Z_{\mathrm{GL}_2}(F)}D^{\mathrm{GL}_2(F)}(t)^2c_{\pi_1}(\begin{pmatrix}t&0\\0&t\end{pmatrix})\theta_{\pi_2}(t)dt.$$

Here $\mathcal{T}_{ell}(\mathrm{GL}_2)$ is the set of all the maximal elliptic tori of $\mathrm{GL}_2(F)$ up to conjugation (specifically, we can realize $\mathcal{T}_{ell}$ as the union of the tori $T_v(F)=\{\begin{pmatrix}a&bv\\b&a\end{pmatrix}\in\mathrm{GL}_2(F)|\ a,b\in F\}$ where v runs over $F^\times/(F^\times)^2$ with $v\neq 1$), $W(\mathrm{GL}_2,T)$ is the Weyl group, Z_{GL_2} is the center of GL_2, dt is the Haar measure on $T(F)/Z_{\mathrm{GL}_2}(F)$ such that the total volume is equal to 1, $D^{\mathrm{GL}_2(F)}(t)$ is the Weyl determinant, θ_{π_i} is the distribution character of π_i, $c_{\pi_1}(1)$ is the regular germ of θ_{π_1} at the identity element, and $c_{\pi_1}(\begin{pmatrix}t&0\\0&t\end{pmatrix})$ is the regular germ of θ_{π_1} at $\begin{pmatrix}t&0\\0&t\end{pmatrix}$. We refer the reader to Section 4.5 of [B15] for the definition of regular germs. Similarly, we can also define the quaternion version of the geometric multiplicity

$$m_{geom}(\pi_D)=\Sigma_{T_D\in\mathcal{T}_{ell}(\mathrm{GL}_1(D))}|W(\mathrm{GL}_1(D),T)|^{-1}\int_{T_D(F)/Z_{\mathrm{GL}_1(D)}(F)}D^{\mathrm{GL}_1(D)}(t_D)^2c_{\pi_{1,D}}(\begin{pmatrix}t_D&0\\0&t_D\end{pmatrix})\theta_{\pi_{2,D}}(t_D)dt_D.$$

Note the for the quaternion case, we don't need to include the regular germ at the identity element because the group is not quasi-split.

The proof of the following theorem follows from a similar (but easier) argument as in the Ginzburg-Rallis model case ([Wana], [Wanb], [Wan17]), we will skip it here. In fact, the argument for the current model is more similar to the argument for the "middle model" (which is a reduced model of the Ginzburg-Rallis model) defined in Appendix B of [Wana]. But it is easier since H is reductive.

Theorem 9.13. *(1) Assume that F is p-adic. Let $\pi=\pi_1\otimes\pi_2$ (resp. $\pi_D=\pi_{1,D}\otimes\pi_{2,D}$) be an irreducible tempered representation of $G(F)$ (resp. $G_D(F)$) with trivial central character. Then we have a multiplicity formula*

$$m(\pi)=m_{geom}(\pi),\ m(\pi_D)=m_{geom}(\pi_D).$$

(2) Assume that F is p-adic. Let $\pi=\pi_1\otimes\pi_2$ be an irreducible tempered representation of $G(F)$ with trivial central character, and let π_D be the Jacquet-Langlands correspondence of π from $G(F)$ to $G_D(F)$ if it exists; otherwise let $\pi_D=0$. Then we have

$$m(\pi)+m(\pi_D)=1.$$

In other words, the summation of the multiplicities over every tempered local Vogan L-packet is equal to 1.

(3) The statement in (2) also holds when $F=\mathbb{R}$.

(4) When $F=\mathbb{C}$, the multiplicity $m(\pi)=1$ for all irreducible tempered representation π of $G(F)$ with trivial central character.

As in the Ginzburg-Rallis model case, we can make the epsilon dichotomy conjecture which gives a relation between the multiplicity and some local epsilon factor. To be specific, let $\pi = \pi_1 \otimes \pi_2$ be an irreducible tempered representation of $G(F)$ with trivial central character, and let π_D be the Jacquet-Langlands correspondence of π from $G(F)$ to $G_D(F)$ if it exists; otherwise let $\pi_D = 0$. Then the conjecture states that

$$m(\pi) = 1 \iff \epsilon(1/2, \pi, \rho_X) = 1, \ m(\pi) = 0 \iff \epsilon(1/2, \pi, \rho_X) = -1.$$

Here $\rho_X = \wedge^2 \otimes std$ is the 12-dimensional representation of ${}^L G = \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ as in the previous subsections.

When $F = \mathbb{C}$, the above relation is trivial since $m(\pi) = \epsilon(1/2, \pi, \rho_X) = 1$ for all irreducible tempered representations of $G(F)$ with trivial central character. When $F = \mathbb{R}$, or when F is p-adic and π is not a discrete series, the relation can be proved by the same arguments as in the Ginzburg-Rallis model case in Section 6.2 (archimedean case) and Section 8.3 (p-adic case) of [Wanb].

Remark 9.14. *In general, we expect the results above hold for all generic representations of $G(F)$.*

10. The model $(\mathrm{Sp}_4, \mathrm{Sp}_2 \times \mathrm{GL}_1)$

10.1. **The global result.** Let $\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} |\ g^t J_{2n} g = J_{2n}\}$ be the symplectic group with $J_{2n} = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix}$ and w_n is the longest Weyl element in GL_n. We let $G = \mathrm{Sp}_4$ and $H = \{diag(a, h, a^{-1}) |\ a \in \mathrm{GL}_1, h \in \mathrm{Sp}_2\}$ be a maximal Levi subgroup of G. Similarly, we let $\underline{G} = \mathrm{Sp}_6$ and $\underline{H} = \{diag(a, h, a^{-1}) |\ a \in \mathrm{GL}_1, h \in \mathrm{Sp}_4\}$. We embed G into $\underline{G}$ via the map $g \in G \mapsto diag(1, g, 1) \in \underline{G}$. We also let B (resp. $\underline{B}$) be the upper triangular Borel subgroup of G (resp. $\underline{G}$), and $B_{\underline{H}} = \{diag(a, b, a^{-1}) |\ a \in \mathrm{GL}_1, b \in B\}$ be a Borel subgroup of $\underline{H}$. We use $\underline{A_0}$ to denote the diagonal elements of $\underline{G}$. It is a maximal split torus of $\underline{G}$ and of $\underline{H}$.

Let $\underline{P}' = \underline{M}'\underline{N}'$ (resp. $\underline{P}'' = \underline{M}'\underline{N}''$) be the upper triangular (resp. lower triangular) parabolic subgroup of $\underline{G}$ with $\underline{M}' = \underline{H}$. In particular, we have $G \subset \underline{M}'$. Then we let $\underline{P} = w\underline{P}'w^{-1}$, $\underline{M} = w\underline{M}'w^{-1}$, $\underline{N} = w\underline{N}'w^{-1}$ with $w = \mathrm{diag}(w_2, I_2, w_2)$. It is easy to see that $\underline{P} \cap \underline{H} = (\underline{M} \cap \underline{H}) \ltimes (\underline{N} \cap \underline{H})$ is a maximal parabolic subgroup of $\underline{H}$ with $\underline{M} \cap \underline{H} = H \times \mathrm{GL}_1$.

Proposition 10.1. *$\mathcal{F}^{\underline{G}}(B_{\underline{H}}, \underline{P}) = \{\underline{P}, \underline{P}', \underline{P}''\}$.*

Proof. This just follows from the definition of the set $\mathcal{F}^{\underline{G}}(B_{\underline{H}}, \underline{P})$ in Definition 2.3. Note that there are only six semistandard parabolic subgroups of $\underline{G}$ that are conjugated to $\underline{P}$. □

Let $\pi = \otimes_{v \in |k|} \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Like in the previous sections, by abusing of language, we also use π to denote the cuspidal automorphic representation $\pi \otimes 1$ of $\underline{M}(\mathbb{A})$. For $\phi \in \mathcal{A}_\pi$ and $s \in \mathbb{C}$, let $E(\phi, s)$ be the Eisenstein series on $\underline{G}(\mathbb{A})$.

Theorem 10.2. *Assume that there exists a local non-archimedean place $v \in |k|$ such that π_v is a generic representation of $G(k_v)$. If the period integral $\mathcal{P}_H(\cdot)$ is nonzero on the space of π, then there exists $\phi \in \mathcal{A}_\pi$ such that the Eisenstein series $E(\phi, s)$ has a pole at $s = 1$.*

Proof. We first compute the constant $s_0 = -c(1 - 2c_{\underline{P}}^{\underline{H}})$. It is easy to see that $c_{\underline{P}}^{\underline{H}} = \frac{2}{3}$. On the other hand, by Proposition 2.1, we have $c = 3$. This implies that $s_0 = 1$. It remains to show that Claim 3.6 holds. It follows from Proposition 10.1 and Corollary 3.4 (note that $\underline{H} \cap \underline{P}' = \underline{H} \cap \underline{P}'' = \underline{M}'$). The theorem follows from Proposition 3.7. □

Remark 10.3. *For $\underline{P}'$ and $\underline{P}''$, we have $c(1 - 2c_{\underline{P}'}^{\underline{H}}) = c(1 - 2c_{\underline{P}''}^{\underline{H}}) = 3(1 - 0) = 3 > s_0 = 1$. This confirms the claim 3.11.*

Proposition 10.4. *Theorem 10.2 implies Corollary 1.3(7).*

Proof. The proof is very similar to the proof of Proposition 5.5, we will skip it here. □

10.2. **The local result.** The proof of Theorem 1.4 for current model is the same as the proof of the model $(\mathrm{SO}_{2n}, \mathrm{SO}_{n+1} \times \mathrm{SO}_{n-1})$ in Section 5.2. We will skip it here.

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Department of Mathematics, University of California, San Diego, La Jolla, CA, 92093, USA
E-mail address: apollack@ucsd.edu

Department of Mathematics&Computer Science, Rutgers UniversityNewark, Newark, NJ 07102, USA
E-mail address: chen.wan@rutgers.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: zydor@umich.edu