THE QUATERNIONIC MAASS SPEZIALSCHAR ON SPLIT $\mathrm{SO}(8)$

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Abstract. The classical Maass Spezialschar is a Hecke-stable subspace of the level one holomorphic Siegel modular forms of genus two, i.e., on Sp_4, cut out by certain linear relations between the Fourier coefficients. It is a theorem of Andrianov, Maass, and Zagier, that the classical Maass Spezialschar is exactly equal to the space of Saito-Kurokawa lifts. We study an analogous space of quaternionic modular forms on split SO_8, and prove the analogue of the Andrianov-Maass-Zagier theorem. Our main tool for proving this theorem is the development of a theory of a Fourier-Jacobi expansion of quaternionic modular forms on orthogonal groups.

Contents

1. Introduction

The classical Saito-Kurokawa subspace of Siegel modular forms on PGSp_4 has many characterizations. One way to define the Saito-Kurokawa subspace is in terms of theta lifts from half-integral weight modular forms on SL_2 to $\mathrm{SO}_5 \simeq \mathrm{PGSp}_4$. The image of this theta lift can be characterized in terms Fourier coefficients, yielding the classical identification of the Saito-Kurokawa subspace with the Maass Spezialschar as recalled in Section 2. With this definition, it becomes an interesting question to ask whether (semi)-classical characterizations, such as the Maass relations, apply to generalizations of the Saito-Kurokawa lifting.

2020 *Mathematics Subject Classification.* 11F03; 11F30.

Key words and phrases. Saito-Kurokawa lift; quaternionic modular forms; Maass relations; Fourier-Jacobi expansion; periods of automorphic forms.

In the present work, we give an affirmative answer to this question for the theta lifting from Sp_4 to SO_8, and present a generalization of the theory of the Maass Spezialschar of Siegel modular forms to the setting of quaternionic modular forms (QMF) on SO_8. In the process, we develop a notion of Fourier-Jacobi coefficient for quaternionic modular forms on groups of type $\mathrm{SO}(4, n+2)$. We now summarize the results of this paper.

1.1. **The Fourier-Jacobi coefficient.** Fix $n \in \mathbf{Z}_{\geq 2}$ and let V be a non-degenerate quadratic space over $\mathbf{Q}$ of signature $(4, n+2)$. Assume V contains an isotropic subspace of dimension 4 and set $G = \mathrm{SO}(V)$. There is a special class of automorphic forms on G, called quaternionic modular forms, corresponding to minimal K-type vectors in the quaternionic discrete series on $G(\mathbf{R})$ [GW96]. Concretely, if $\ell \in \mathbf{Z}_{\geq 1}$ then a quaternionic modular form of weight ℓ is a vector valued automorphic function $\varphi\colon G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{V}_\ell$ where $\mathbf{V}_\ell \simeq \mathrm{Sym}^{2\ell}\mathbf{C}^2$ is the $(2\ell+1)$-dimensional irreducible representation of $\mathrm{SU}(2)$.

In analogy with the case of holomorphic modular forms, a weight ℓ quaternionic modular form φ satisfies a specific "Caucy-Riemann type" equation $D_\ell\varphi \equiv 0$ (see Definition 4.2). This differential equation leads to a semi-classical theory of Fourier coefficients which we now recall. Let $P = M_P N_P$ denote Heisenberg parabolic of G, defined as the stabilizer in G of a fixed isotropic two plane $U \subseteq V$. The Levi factor $M_P \simeq \mathrm{GL}(U) \times \mathrm{SO}(V_{2,n})$ where $V_{2,n} \subseteq V$ is a non-degenerate subspace of signature $(2, n)$. The characters of $N_P(\mathbf{Q})\backslash N_P(\mathbf{A})$ are indexed by ordered pairs $[T_1, T_2]$ consisting of two elements $T_1, T_2 \in V_{2,n}$. Given a quaternionic modular form φ, the results of [Wal03] and [Pol20] give a family of complex numbers

$$\{a_\varphi([T_1, T_2])\colon T_1, T_2 \in V_{2,n}^{\oplus 2}\} \subseteq \mathbf{C}$$

which we refer to as the *quaternionic Fourier coefficients of* φ (see Definition 4.7).

Our first main result gives an analogue of the classical Fourier Jacobi expansion of Siegel modular forms (see for example [PS69, Ch.4, §1]) in the setting of quaternionic modular forms on the group G. To be more precise, fix a line $\mathbf{Q}b_1 \subseteq U$ and let $Q = M_Q N_Q$ be the parabolic subgroup in G stabilizing $\mathbf{Q}b_1$. Given a quaternionic modular form φ on G, we define *the first Fourier Jacobi coefficient* $\mathrm{FJ}_\varphi(h)$ *of* φ via an integral

$$\mathrm{FJ}_\varphi(h) = \overline{\left\{ \int_{[N_Q]} \chi_{y_0}^{-1}(n)\varphi(nh)\, dn, (-ix+y)^{2\ell} \right\}_{K^0}}. \tag{1.1}$$

Here $\chi_{y_0} : N_Q(\mathbf{Q})\backslash N_Q(\mathbf{A}) \to \mathbf{C}^\times$ is a specific non-degenerate character of $N_Q(\mathbf{Q})\backslash N_Q(\mathbf{A})$, $\{x, y\}$ is a particular basis of $\mathbf{C}^2$ (so that $(-ix+y)^{2\ell} \in \mathbf{V}_\ell$), and $\{\,,\,\}_{K^0}$ is the (unique up to scaling) $\mathrm{SU}(2)$-invariant bilinear pairing on $\mathbf{V}_\ell$. The formula (1.1) naturally defines FJ_φ as an automorphic function on the stabilizer $H = \mathrm{Stab}_{M_Q}\{\chi_{y_0}\}$. Moreover, for our particular choice of χ_{y_0}, we have $H(\mathbf{R}) \simeq \mathrm{SO}(2, n+1)$, and so it makes sense to ask whether FJ_φ is (the automorphic function associated to) a holomorphic modular form on H. Our first main theorem settles this question in the affirmative.

Theorem 1.1. *Let the notation be as above, so that φ is a weight ℓ quaternionic modular form on G. Then FJ_φ is the automorphic function corresponding to a weight ℓ holomorphic modular form on H. Moreover, the classical Fourier coefficients of FJ_φ are finite sums of the quaternionic Fourier coefficients of φ.*

1.2. D_4 **modular forms.** The remaining theorems in this paper are specialized to the case of $n = 2$. In this case $G \simeq \mathrm{SO}_8$ is split and $H \simeq \mathrm{PGSp}_4$ is also split. Moreover, the quadratic space $V_{2,n}$ can be identified with the space of 2-by-2 rational matrices $M_2(\mathbf{Q})$. For a level one form φ, the Fourier coefficients $a_\varphi([T_1,T_2])$ can be non-zero only if $[T_1,T_2]$ is a pair consisting of 2-by-2 matrices T_1 and T_2 with integer entries.

In [Wei06], Weissman develops a theory of Fourier coefficients for quaternionic modular forms on the two-fold simply connected covering group $\mathrm{Spin}_8 \to G$. Theorem 1.1, while stated for orthogonal groups, applies equally well to the group Spin_8. Specializing Theorem 1.1 to this case, and applying a triality automorphism on Spin_8, we obtain the following. (The reason we apply triality is for application to Theorem 1.3, and is explained below.)

Theorem 1.2. *Suppose φ is a level one, weight ℓ cuspidal quaternionic modular form on Spin_8, with Fourier coefficients $a_\varphi([T_1,T_2])$. For integers a,b,c, define*

$$b_\varphi\left([a,b,c]\right) = \overline{a_\varphi\left(\left[\left(\begin{smallmatrix} a & 0 \\ b & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ c & 0\end{smallmatrix}\right)\right]\right)}.$$

Then the q-series

$$\sum_{a,b,c\in\mathbf{Z}} b_\varphi\left([a,b,c]\right) e^{2\pi i \,\mathrm{tr}\left(\left(\begin{smallmatrix} a & b/2 \\ b/2 & c\end{smallmatrix}\right) Z\right)}$$

is the Fourier expansion of a weight ℓ, level one cuspidal Siegel modular form on Sp_4.

1.3. **The quaternionic Maass spezialschar.** In [Pol21], a special lift $f \mapsto \theta^*(f)$ from cuspidal holomorphic Siegel modular forms f on Sp_4 to cuspidal quaternionic modular forms (QMFs) on split $\mathrm{SO}_8 =: G$ is developed. For an integer $\ell \geq 1$, we write $\mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ for the space of weight ℓ, level one QMFs on G (see Section 4.2). We define the *weight ℓ quaternionic Saito-Kurokawa subspace* SK_ℓ as the subspace of $\mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ spanned by $\theta^*(f)$ as f ranges over the space of level one cuspidal holomorphic Siegel modular forms on Sp_4.

Our main application of Theorem 1.2 is to characterize the space SK_ℓ in terms of Fourier coefficients. More precisely, we define a subspace MS_ℓ of $\mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ as consisting of φ for which the Fourier coefficients satisfy infinitely many linear relations, described below in Section 5. We call MS_ℓ the quaternionic Maass Spezialschar of weight ℓ. The definition of MS_ℓ is analogous to the classical definition of the Maass Spezialschar on PGSp_4.

Now, it follows easily from [Pol21] that, with our definition of MS_ℓ, there is an inclusion $\mathrm{SK}_\ell \subseteq \mathrm{MS}_\ell$. Using Theorem 1.2, we are able to prove that this inclusion is an equality. Specifically, given an element $\varphi \in \mathrm{MS}_\ell$, if we want to prove that $\varphi \in \mathrm{SK}_\ell$, then we must produce a level one Siegel modular form f on Sp_4 such that $\varphi = \theta^*(f)$. This f is exactly given by the q-series of Theorem 1.2. One arrives at the main theorem of this paper.

Theorem 1.3. *Suppose $\ell \geq 16$ is even. Then the inclusion $\mathrm{SK}_\ell \subseteq \mathrm{MS}_\ell$ is an equality.*

Let us remark explicitly that the composite map $f \mapsto \theta^*(f) \mapsto \mathrm{FJ}_{\theta^*(f)}$, from level one Siegel modular forms f on Sp_4 to itself is not the identity map. In order to recover f from $\varphi = \theta^*(f)$, we instead:

(1) pullback φ to Spin_8, to obtain a QMF φ_1;
(2) apply a triality automorphism $\sigma \in C_3 \subseteq S_3$ to φ_1 to obtain $\varphi_2 = \sigma(\varphi_1)$;
(3) take a Fourier-Jacobi coefficient of φ_2, $f = \mathrm{FJ}_{\varphi_2}$.

One could also theta lift from Siegel modular forms on Sp_4 to quaternionic modular forms on $\mathrm{SO}(4,n+2)$ with $n > 2$; this is partially developed in [Pol21]. However, in this more general setting, we do not know how to characterize the image of the theta lift in terms of Fourier coefficients.

1.4. **Periods of quaternionic modular forms.** Theorem 1.3 characterizes the Saito-Kurokawa subspace in terms of Fourier coefficients. This subspace can also be characterized in terms of periods. We now setup this result.

Let $L \subseteq V$ be a fixed choice of a split, unimodular lattice. For $v_1, v_2 \in L$ spanning a positive-definite two plane, define $H_{v_1,v_2} \subseteq G$ as the pointwise stabilizer of the vectors v_1 and v_2. Given a quaternionic modular form φ on G of weight ℓ, we define

$$\mathcal{P}_{v_1,v_2}(\varphi) = \int_{H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})} \varphi(h)\, dh$$

the period of φ over the $H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})$. Let $D(v_1,v_2)$ be the integer $(v_1,v_2)^2 - 4(v_1,v_1)(v_2,v_2)$.

Finally, let $k_0 \in O(L)(\mathbf{Z})$ have determinant -1. Say that φ is in the plus subspace if $\varphi(g) = \varphi(k_0 g k_0^{-1})$. (The right-hand side of this expression is independent of the choice of k_0.) One has that SK_ℓ is contained in the plus subspace.

Theorem 1.4. *Suppose $\ell \geq 22$ is even, and $\varphi \in \mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ is a Hecke eigenform in the plus subspace. Then $\varphi \in \mathrm{SK}_\ell$ if and only if there exists $v_1, v_2 \in L$ such that $\mathcal{P}_{v_1,v_2}(\varphi) \neq 0$ and $D(v_1,v_2)$ is odd and square-free.*

Characterization of theta lifts in terms of periods are well-studied. The contribution of Theorem 1.4 is simply its precision. In more detail, let $L(\varphi)$ be a "lift" of φ from the special orthogonal group $\mathrm{SO}(V)$ to the orthogonal group $O(V)$ as in Proposition 8.2, and let $\Pi_{L(\varphi)}$ be the automorphic representation generated by $L(\varphi)$. In Theorem 8.10 we verify that φ is a quaternionic Saito-Kurokawa lift if and only if the representation $\Pi_{L(\varphi)}$ has a nonzero theta lift to Sp_4. Now, it is standard that $\Pi_{L(\varphi)}$ has a nonzero theta lift to Sp_4 if and only the representation $\Pi_{L(\varphi)}$ is distinguished by a subgroup of the form H_{v_1,v_2}, for some v_1, v_2. Theorem 1.4 refines this standard computation by showing that it is enough to check if the function φ itself is distinguished.

We remark that one should also be able to characterize the Hecke eigenforms in the quaternionic Maass Spezialschar via a pole at $s = 2$ in their standard L-functions. See [GJS09], [GT11], [Yam14].

The remainder of the paper is organized as follows. In Section 2, we recall some well known facts about the classical Saito-Kurokawa lift and Maass Spezialschar. In Section 3, we define various notations that we use throughout the paper. In Section 4, we define quaternionic modular forms on G and review some results about the Fourier expansion of quaternionic modular forms. In Section 5, we define the quaternionic Saito-Kurokawa lifting and quaternionic Maass Spezialschar, and show that the quaternionic Saito Kurokawa subspace SK_ℓ is contained in the quaternionic Maass Spezialschar MS_ℓ. In Section 6, we discuss a conjectural Dirichlet series for the L-functions of irreducible, cuspidal, quaternionic automorphic representations of G and show that this conjecture is satisfied by the Saito-Kurokawa lifts. In Section 7, we show that the Fourier Jacobi coefficient of a quaternionic modular form on G is a Siegel modular form on Sp_4 of the correct weight (see Corollary 7.6), and using this result we obtain $\mathrm{MS}_\ell \subseteq \mathrm{SK}_\ell$. In Section 8, we collect results from the literature on theta lifts that we use in later sections. In Section 9, we prove Theorem 1.4. The paper ends with three appendices: Appendix A contains results about triality on Spin_8 and Appendices B and C contain technical results used in the proof of Theorem 1.4.

1.5. **Acknowledgments.** This project began at Research Innovations and Diverse Collaborations workshop at the University of Oregon, which was supported by NSF CAREER grant DMS-1751281 (E. Eischen) and NSA MSP conference grant Grant H98230-21-1-0029 (E. Eischen). The authors thank the organizers of this workshop for their generous support in establishing this collaboration. Work was also done at a SQuaRE hosted by the American Institute for Mathematics. The authors thank AIM for providing a supportive and mathematically rich environment.

JJL was partially supported by the Renfrew Faculty Fellowship from the University of Idaho College of Science. IN was partially supported by a FRQNT Scholarship by Fonds de Recherche du Québec and by NSF Grant No. DMS-1928930 while the author was in residence at MSRI in Berkeley, California, during the Spring 2023 semester. AP has been supported by the Simons Foundation via Collaboration Grant number 585147 and by the NSF via grant numbers 2101888 and 2144021. MR was partially supported by the AMS Simons Travel Grant program.

2. The classical Saito-Kurokawa lift and Maass Spezialschar

In this introductory section, we define the classical Maass Spezialschar and give different characterizations for it, including one via the Saito-Kurokawa lift. Most of the characterizations given here for the Maass space of Siegel modular forms will be generalized to quaternionic forms on SO(8) in subsequent sections.

Let $\mathcal{M}_k^g(\Gamma)$ denote the space of Siegel modular forms of even weight k and level $\Gamma \subset \mathrm{Sp}_{2g}(\mathbf{Z})$. Recall that $F \in \mathcal{M}_k^g(\mathrm{Sp}_{2g}(\mathbf{Z}))$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in S_g(\mathbf{Z})^\vee} a_F(T) e^{2\pi i \operatorname{Tr}(ZT)},$$

where

$$\begin{aligned} S_g(\mathbf{Z})^\vee &= \{T \in S_g(\mathbf{R}) : \operatorname{Tr}(XT) \in \mathbf{Z} \text{ for all } X \in S_g(\mathbf{Z})\} \\ &= \left\{ T \in S_g(\mathbf{Q}) : t_{ii} \in \mathbf{Z},\ t_{ij} \in \frac{1}{2}\mathbf{Z} \text{ for } i \neq j \right\}. \end{aligned} \tag{2.1}$$

Here $S_g(\mathbf{R})$ (resp. $S_g(\mathbf{Q})$) denotes symmetric $g \times g$ matrices with coefficients in $\mathbf{R}$ (resp. $\mathbf{Q}$). The Fourier coefficients $a_F(T)$ are 0 if T is not positive semi-definite. If $g = 2$ and $T \in S_2(\mathbf{Z})^\vee$ is positive semi-definite, we can write

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \ \ a, b, c \in \mathbf{Z}, \quad a, c \geq 0, \ \ b^2 \leq 4ac.$$

Following the notation of [EZ85], we let $A_F(a, b, c) := a_F(T)$. We will now define a subspace of $\mathcal{M}_k^2(\mathrm{Sp}_4(\mathbf{Z}))$ called the *Maass Spezialschar*, which we will also call "Maass subspace".

Definition 2.1. The Maass Spezialschar, alternatively Maass subspace, is the subspace $\mathrm{MS}_k(\mathrm{Sp}_4(\mathbf{Z}))$ of $S_k^2(\mathrm{Sp}_4(\mathbf{Z}))$ given by all cusp forms F such that

$$a_F \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \sum_{d \mid \gcd(a,b,c)} d^{l-1} a_F \begin{pmatrix} ac/d^2 & b/(2d) \\ b/(2d) & 1 \end{pmatrix}.$$

This space was first studied by Maass [Maa78, Maa79] building on experimental evidence found by Resnikoff and Saldana [RSn74].

The Maass spezialschar is isomorphic to the *Kohnen plus space* $S^+_{k-1/2}$, which for k even is the space of cusp forms of weight $k-\frac{1}{2}$ on $\Gamma_0(4)$ having a Fourier development of the form

$$g(z) = \sum_{n\geq 1} c(n)q^n, \text{ with } \qquad c(n) = 0 \text{ unless } n \equiv 0 \text{ or } 3 \pmod 4.$$

Proposition 2.2. *(Eichler, Zagier). There is a Hecke-equivariant linear isomorphism from* $S^+_{k-1/2}$ *to* $\mathrm{MS}_k(\mathrm{Sp}_4(\mathbf{Z}))$ *sending*

$$g(z) = \sum_{n\geq 1} c(n)q^n \in S^+_{k-1/2}$$

to

$$\sum_{T>0} A_F(a,b,c)e^{2\pi i\,\mathrm{Tr}(ZT))},$$

where

$$A_F(a,b,c) = \sum_{d|gcd(a,b,c)} d^{l-1}c\Big(\frac{4ac-b^2}{d^2}\Big).$$

The map from the Proposition is called the *Saito-Kurokawa lift*. It has the kernel function

$$\Omega_k(Z,\tau) = \sum_{N>0} N^{3/2-k}w_{N,k}(Z)e^{2\pi iN\tau}, \quad \text{with } Z \in \mathcal{H}_2,\ \tau \in \mathcal{H},$$

where

$$w_{N,k}(Z) = \sum_{\substack{(a,b,c,d,e)\in\mathbf{Z}^5 \\ 4bd-c^2-4ae=N}} \frac{1}{(a(\tau\tau'-z^2)+b\tau+cz+d\tau'+e)^k}, \qquad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}.$$

More precisely, for $k \geq 8$ even, $\Omega_k(Z,\tau)$ belongs to $\mathcal{M}^2_k(\mathrm{Sp}_4(\mathbf{Z}))$ as a function of Z, and to $S_{k-1/2}(\Gamma_0(4))$ as a function of τ. As explained in Section 3 of [Zag81], the Saito-Kurokawa lift $\mathcal{SK} : S_{k-1/2}(\Gamma_0(4)) \to \mathcal{M}^2_k(\mathrm{Sp}_4(\mathbf{Z}))$ is given via the Petersson inner product by

$$\mathcal{SK}(h)(Z) = \int_{\Gamma_0(4)\backslash\mathcal{H}} h(u+iv)\Omega_k(Z,u-iv)v^{k-1/2}\frac{dudv}{v^2}.$$

The function Ω_k is the holomorphic projection of a theta series attached to $\mathrm{SO}(2,3)$ [Zag81], and the Saito-Kurokawa lift is the theta lift from $\widetilde{\mathrm{SL}_2}$ to $\mathrm{SO}(2,3) \simeq \mathrm{PGSp}_4$.

Let $F \in S^2_k(\mathrm{Sp}_4(\mathbf{Z}))$. From the point of view of general theta correspondences, the fact that $F = \mathcal{SK}(h)$ for some $h \in S_{k-1/2}(\Gamma_0(4))$ is equivalent to the fact that the theta lift of F to $\mathrm{SL}_2(\mathbf{Z})$ is non-zero (see [PS83, Theorem 2.2]). The space $S^+_{k-1/2}$ is related to the space $S_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ of weight $2k-2$ cusp forms for $\mathrm{SL}_2(\mathbf{Z})$ by the Shimura lifting $\mathcal{S} : S^+_{k-1/2} \to S_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$, which can be given as

$$\mathcal{S} : \sum_{n\geq 1} c(n)q^n \mapsto \sum_{n\geq 1}(\sum_{d|n} d^{k-1}c(n^2/d^2))q^n.$$

Moreover, the isomorphism between $S_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ and $S^+_{k-1/2}$ is Hecke equivariant [Koh80].

The fact that $S_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ and $S^+_{k-1/2}$ are isomorphic as Hecke modules can be used, together with Proposition 2.2, to prove the following theorem, which is proved in[Koh07, Theorem 16] and gives an alternate characterization of the Maass subspace.

Theorem 2.3. *Let k be even. Then for each normalized Hecke eigenform $f \in S_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ there exists a cuspidal Hecke eigenform $F \in S_k^2(\mathrm{Sp}_4(\mathbf{Z}))$ (uniquely determined up to a non-zero scalar) such that*

$$L_{spin}(F,s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s),$$

where $L(f,s)$ is the L-function of f and both L-functions are given in the classical normalization. The space generated by these F's is the Maass Spezialschar.

For k even the Maass Spezialschar $\mathrm{MS}_k(\mathrm{Sp}_4(\mathbf{Z}))$ is also isomorphic to the space $J_{k,1}$ of Jacobi forms of weight k and index 1. We can see this in two ways. The first way follows from the theorem below [EZ85, Theorem 5.4], which states that $J_{k,1}$ is isomorphic to the Kohnen plus space. But by Proposition 2.2 this space is isomorphic to the Maass Spezialschar, hence the Maass Spezialschar is isomorphic to $J_{k,1}$.

Theorem 2.4. *For k even, the assignment sending*

$$g(z) = \sum_{n\geq 1} c(n)q^n \in S^+_{k-1/2}$$

to

$$\sum_{\substack{n,r\in\mathbf{Z}\\ 4n\geq r^2}} c(4n-r^2)q^n\zeta^r \in J_{k,1}$$

gives a Hecke-equivariant isomorphism between $S^+_{k-1/2}$ and $J_{k,1}$.

Alternatively, one can associate to a Siegel modular form in the Maass space its first Fourier-Jacobi coefficient, and this gives an isomorphism between $\mathrm{MS}_k(\mathrm{Sp}_4(\mathbf{Z}))$ and $J_{k,1}$. The inverse of the Saito-Kurokawa lift is actually given by composing this assignment with the one of Theorem 2.4 (see [EZ85] for more details).

3. Preliminaries

In this section, we set up explicit notation for working with quaternionic modular forms on special orthogonal groups.

3.1. **The underlying quadratic space.** Let $n \in \mathbf{Z}_{\geq 2}$ and let (V,q) denote a non-degenerate quadratic space over $\mathbf{Q}$ of signature $(4, n+2)$. Write $(x,y) = q(x+y) - q(x) - q(y)$ for the bilinear form associated to q and assume that V contains an isotropic subspace of dimension 4. We fix a two-dimensional isotropic subspace $U \subseteq V$. Since (V,q) is non-degenerate we may fix a second isotropic 2-plane $U^\vee$ such that $(\cdot,\cdot)$ defines a perfect pairing between U and $U^\vee$. We write $V_{2,n}$ for the orthogonal complement of $U + U^\vee$ in V so that $(V_{2,n}, q)$ is a quadratic space of signature $(2,n)$ and

$$V = U \oplus V_{2,n} \oplus U^\vee. \tag{3.1}$$

Let $\{b_1, b_2\}$ denote a fixed basis of U and write $\{b_{-1}, b_{-2}\}$ for the corresponding dual basis of $U^\vee$ so that $(b_i, b_{-j}) = \delta_{i,j}$ for $i,j = 1,2$. Then V admits an orthogonal decomposition

$$V = \mathbf{Q}b_1 \oplus V_{3,n+1} \oplus \mathbf{Q}b_{-1} \tag{3.2}$$

where $V_{3,n+1} = (\mathbf{Q}b_1 + \mathbf{Q}b_{-1})^\perp$ is a quadratic space of signature $(3, n+1)$. Our assumption that V contains an isotropic 4-plane implies that there exists vectors $b_3, b_4, b_{-3}, b_{-4} \in V_{2,n}$ such that $\{b_1, b_2, b_3, b_4\}$ and $\{b_{-1}, b_{-2}, b_{-3}, b_{-4}\}$ give dual bases for a pair of isotropic 4-planes which are in perfect duality under $(\cdot,\cdot)$.

Let $y_0 = b_3 + b_{-3}$ and $y_1 = b_4 + b_{-4}$ and write V_2^+ for the $\mathbf{Q}$-rational subspace of $V_{2,n}$ spanned by y_0 and y_1. Let $V_n^-(\mathbf{R})$ denote the orthogonal complement of $V_2^+(\mathbf{R}) := V_2^+ \otimes_{\mathbf{Q}} \mathbf{R}$ in $V_{2,n}(\mathbf{R}) := V_{2,n} \otimes_{\mathbf{Q}} \mathbf{R}$ so that

$$V_{2,n}(\mathbf{R}) = V_2^+(\mathbf{R}) \oplus V_n^-(\mathbf{R}). \tag{3.3}$$

For $j = 1, 2$ we set $u_j = (b_j + b_{-j})/\sqrt{2} \in V(\mathbf{R})$ and $u_{-j} = (b_j - b_{-j})/\sqrt{2} \in V(\mathbf{R})$. Thus $\{u_1, u_2, u_{-1}, u_{-2}\}$ is an orthonormal basis of $V_{2,n}(\mathbf{R})^\perp$. Similarly we set $v_1 = (b_3 + b_{-3})/\sqrt{2}$, $v_2 = (b_4 + b_{-4})/\sqrt{2}$, $v_{-1} = (b_3 - b_{-3})/\sqrt{2}$ and $v_{-2} = (b_4 - b_{-4})/\sqrt{2}$.

For much of the paper, we will be working in the case that V is split over $\mathbf{Q}$ of dimension 8. In this case, we assume that V is endowed with an integral structure, that is unimodular for the pairing $(\cdot, \cdot)$. More specifically, we assume that V_0 is endowed with a unimodular integral structure, and that $V(\mathbf{Z}) = \mathbf{Z}b_1 \oplus \mathbf{Z}b_2 \oplus V_0(\mathbf{Z}) \oplus \mathbf{Z}b_{-1} \oplus \mathbf{Z}b_{-2}$.

3.2. **Octonions.** In this case that $\dim(V) = 8$, we will also have occasion to identify V with the split octonions over $\mathbf{Q}$.

Definition 3.1. An *octonion algebra* Θ is an eight dimensional algebra over a field F, neither associative nor commutative, such that:

1. the multiplication map $\cdot : \Theta \times \Theta \to F$ is distributive,
2. there exists an element $1 \in \Theta$ with $1 \cdot x = x \cdot 1$ for each $x \in \Theta$,
3. there exists a non-degenerate quadratic form $n_\Theta : \Theta \to F$,
4. $n_\Theta(x \cdot y) = n_\Theta(x) n_\Theta(y)$ for all $x, y \in \Theta$.

We will refer to the map n_Θ as the *norm map*. One can show that it is possible to define an involution $*$ on Θ, i.e. a map $* : \Theta \to \Theta$ such that $(x \cdot y)^* = y^* \cdot x^*$. Moreover, $x + x^* \in F \cdot 1$ for all $x \in \Theta$. We can then define the *trace map*

$$tr_\Theta : \Theta \to F$$

by $tr_\Theta(x) \cdot 1 = x + x^*$. We also have a trilinear form

$$(\cdot, \cdot, \cdot) : \Theta \times \Theta \times \Theta \to F$$

given by

$$(x_1, x_2, x_3) := tr_\Theta(x_1 \cdot (x_2 \cdot x_3)).$$

Remark 3.2. Some properties of the trilinear form are:

1. $tr_\Theta(x_1 \cdot (x_2 \cdot x_3)) = tr_\Theta((x_1 \cdot x_2) \cdot x_3)$, even if $x_1 \cdot (x_2 \cdot x_3)$ is not necessarily equal to $(x_1 \cdot x_2) \cdot x_3$;
2. $(x_1, x_2, x_3) = (x_3, x_1, x_2) = (x_2, x_3, x_1)$, i.e. the trilinear form is invariant under cyclic permutations;
3. $(x_1, x_2, x_3) = tr_\Theta(x_1 \cdot (x_2 \cdot x_3)) = tr_\Theta((x_3^* \cdot x_2^*) \cdot x_1^*) = tr_\Theta(x_3^* \cdot (x_2^* \cdot x_1^*)) = (x_3^*, x_2^*, x_1^*)$.

We now construct the so-called *Zorn model* $\mathbb{O}$, which gives an example of an octonion algebra. Let V_3 be the three-dimensional defining representation of SL_3 and let $V_3^\vee$ be its dual representation. Then let

$$\mathbb{O} := \left\{ \begin{pmatrix} a & v \\ \phi & d \end{pmatrix} : \ a, d, \in F; \ v \in V_3, \ \phi \in V_3^\vee \right\},$$

with multiplication

$$\begin{pmatrix} a & v \\ \phi & d \end{pmatrix} \cdot \begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix} := \begin{pmatrix} aa' + \phi'(v) & av' + d'v - \phi \wedge \phi' \\ a'\phi + d\phi' + v \wedge v' & \phi(v') + dd' \end{pmatrix}.$$

Here an explicit identification $\wedge^2 V_3 \simeq V_3^\vee$ and $\wedge^2 V_3^\vee \simeq V_3$ is being made.

The norm form is

$$n_{\mathbb{O}}\left(\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)\right) := ad - \phi(v).$$

The involution on $\mathbb{O}$ is

$$\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}^* := \begin{pmatrix} d & -v \\ -\phi & a \end{pmatrix},$$

hence $tr_{\mathbb{O}}(\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)) = a + d$.

Let e_1, e_2, e_3 be the standard basis of V_3 and e_1^*, e_2^*, e_3^* the dual basis of $V_3^\vee$. A basis for $\mathbb{O}$ is given by the following matrices:

$$e_j := \begin{pmatrix} 0 & e_j \\ 0 & 0 \end{pmatrix},\ j = 1, 2, 3,$$

where we are abusing notation;

$$e_j^* := \begin{pmatrix} 0 & 0 \\ e_j^* & 0 \end{pmatrix},\ j = 1, 2, 3,$$

where we again abuse notation; and the matrices

$$\epsilon_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \epsilon_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

To obtain an eight-dimensional quadratic space (V, q) from Θ, we identify V with Θ and set $q : V \to \mathbf{Q}$ the quadratic form as $q(x) = -n_{\mathbb{O}}(x)$. We define the b_i via the identification of lists

$$(b_1, b_2, b_3, b_4, b_{-4}, b_{-3}, b_{-2}, b_{-1}) = (e_1, e_3^*, \epsilon_2, e_2^*, e_2, -\epsilon_1, e_3, e_1^*). \tag{3.4}$$

3.3. **Algebraic groups, their Lie algebras, and parabolic subgroups.** Now let n be arbitrary and let G denote the special orthogonal group of (V, q), which we assume acts on the left of V. Let Q denote the parabolic subgroup of G defined as the stabilizer in G of the line $\mathbf{Q}b_1$. The *Heisenberg parabolic* P is defined as the subgroup of G stabilizing the space $U = \mathrm{Span}\{b_1, b_2\}$. Define M_Q to be the Levi subgroup of Q that stabilizes $\mathbf{Q}b_{-1}$ and define the Levi factor $M_P \le P$ as the subgroup stabilizing the subspace $U^\vee = \mathrm{Span}\{b_{-1}, b_{-2}\}$. Through its action on the decomposition (3.1) the subgroup M_P is identified with the product $\mathrm{GL}(U) \times \mathrm{SO}(V_{2,n})$. Similarly the action of M_Q on the decomposition (3.2) yields an identification $M_Q \simeq \mathrm{GL}(\mathbf{Q}b_1) \times \mathrm{SO}(V_{3,n+1})$. Let N_P and N_Q denote the unipotent radical of P and Q, respectively. We set $Z = [N_P, N_P]$, which is one-dimensional and the center of N_P.

The Lie algebra of G can be identified with $\wedge^2 V$, where $\wedge^2 V$ acts on V via

$$(v \wedge w) \cdot x = (w, x)v - (v, x)w. \qquad (v, w, x \in V)$$

We note that $\mathrm{Lie}(Z) = \mathbf{Q}\text{-span}\{b_1 \wedge b_2\}$. As an M_P module, the Lie algebra of the quotient $N_P^{\mathrm{ab}} = N_P/Z$ is identified as

$$U \otimes_{\mathbf{Q}} V_{2,n} \xrightarrow{\sim} \mathrm{Lie}(N_P^{\mathrm{ab}}), \quad b_1 \otimes w_1 + b_2 \otimes w_2 \mapsto b_1 \wedge w_1 + b_2 \wedge w_2 + \mathrm{Lie}(Z). \tag{3.5}$$

Let $\psi : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}^\times$ denote the standard additive character. For $y, y' \in V_{2,n}$, let $\varepsilon_{[y,y']}$ denote the unique automorphic character of $N_P(\mathbf{A})$ satisfying $\varepsilon_{[y,y']}(\exp(b_1 \wedge w + b_2 \wedge w')) = \psi((y, w) + (y', w'))$ for all $w, w' \in V_{2,n}(\mathbf{A})$. We have an M_P-equivariant identification

$$U^\vee \otimes_{\mathbf{Q}} V_{2,n} \xrightarrow{\sim} \mathrm{Hom}(N_P(\mathbf{Q})\backslash N_P(\mathbf{A}), \mathbf{C}^1), \qquad b_{-1} \otimes y + b_{-2} \otimes y' \mapsto \varepsilon_{[y,y']} \tag{3.6}$$

Similarly, the abelian Lie algebra $\mathrm{Lie}(N_Q)$ is identified as an M_Q-module via the map

$$\mathbf{Q}b_1 \otimes_{\mathbf{Q}} V_{3,n+1} \xrightarrow{\sim} \mathrm{Lie}(N_Q), \quad b_1 \otimes w \mapsto b_1 \wedge w.$$

Let $\chi_y \colon N_Q(\mathbf{A}) \to \mathbf{C}^\times$ denote the automorphic character satisfying $\chi_y(\exp(b_1 \wedge v)) = \psi((v, y))$ for all $v \in V_{3,n+1}(\mathbf{A})$. We have an M_Q-equivariant identification

$$\mathbf{Q}b_{-1} \otimes_{\mathbf{Q}} V_{3,n+1} \xrightarrow{\sim} \mathrm{Hom}(N_Q(\mathbf{Q})\backslash N_Q(\mathbf{A}), \mathbf{C}^1), \qquad b_{-1} \otimes y \mapsto \chi_y. \tag{3.7}$$

We denote by G' the spin group, defined as follows:

$$G' = \{g = (g_1, g_2, g_3) \in \mathrm{SO}(\mathbb{O}, n_{\mathbb{O}})^3 : (g_1x_1, g_2x_2, g_3x_3) = (x_1, x_2, x_3) \forall x_j \in \mathbb{O}\}.$$

The association $g \mapsto g_1$ defines a map $G' \to G$ when $\dim(V) = 8$. We set $P' = M_{P'}N_{P'}$ the Heisenberg parabolic of G' defined as the inverse image of P under the map $G' \to G$. See Appendix A for more on $M_{P'}$ and $N_{P'}$.

3.4. **Compact subgroups.** We now define Cartan involutions and maximal compact subgroups of the groups with which we work.

In the case of $G = \mathrm{SO}(V)$, we proceed as follows. Let $\iota_{2,n} : V_{2,n}(\mathbf{R}) \to V_{2,n}(\mathbf{R})$ be the involution that is $+1$ on $V_2^+(\mathbf{R})$ and -1 on $V_n^-(\mathbf{R})$. Let $\iota : V(\mathbf{R}) \to V(\mathbf{R})$ be the involution with $\iota(b_j) = b_{-j}$ for $j \in \{1, 2, -1, -2\}$ and ι restricted to $V_{2,n}(\mathbf{R})$ is $\iota_{2,n}$. Note that $\iota \in O(V)(\mathbf{R})$. If V^+ is the $\iota = 1$ subspace, and V^- is the $\iota = -1$ subspace, then $V(\mathbf{R}) = V^+ \oplus V^-$, and we define $K \subseteq G(\mathbf{R})$ the subgroup that preserves the decomposition $V(\mathbf{R}) = V^+ \oplus V^-$. Note that $K = S(O(V^+) \times O(V^-))$ and has identity component $K^0 = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$. The corresponding Cartan involution Θ_ι on $\mathrm{Lie}(G(\mathbf{R})) \simeq \wedge^2 V(\mathbf{R})$ is defined as $\Theta_\iota(u \wedge v) = \iota(u) \wedge \iota(v)$.

To define quaternionic modular forms on G, we need a distinguished map

$$K^0 \to \mathrm{SU}(2)/\mu_2. \tag{3.8}$$

To see that we have such a map, recall that V^+ is four-dimensional, and thus there is an exceptional isomorphism $\mathrm{SO}(V^+) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$. Mapping to the first $\mathrm{SU}(2)/\mu_2$ in this pair, we obtain a map $K^0 \to \mathrm{SO}(V^+) \to \mathrm{SU}(2)/\mu_2$. To pick out this "first" $\mathrm{SU}(2)$, and its irreducible 3-dimensional representation, we make a Lie algebra argument. Namely, we have $V^+ = \mathrm{Span}\{u_1, u_2, v_1, v_2\}$, and we set (as in [Pol22a, Ch. 8])

- $e^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 - iv_2)$
- $h^+ = i(u_1 \wedge u_2 + v_1 \wedge v_2) = \frac{1}{2}(u_1 - iu_2) \wedge (u_1 + iu_2) + \frac{1}{2}(v_1 - iv_2) \wedge (v_1 + iv_2)$
- $f^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 + iv_2)$.

This gives us our "first" $\mathfrak{sl}_2$. The "second" $\mathfrak{sl}_2$ is obtained by replacing v_2 with $-v_2$ in the above formulas: That is, it has basis

- $e'^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 + iv_2)$
- $h'^+ = i(u_1 \wedge u_2 - v_1 \wedge v_2)$
- $f'^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 - iv_2)$.

The adjoint action of K^0 on the first $\mathfrak{sl}_2$ defines our distinguished three-dimensional representation of K^0. We choose a basis x, y of $\mathbf{C}^2 = V_2$ so that $\mathrm{Sym}^2(V_2)$ is identified with the first $\mathfrak{sl}_2$ via $e^+ = -x^2$, $h^+ = 2xy$, $f^+ = y^2$. This gives our distinguished map $K^0 \to \mathrm{SU}(2)/\mu_2$. For an integer ℓ at least 1, we write $\mathbf{V}_\ell$ for the highest weight quotient of the ℓ^{th} symmetric power of this three-dimensional representation of K^0. In other words, $\mathbf{V}_\ell = \mathrm{Sym}^{2\ell}(V_2)$ as a representation of $\mathrm{SU}(2)/\mu_2$, pulled back to K^0. It has a basis $x^{2\ell}, x^{2\ell-1}y, \ldots, xy^{2\ell-1}, y^{2\ell}$.

In the case of G', the spin group, we define a maximal compact subgroup $K' \subseteq G'(\mathbf{R})$ to be inverse image of K or K^0 in $G'(\mathbf{R})$. In this case, the Cartan involution on $\wedge^2\mathbb{O}$ is given by Θ_ι, where $\iota(b_j) = b_{-j}$ and $\Theta_\iota(u \wedge v) = \iota(u) \wedge \iota(v)$. We have an S_3 action on G', and K' is stable by this S_3 action: this follows from the work in Theorem A.1. Moreover, if $\mathfrak{sl}_2$ denotes the "first" $\mathfrak{sl}_2$ in $\mathrm{Lie}(G'(\mathbf{R})) \simeq \wedge^2\mathbb{O}$, then the S_3-action is trivial on this $\mathfrak{sl}_2$. This follows from Theorem A.8 and a statement in the last paragraph of [cDD+22, Section 2.3].

4. Holomorphic and Quaternionic Modular Forms

In this section, we define holomorphic and quaternionic modular forms on orthogonal groups. In Theorem 4.5, we recall a result of Pollack on generalized Whittaker functions necessary for the existence of Fourier expansions of QMF and show that the Fourier expansions of quaternionic cusp forms are supported on positive-definite indices in Corollary 4.10.

4.1. **Holomorphic modular forms on orthogonal groups.** Recall that $n \geq 2$ and fix $V_{2,n+1}$ to be a $\mathbf{Q}$-vector space equipped with a non-degenerate quadratic form q of signature $(2, n+1)$. Write $(x, y) = q(x + y) - q(x) - q(y)$ for the symmetric bilinear pairing on $V_{2,n+1}$ associated to q. In this subsection, we discuss holomorphic modular forms on $H := \mathrm{SO}(V_{2,n+1})$. We will later consider $V_{2,n+1}$ as a subspace of $V_{4,n+2}$, and our notation is chosen to be compatible with this embedding.

According to the Hasse principal, there exists a non-zero isotropic vector $b_2 \in V_{2,n+1}$. Since q is non-degenerate, we may fix a second istropic vector $b_{-2} \in V_{2,n+1}$ satisfying $(b_2, b_{-2}) = 1$.

Let $V_{1,n} := (\mathbf{Q}b_2 + \mathbf{Q}b_{-2})^\perp$ be the orthogonal complement of $\mathbf{Q}b_2 + \mathbf{Q}b_{-2}$ in $V_{2,n+1}$. Then

$$V_{2,n+1} = \mathbf{Q}b_2 \oplus V_{1,n} \oplus \mathbf{Q}b_{-2}. \tag{4.1}$$

The Siegel parabolic $R \leq H$ is the parabolic subgroup of H defined as the stabilizer of the line $\mathbf{Q}b_2 \subseteq V_{2,n+1}$. Let $M_R \leq R$ denote the Levi subgroup of R stabilizing the line spanned by b_{-2}. The unipotent radical $N_R \trianglelefteq R$ is identified as an $\mathrm{SO}(V_{1,n})$ module via the map

$$\kappa \colon V_{1,n} \xrightarrow{\sim} N_R, \qquad v \mapsto \exp(b_2 \wedge v). \tag{4.2}$$

Fix $y_1 \in V_{1,n}$ satisfying $(y_1, y_1) = 2$. The symmetric space for the group $H(\mathbf{R})$ is

$$\mathfrak{h} = \{Z = X + iY \in V_{1,n} \otimes \mathbf{C} : (Y, -y_1) > 0 \text{ and } (Y, Y) > 0\}.$$

We have an identification of $\mathfrak{h}$ with the set of isotropic elements in $V_{2,n+1} \otimes \mathbf{C}$ via the map

$$\mathfrak{h} \to V_{2,n} \otimes_{\mathbf{R}} \mathbf{C}, \qquad Z \mapsto r(Z) := -q(Z)b_2 + Z + b_{-2}.$$

This identification yields an action of the identity component $H(\mathbf{R})^0$ on $\mathfrak{h}$ as follows: If $g \in H(\mathbf{R})^0$, then there exists a unique nonzero complex number $j_H(g, Z)$ and a unique element $gZ \in \mathfrak{h}$ so that $gr(Z) = j_H(g, Z)r(gZ)$. Observe that $j_H(g, Z) = (gr(Z), b_2)$.

We can now define classical holomorphic modular forms on $\mathfrak{h}$.

Definition 4.1. Suppose $\ell \in \mathbf{Z}$, and $\Gamma \subseteq H(\mathbf{R})^0$ is a congruence subgroup. Then $f : \mathfrak{h} \to \mathbf{C}$ is a *holomorphic modular form* of weight ℓ and level Γ if

1. f is holomorphic
2. $f(\gamma Z) = j(\gamma, Z)^\ell f(Z)$ for all $Z \in \mathfrak{h}$ and $\gamma \in \Gamma$
3. the function $\varphi_f(g) := j_H(g, iy_1)^{-\ell} f(g \cdot iy_1) : H(\mathbf{R}) \to \mathbf{C}$ is of moderate growth.

If $T \in V_{1,n}$, say that T is *positive definite* if $(T, -y_1) > 0$ and $(T, T) > 0$. We say that T is *positive semi-definite* if $(T, -y_1) \geq 0$ and $(T, T) \geq 0$. Write $T > 0$ (resp. $T \geq 0$) if and only if T is positive definite (resp. semi-definite). If $f\colon \mathfrak{h} \to \mathbf{C}$ is a holomorphic modular form of level Γ then $f(n \cdot Z) = f(Z)$ for $n \in N_R \cap \Gamma$. Therefore f admits a Fourier expansion

$$f(Z) = \sum_{T \in \Lambda\colon T \geq 0} a_f(T) e^{2\pi i (T,Z)}.$$

Here we make use of (4.2) to define the lattice $\Lambda \leq V_{1,n}$ according to the formula $\Lambda := \kappa^{-1}(\Gamma \cap N_R)$. If f is cuspidal, so that $|\varphi_f(g)|$ is bounded, then $a_f(T) \neq 0$ implies that $T > 0$.

We define $K_H = \mathrm{Stab}(\mathrm{Span}(b_2 + b_{-2}, y_1)) \subseteq H(\mathbf{R})$ and $K_H^0 = K_H \cap H(\mathbf{R})^0$. Then K_H is a maximal compact subgroup of $H(\mathbf{R})$, and one can verify that K_H^0 is the stabilizer of $i(-y_1)$ in $H(\mathbf{R})^0$. Observe that $j_H(\cdot, -iy_1) : K_H^0 \to \mathbf{C}^\times$ is a character.

Suppose that $\varphi : \Gamma \backslash H(\mathbf{R})^0 \to \mathbf{C}$ is function satisfying $\varphi(hk) = j_H(k, -iy_1)^\ell \varphi(h)$ for all $h \in H(\mathbf{R})^0$ and $k \in K_H^0$. Then the function $f_\varphi : \mathfrak{h} \to \mathbf{C}$,

$$f_\varphi(Z) = j(g, -iy_1)^\ell \varphi(g), \text{ if } g \cdot (-iy_1) = Z$$

is well-defined. If $f_\varphi(Z)$ is holomorphic, and φ is of moderate growth, we say that φ is the automorphic function associated to a holomorphic modular form of weight ℓ.

4.2. **Two Definitions of Quaternionic Modular Forms.** Quaternionic modular forms were introduced by Gross-Wallach [GW96], and Gan-Gross-Savin [GGS02]. They have been studied in several papers by Pollack. We present two related definitions of them. The first uses differential operators, while the second uses representation theory.

Definition 4.2. Suppose $\varphi : G(\mathbf{Q}) \backslash G(\mathbf{A}) \to \mathbf{V}_\ell$ is an automorphic form, so that in particular φ is smooth and of moderate growth. We say that φ is a quaternionic modular form of weight ℓ if

(1) φ is K^0-equivariant, i.e., $\varphi(gk) = k^{-1}\varphi(g)$ for all $g \in G(\mathbf{A})$ and $k \in K^0$;
(2) $D_\ell \varphi \equiv 0$, for a certain specific linear first order differential operator D_ℓ.

We denote $\mathcal{A}(G, \ell)$ the space of quaternionic modular forms on G of weight ℓ, $\mathcal{A}_0(G, \ell)$ the space of quaternionic modular cusp forms. When G is split $\mathrm{SO}(8)$, we write $\mathcal{A}_{0,\mathbf{Z}}(G, \ell)$ for the quaternionic modular cusp forms that are right $G(\widehat{\mathbf{Z}})$ invariant.

To specify the differential operator D_ℓ, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ of $G(\mathbf{R})$ with respect to the compact subgroup K^0. Let $\{X_\alpha\}$ be a basis of $\mathfrak{p}$ and $\{X_\alpha^\vee\}$ the dual basis of $\mathfrak{p}^\vee$. One has that, as a representation of K^0, $\mathfrak{p} \simeq \mathfrak{p}^\vee \simeq V_2 \boxtimes W$, where the distinguished SU_2 acts trivially on W. For a K^0-equivariant function $F : G(\mathbf{R}) \to \mathbf{V}_\ell$, set $\widetilde{D}_\ell F = \sum_\alpha X_\alpha F \otimes X_\alpha^\vee$. The sum is independent of the choice of basis and $\widetilde{D}_\ell F$ takes values in $\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \simeq \mathrm{Sym}^{2\ell-1}(V_2) \otimes W \oplus \mathrm{Sym}^{2\ell+1}(V_2) \otimes W$. Let pr be the projection $\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \to \mathrm{Sym}^{2\ell-1}(V_2) \otimes W$. Then $D_\ell = \mathrm{pr} \circ \widetilde{D}_\ell$.

Definition 4.3. For an integer $\ell \geq 4$, we let π_ℓ^0 be the quaternionic discrete series representation of $G(\mathbf{R})^0$ with minimal K^0-type $S^{2\ell}(\mathbf{C}^2) \otimes \mathbf{1} \otimes \mathbf{1}$, as a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(4)$. It has the same infinitesimal character as the highest weight submodule of $\mathrm{Sym}^{\ell-4}(\wedge^2 V)$. We define $\mathcal{A}_0^{rep}(G, \ell)$ to be the space of $(\mathfrak{g}, K^0)$ homomorphisms from π_ℓ^0 to $\mathcal{A}_0(G)$, and similarly $\mathcal{A}_{0,\mathbf{Z}}^{rep}(G, \ell)$ for the subspace of maps that land in the $G(\widehat{\mathbf{Z}})$-invariant functions (when $\dim(V) = 8$.)

There is a canonical injection $\mathcal{A}_0^{rep}(G,\ell) \to \mathcal{A}_0(G,\ell)$ given as follows. Suppose $\alpha : \pi_\ell^0 \to \mathcal{A}_0(G)$ is in $\mathcal{A}_0^{rep}(G,\ell)$. Let $\{v_j\}_j$ be a basis of $\mathbf{V}_\ell$ and $\{v_j^\vee\}$ the dual basis of $\mathbf{V}_\ell^\vee \simeq \mathbf{V}_\ell$. Then $\sum_j \alpha(v_j) \otimes v_j^\vee$ is an element of $\mathcal{A}_0(G,\ell)$.

We will take Definition 4.2 as our definition of quaternionic modular forms, because it is *a priori* broader. In section 8, we will also need Definition 4.3.

4.3. **The Explicit Fourier Expansion of Quaternionic Modular Forms.** Let dz denote the standard right $Z(\mathbf{A})$-invariant measure on $Z(\mathbf{Q})\backslash Z(\mathbf{A}) \simeq \mathbf{Q}\backslash\mathbf{A}$. Suppose φ is a quaternionic modular form on $G(\mathbf{A})$. The constant term of φ along Z is defined as the function on $G(\mathbf{A})$ given by

$$\varphi_Z(g) = \int_{Z(\mathbf{Q})\backslash Z(\mathbf{A})} \varphi(zg)dz. \tag{4.3}$$

As a general piece of notation, suppose $\mathcal{G}$ is an algebraic group over $\mathbf{Q}$ and let $\psi\colon \mathcal{G}(\mathbf{Q})\backslash\mathcal{G}(\mathbf{A}) \to \mathbf{C}$ denote an automorphic function. If $\mathcal{N}$ is a unipotent subgroup of $\mathcal{G}$ equipped with a character $\chi\colon \mathcal{N}(\mathbf{Q})\backslash\mathcal{N}(\mathbf{A}) \to \mathbf{C}^\times$ then we define the χ-th Fourier of ψ along $\mathcal{N}$ via the integral

$$\psi_{\mathcal{N},\chi}(g) = \int_{\mathcal{N}(\mathbf{Q})\backslash\mathcal{N}(\mathbf{A})} \psi(ng)\chi(n)^{-1}dn. \tag{4.4}$$

In this notation, the Fourier expansion of φ_Z along the compact abelian group $Z(\mathbf{A})N_P(\mathbf{Q})\backslash N(\mathbf{A})$ takes the form

$$\varphi_Z(g) = \sum_{[T_1,T_2]\in V_{2,n}\times V_{2,n}} \varphi_{N_P,\varepsilon_{[T_1,T_2]}}(g). \tag{4.5}$$

The main result of [Pol20] fully explicates (4.5) in the case when φ is a quaternionic modular form. We use a version of this result from [Pol22a]. To state the theorem, we write B to denote the natural $\mathrm{GL}(U)\times\mathrm{SO}(V_{2,n})$-equivariant pairing between $U^\vee \otimes_{\mathbf{Q}} V_{2,n}$ and $U \otimes_{\mathbf{Q}} V_{2,n}$. So if $T_1, T_1', T_2, T_2' \in V_{2,n}$ then

$$B(b_{-1}\otimes T_1 + b_{-2}\otimes T_2, b_1\otimes T_1' + b_2 \otimes T_2') = (T_1,T_1') + (T_2,T_2'). \tag{4.6}$$

We express a general element $r \in M_P(\mathbf{R})$ as $r = (m,h)$ with the understanding that $m \in \mathrm{GL}_2(U)(\mathbf{R})$ and $h \in \mathrm{SO}(V_{2,n})(\mathbf{R})$. Recall the elements $v_1 = y_0/\sqrt{2}$ and $v_2 = y_1/\sqrt{2}$ defined in (3.1).

Definition 4.4. Let $[T_1,T_2] \in V_{2,n}\times V_{2,n}$ denote an ordered pair. Define $\beta_{[T_1,T_2]}\colon M_P(\mathbf{R}) \to \mathbf{C}$ by the formula

$$\beta_{[T_1,T_2]}(r) := \sqrt{2}iB(r^{-1}\cdot(b_{-1}\otimes T_1 + b_{-2}\otimes T_2), b_1\otimes(v_1+iv_2) + b_2\otimes i(v_1+iv_2)). \tag{4.7}$$

The ordered pair $[T_1,T_2]$ is *positive semi-definite* if $\beta_{[T_1,T_2]}(r) \neq 0$ for all $r \in M_P(\mathbf{R})^0$. We write $[T_1,T_2] \succeq 0$ to mean that the pair $[T_1,T_2]$ is positive semi-definite. We write $[T_1,T_2] \succ 0$ if $[T_1,T_2] \succeq 0$ and $(T_1,T_1)(T_2,T_2) - (T_1,T_2)^2 > 0$.

Theorem 4.5. [Pol22a, Theorem 8.2.2] *Fix $\ell \in \mathbf{Z}_{\geq 1}$ and suppose $[T_1,T_2] \in V_{2,n}^{\oplus 2}$. Then up to multiplying by a constant, there is a unique moderate growth function $\mathcal{W}_{[T_1,T_2]}\colon G(\mathbf{R}) \to \mathbf{V}_\ell$ satisfying the conditions (i)-(iii) below:*

(i) If $g \in G(\mathbf{R})$ and $k \in K^0$ then $\mathcal{W}_{[T_1,T_2]}(gk) = k^{-1}\mathcal{W}_{[T_1,T_2]}(g)$.

(ii) If $g \in G(\mathbf{R})$ and $n \in N_P(\mathbf{R})$ satisfies $log(n) = b_1\wedge w_1 + b_2\wedge w_2 + zb_1\wedge b_2$, then $\mathcal{W}_{[T_1,T_2]}(ng) = e^{i(T_1,w_1)+i(T_2,w_2)}\mathcal{W}_{[T_1,T_2]}(g)$.

(iii) If $g \in G(\mathbf{R})$ then $D_\ell\mathcal{W}_{[T_1,T_2]}(g) = 0$.

Moreover $\mathcal{W}_{[T_1,T_2]}(g) \equiv 0$ *unless* $[T_1,T_2] \succeq 0$*, and if* $[T_1,T_2] \succeq 0$ *then the function* $\mathcal{W}_{[T_1,T_2]}(g)$ *is uniquely characterized by requiring that for all* $r = (m,h) \in M_P(\mathbf{R})^0$,

$$\mathcal{W}_{[T_1,T_2]}(r) = \det(m)^{\ell}|\det(m)| \sum_{-\ell \le v \le \ell} \left(\frac{\beta_{[T_1,T_2]}(r)}{|\beta_{[T_1,T_2]}(r)|}\right)^{v} K_v(|\beta_{[T_1,T_2]}(r)|)\frac{x^{\ell+v}y^{\ell-v}}{(\ell+v)!(\ell-v)!}. \quad (4.8)$$

Here $K_v\colon \mathbf{R}_{>0} \to \mathbf{R}$ *denotes the modified K-Bessel function* $K_v(x) = \frac{1}{2}\int_0^\infty t^{v-1}e^{-x(t+t^{-1})}dt$.

If $\ell \ge n+2$ and one adds the assumption that $(T_1,T_1)(T_2,T_2) - (T_1,T_2)^2 \neq 0$ then Theorem 4.5 follows from earlier work of Wallach [Wal03, Theorem 16] with the exception of the explicit formula (4.8). As a corollary to Theorem 4.5 we deduce the following.

Corollary 4.6. *Suppose* $\ell \in \mathbf{Z}_{\ge 1}$ *and let* $\varphi\colon G(\mathbf{R}) \to \mathbf{V}_\ell$ *be a weight* ℓ *quaternionic modular form on* $G(\mathbf{A})$*. Then there exists a family of locally constant functions*

$$\{a_{[T_1,T_2]}(\varphi,\cdot)\colon G(\mathbf{A}_f) \to \mathbf{C}\colon [T_1,T_2] \in V_{2,n} \times V_{2,n} \text{ such that } [T_1,T_2] \succeq 0\}$$

such that the Fourier expansion of φ_Z *along* $Z(\mathbf{A})N_P(\mathbf{Q})\backslash N_P(\mathbf{A})$ *takes the form*

$$\varphi_Z(g_f g_\infty) = \varphi_{N_P}(g_f g_\infty) + \sum_{[T_1,T_2] \in V_{2,n}\times V_{2,n}\colon [T_1,T_2]\succeq 0} a_{[T_1,T_2]}(\varphi, g_f)\mathcal{W}_{[2\pi T_1, 2\pi T_2]}(g_\infty). \quad (4.9)$$

Definition 4.7. With hypotheses as in Corollary 4.6, let $\lambda = [T_1,T_2]$ denote an element in $V_{2,n} \times V_{2,n}$ and suppose $[T_1,T_2] \succeq 0$. Define the *λ-th Fourier coefficient of φ* to be the complex number $a_\varphi(\lambda) := a_{[T_1,T_2]}(\varphi, 1)$.

4.4. **The Vanishing of $\beta_{[T_1,T_2]}$.** In this subsection we develop some properties of the the functions $\beta_{[T_1,T_2]}$ defined by (4.7). We write $\mathrm{SO}(V_{2,n})(\mathbf{R})^0$ to denote the identity component of $\mathrm{SO}(V_{2,n})(\mathbf{R})$. The reader may verify the following lemma.

Lemma 4.8. *Suppose* $h \in \mathrm{SO}(V_{2,n})(\mathbf{R})$ *and let* $\{w_1,w_2\}$ *denote set of vectors in* $V_{2,n} \otimes_{\mathbf{Q}} \mathbf{R}$ *satisfying* $(w_i,w_j) = \delta_{ij}$ *for* $i,j = 1,2$*. Then* $h \in \mathrm{SO}(V_{2,n})(\mathbf{R})^0$ *if and only if*

$$\chi_{w_1,w_2}(h) := \det\begin{pmatrix} (w_1,hw_1) & (w_1,hw_2) \\ (w_2,hw_1) & (w_2,hw_2)\end{pmatrix} > 0. \quad (4.10)$$

The proposition below is an analogue of [Pol20, Proposition 10.0.01]. Write $\mathrm{GL}(U)(\mathbf{R})^0$ to denote the identity component of $\mathrm{GL}(U)(\mathbf{R})$ then $M_P(\mathbf{R})^0 = \mathrm{GL}_2(\mathbf{R})^0 \times \mathrm{SO}(V_{2,n})(\mathbf{R})^0$.

Proposition 4.9. *Suppose* $[T_1,T_2] \in V_{2,n} \times V_{2,n}$.

(i) If $\mathbf{R}$*-span*$\{T_1,T_2\}$ *is an indefinite or negative definite two plane in* $V_{2,n}(\mathbf{R})$ *then there exists* $r \in M_P(\mathbf{R})^0$ *such that* $\beta_{[T_1,T_2]}(r) = 0$.

(ii) If $\mathbf{R}$*-span*$\{T_1,T_2\}$ *is a positive definite two plane in* $V_{2,n}(\mathbf{R})$ *then exactly one of* $\beta_{[T_1,T_2]}$ *and* $\beta_{[T_2,T_1]}$ *has a zero on* $M_P(\mathbf{R})^0$.

(iii) If $|\beta_{[T_1,T_2]}(r)|$ *is bounded away from zero on the set* $\{(m,h) \in M_P(\mathbf{R})^0\colon \det(m) = 1\}$ *then* $(T_1,T_1)(T_2,T_2) - (T_1,T_2)^2 > 0$*. In particular* T_1 *and* T_2 *span a two plane in* $V_{2,n}(\mathbf{R})$.

Proof. In the proof below we adopt the temporary notation $W = \mathbf{R}$-span$\{T_1,T_2\}$. Recall the fixed positive definite two plane $V_2^+(\mathbf{R}) = \mathbf{R}$-span$\{v_1,v_2\}$ defined in §3.1.

(i) If W is a negative definite two plane then there exists a positive definite two plane P^+ such that P^+ is orthogonal to W. By the Witt extension theorem there exists $h \in \mathrm{O}(V_{2,n})(\mathbf{R})$ such that hv_1 and hv_2 give a basis for P^+. If necessary we pre-compose h with any isometry of $V_{2,n}(\mathbf{R})$ which acts as the identity on $V_2^+(\mathbf{R})$ and acts by an orthogonal transformation of determinant -1 on $V_2^+(\mathbf{R})^\perp$. In this way we may ensure that $h \in \mathrm{SO}(V_{2,n})(\mathbf{R})$. Since P^+ and W are orthogonal, the element $r = (1, h) \in M_P(\mathbf{R})$ satisfies $\beta_{[T_1,T_2]}(r) = 0$. To ensure that $r \in M_P(\mathbf{R})^0$ it may be necessary to pre-compose h with an isometry of $V_{2,n}(\mathbf{R})$ which interchanges v_1 and v_2 and acts by an orthogonal transform of determinant -1 on $V_2^+(\mathbf{R})^\perp$.

Next we address the case when W is a 2 plane of indefinite signature $(1, 1)$. Fix a basis $\{e, f\}$ for W such that $(e, e) = (f, f) = 0$ and $(e, f) = 1$. Without loss of generality we may assume that there exists $m \in \mathrm{GL}(U)(\mathbf{R})^0$ such that $m^{-1}(b_{-1}\otimes T_1+b_{-2}\otimes T_2) = b_{-1}\otimes e+b_{-2}\otimes f$. Let $u_+ \in W^\perp$ be such that $(u_+, u_+) = 2$. Write P^+ to denote the positive definite two plane spanned by the vectors $p_1 = \frac{1}{\sqrt{2}}(e + f)$ and $p_2 = u_+ + \frac{1}{\sqrt{2}}(e - f)$. Reasoning as we did in the case of W negative definite, we conclude that there exists $h \in \mathrm{SO}(V_{2,n})(\mathbf{R})$ such that $hv_1 = p_1$ and $hv_2 = p_2$. Now a short computation shows that if $r = (m, h)$ then $\beta_{[T_1,T_2]}(r) = 0$. It remains to show that we can choose $r \in M_P(\mathbf{R})^0$. Thus suppose $r \notin M_P(\mathbf{R})^0$ which implies $h \notin \mathrm{SO}(V_{2,n})(\mathbf{R})^0$. Define $p_1' = \frac{1}{\sqrt{2}}(e + f)$, $p_2' = -u_+ + \frac{1}{\sqrt{2}}(e - f)$ and let $h' \in \mathrm{SO}(V_{2,n})(\mathbf{R})$ be such that $h'p_1 = p_1'$ and $h'p_2 = p_2'$. Then $\chi_{p_1,p_2}(h') = -1$ and so Lemma 4.8 implies $h' \notin \mathrm{SO}(V_{2,n})(\mathbf{R})^0$. Since $\#\left(\mathrm{SO}(V_{2,n})(\mathbf{R})/\mathrm{SO}(V_{2,n})(\mathbf{R})^0\right) = 2$ it follows that $h'h \in \mathrm{SO}(V_{2,n})(\mathbf{R})^0$. Thus if $h \notin \mathrm{SO}(V_{2,n})(\mathbf{R})^0$ then $r := (m, h'h) \in M_P(\mathbf{R})^0$ satisfies $\beta_{[T_1,T_2]}(r) = 0$ as required.

(ii) Assume W is a positive definite 2 plane in $V_{2,n}(\mathbf{R})$. Reasoning as in the case of W negative definite, there exists $r \in M_P(\mathbf{R})$ such that $r^{-1}(b_{-1}\otimes T_1+b_{-2}\otimes T_2) = b_{-1}\otimes v_1+b_{-2}\otimes v_2$. Without loss of generality we may assume that $r = (m, h)$ with $m \in \mathrm{GL}(U)(\mathbf{R})^0$. Define $T_1', T_2' \in V_{2,n}(\mathbf{R})$ by the equality $m^{-1}(b_{-1}\otimes T_1+b_{-2}\otimes T_2) = b_{-1}\otimes T_1'+b_{-2}\otimes T_2'$. If $\chi_{T_1',T_2'}(h) > 0$ then $r \in M_P(\mathbf{R})^0$ and the equality $\beta_{[T_1,T_2]}(r) = \sqrt{2}i(hv_1 + ihv_2, hv_2 + ihv_2) = 0$ implies that $\beta_{[T_1,T_2]}$ has a zero on $M_P(\mathbf{R})^0$. Conversely if $\chi_{T_1',T_2'}(h) < 0$ then Lemma 4.8 implies $\chi_{T_1',T_2'}(h')$ is negative for all $h' \in M_P(\mathbf{R})^0$. Thus if $r = (m, h') \in M_P(\mathbf{R})^0$ then

$$|\beta_{[T_1,T_2]}(r)|^2 = (T_1', h'v_1)^2 + (T_2', h'v_2)^2 + (T_1', h'v_2)^2 + (T_2', h'v_1)^2 - 2\chi_{T_1',T_2'}(h') > 0.$$

Hence $\beta_{[T_1,T_2]}$ vanishes on exactly one of the two components of $\mathrm{GL}(U)(\mathbf{R})^0 \times \mathrm{SO}(V_{2,n})(\mathbf{R})$

(iii) Assume the quantity $|\beta_{T_1,T_2}(r)|$ is bounded away from zero on the subset of pairs $(m, h) \in M_P(\mathbf{R})^0$ with $\det(m) = 1$. Then by what we proved in part (i), either $\dim(W) < 2$ or W is a positive definite 2-plane. If W is a positive definite two plane then there exists $\theta \in (0, \pi)$ such that $(T_1, T_2) = \|T_1\|\|T_2\|\cos(\theta)$ and so $(T_1, T_1)(T_2, T_2) - (T_1, T_2)(T_2, T_1) = \|T_1\|^2\|T_2\|^2(1-\cos\theta) > 0$. The case $\dim(W) < 2$ does not occur when $\beta_{[T_1,T_2]}$ is bounded away from zero. Indeed if T_1 and T_2 are colinear and $\alpha \in \mathbf{R}_{>0}$ then there exists $m_\alpha \in \mathrm{SL}(U)(\mathbf{R})$ such that m_α acts by multiplication by α on both T_1 and T_2. It follows that $\beta_{[T_1,T_2]}(m_\alpha, 1) \to 0$ as $\alpha \to 0^+$.

□

Corollary 4.10. *Suppose $\ell \in \mathbf{Z}_{\geq 1}$ and $\varphi \colon G(\mathbf{A}) \to \mathbf{V}_\ell$ is a weight ℓ quaternionic modular form. If φ is cusp form then the Fourier expansion (4.9) takes the form*

$$\varphi_Z(g_f g_\infty) = \sum_{[T_1,T_2]\in V_{2,n}\times V_{2,n}\colon [T_1,T_2]\succ 0} a_{[T_1,T_2]}(\varphi, g_f)\mathcal{W}_{[2\pi T_1,2\pi T_2]}(g_\infty). \tag{4.11}$$

Proof. Suppose φ is cuspidal and let $[T_1, T_2] \succeq 0$. Fix $g_f \in G(\mathbf{A}_f)$. On one hand, the form φ is rapidly decaying relative to the K^0-invariant norm $(\cdot, \cdot)_{K^0}$ on $\mathbf{V}_\ell$. In particular φ is bounded and the function

$$\int_{N_P(\mathbf{Q})\backslash N_P(\mathbf{A})} \varphi(ng_f g_\infty)\varepsilon_{[T_1,T_2]}(n)^{-1}dn = a_{[T_1,T_2]}(\varphi, g_f)\mathcal{W}_{[2\pi T_1, 2\pi T_2]}(g_\infty)$$

is the integral of the bounded function $g_\infty \mapsto \varphi(ng_f g_\infty)\varepsilon_{[T_1,T_2]}(n)^{-1}$ over the compact domain $N_P(\mathbf{Q})\backslash N_P(\mathbf{A})$. As such, either $a_{[T_1,T_2]}(\varphi, g_f) \equiv 0$, or the function $\mathcal{W}_{[2\pi T_1, 2\pi T_2]}(g_\infty)$ is bounded relative to the K^0-invariant norm on $\mathbf{V}_\ell$. On the other hand, the function $K_{-v}(x)$ has a pole (and is thus unbounded in absolute value) as $x \to 0^+$. Applying Proposition 4.9)(iii), it follows that if $-\ell \le v \le \ell$ and $\{T_1, T_2\}$ spans a one plane in $V_{2,n}(\mathbf{R})$, then

$$M_P(\mathbf{R})^0 \to \mathbf{R}_{\ge 0}, \qquad m_\infty \mapsto |\left(\mathcal{W}_{[2\pi T_1, 2\pi T_2]}(m_\infty), x^{\ell-v}y^{\ell+v}\right)_{K^0}|$$

is unbounded in absolute value. We conclude that $a_{[T_1,T_2]}(\varphi, g_f) \equiv 0$ as required. □

5. The quaternionic Saito-Kurokawa lifting and quaternionic Maass Spezialschar

In this section we continue with the notation of §3.1, specialized to the case $n = 2$. So V is a quadratic space of signature $(4,4)$ and $G = \mathrm{SO}(V)$. We let $V(\mathbf{Z})$ denote the $\mathbf{Z}$-span of the $b_{\pm j}$, $j = 1,2,3,4$ and $V_{2,2}(\mathbf{Z})$ denote the $\mathbf{Z}$-span of $b_{\pm j}$, $j = 3,4$. We have $V_{2,2} = M_2(\mathbf{Q})$, the space of 2-by-2 rational matrices equipped with the determinant quadratic form, via the identification

$$m_{11}b_3 - m_{21}b_4 + m_{12}b_{-4} + m_{22}b_{-3} \mapsto \left(\begin{array}{cc} m_{11} & m_{12} \\ m_{21} & m_{22} \end{array}\right).$$

In this notation, $V_{2,2}(\mathbf{Z}) = M_2(\mathbf{Z})$, the space of 2-by-2 integral matrices. We write $K_f \le \mathrm{SO}(V)(\mathbf{A}_f)$ for the level subgroup corresponding to $V(\mathbf{Z})$; automorphic forms invariant by K_f are said to be level one. Fourier coefficient of elements in $\mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ are indexed by pairs $[T_1, T_2] \in V_{2,2}(\mathbf{Z})^{\oplus 2}$ satisfying $[T_1, T_2] \succ 0$.

5.1. **The Quaternionic Saito-Kurokawa Lift.**

In this subsection we recall the statement of [Pol21, Theorem 4.1.1], in the case $n = 2$. Before we give the statement of (loc. cite), suppose $\lambda = [T_1, T_2] \in V^{\oplus 2}$ and define a 2-by-2 matrix with entries in $\mathbf{Q}$ via the formula

$$S(\lambda) = \frac{1}{2}\begin{pmatrix} (T_1,T_1) & (T_1,T_2) \\ (T_2,T_1) & (T_2,T_2) \end{pmatrix}. \tag{5.1}$$

Theorem 5.1. [Pol21, Theorem 4.1.1] *Suppose $F(Z) = \sum_{T>0} a_F(T)q^T$ is a weight ℓ cuspidal Siegel modular form on Sp_4 of level one. There exists a unique element $\theta^*(F) \in \mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ such that if $[T_1, T_2] \in V_{2,2}(\mathbf{Z}) \times V_{2,2}(\mathbf{Z})$ and $[T_1, T_2] \succ 0$ then*

$$a_{\theta^*(F)}(\lambda) = \sum_{r \in \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2(\mathbf{Z})^{\det\neq 0} \ s.t.\ \lambda r^{-1} \in V_{2,2}(\mathbf{Z})^{\oplus 2}} |\det(r)|^{w-1}\overline{a_F({}^t r^{-1}S(\lambda)r^{-1})}. \tag{5.2}$$

Remark 5.2. The construction of $\theta^*(F)$ recovers the theta lifting from Sp_4 to $\mathrm{SO}(V)$. It bears a close analogy to the construction of a family of holomorphic modular forms given in [Oda78] and [RS81].

5.2. **Poincare Lifts on G.** Let T denote a half integral, 2-by-2, positive definite matrix. Write $\mathcal{H}_2$ for the Siegel upper half space of degree 2, and define $S_\ell(\mathrm{Sp}_4(\mathbf{Z}))$ to be the space of level 1, weight ℓ, holomorphic modular forms on $\mathcal{H}_2$. Write $j(\cdot,\cdot)\colon \mathrm{Sp}_4(\mathbf{R}) \times \mathcal{H}_2 \to \mathbf{C}$ for the standard factor of automorphy and recall the classical Poincaré series

$$P_{T,\ell}(Z) = \sum_{\gamma \in \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} \backslash \mathrm{Sp}_4(\mathbf{Z})} j(\gamma, Z)^{-\ell} e^{2\pi i \mathrm{Tr}(T\gamma(Z))} \tag{5.3}$$

If ℓ is sufficiently large then the sum (5.3) converges to an element in $S_\ell(\mathrm{Sp}_4(\mathbf{Z}))$.

We now set-up notation to study the theta lift $\theta^*(P_{T,\ell})$. Recall that in §3.4 we introduced a distinguished map $K^0 \to \mathrm{SU}_2/\mu_2$ together with an identification $\mathrm{Lie}(\mathrm{SU}_2/\mu_2) \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathrm{Sym}^2(\mathbf{C}^2)$. We thus have a K^0-equivariant projection

$$\mathrm{pr}_K\colon \mathrm{Lie}(\mathrm{SO}(V)) \to \mathrm{Lie}(K^0) \to \mathrm{Lie}(\mathrm{SU}_2/\mu_2) \to \mathrm{Sym}^2(\mathbf{C}^2). \tag{5.4}$$

Given $X \in \mathrm{Sym}^2(V_2)$, let X^ℓ denote the element of $\mathbf{V}_\ell$ obtained by raising X to the ℓ-th power. We write $\|\cdot\|^2$ to denote the unique K^0-invariant quadratic norm on $\mathrm{Sym}^2(\mathbf{C}^2)$. If $[v_1, v_2] \in V(\mathbf{Z}) \times V(\mathbf{Z})$ is such that v_1 and v_2 span a positive definite two plane then $\mathrm{pr}_K(v_1 \wedge v_2) \neq 0$ [Pol21, Lemma 3.2.2], thus we may define

$$B_{[v_1,v_2]}\colon \mathrm{SO}(V)(\mathbf{R}) \to \mathbf{V}_\ell, \quad g \mapsto \frac{\mathrm{pr}_K(\mathrm{Ad}(g^{-1})\cdot v_1 \wedge v_2)^\ell}{\|\mathrm{pr}_K(\mathrm{Ad}(g^{-1})\cdot v_1 \wedge v_2))\|^{2\ell+1}}.$$

Proposition 5.3. [Pol21, Corollary 3.3.7] *Suppose $\ell \geq 16$ is even. Let T denote a half integral, 2-by-2, positive definite matrix. With notation as in (5.1), suppose $[v_1, v_2] \in V(\mathbf{Z})^{\oplus 2}$ satisfies $S(v_1, v_2) = T$. Write $\Gamma = \mathrm{SO}(V)(\mathbf{Q}) \cap K_f$ and consider the identification*

$$\Gamma\backslash \mathrm{SO}(V)(\mathbf{R}) \xrightarrow{\sim} \mathrm{SO}(V)(\mathbf{Q})\backslash \mathrm{SO}(V)(\mathbf{A})/K_f, \quad \Gamma g_\infty \mapsto \mathrm{SO}(V)(\mathbf{Q}) g_\infty K_f.$$

Then as a function on $\Gamma\backslash\mathrm{SO}(V)(\mathbf{R})$, the theta lift $\theta^*(P_{T,\ell})$ is proportional to the function

$$Q_{T,\ell}\colon \Gamma\backslash \mathrm{SO}(V)(\mathbf{R}) \to \mathbf{V}_\ell, \qquad \Gamma g_\infty \mapsto \sum_{[v_1,v_2]\in V(\mathbf{Z})\times V(\mathbf{Z})\colon S([v_1,v_2])=T} B_{[v_1,v_2]}(g_\infty). \tag{5.5}$$

5.3. **The Quaternionic Maass Spezialschar and Saito-Kurokawa Subspace.** By Theorem 5.1 we have a lifting

$$\theta^*\colon S_\ell(\mathrm{Sp}_4(\mathbf{Z})) \to \mathcal{A}_{0,\mathbf{Z}}(G,\ell), \qquad F \mapsto \theta^*(F).$$

The *quaternionic Saito-Kurokawa subspace* $\mathrm{SK}_\ell \subseteq \mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ is defined as the image of θ^*.

Definition 5.4. Suppose $\lambda \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$. We say that λ is *strongly primitive* if

$$\{r \in \mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})\colon \lambda r^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}\} = \mathrm{GL}_2(\mathbf{Z}).$$

Remark 5.5. Note that if $\lambda = [T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ is strongly primitive and $F \in S_\ell(\mathrm{Sp}_4(\mathbf{Z}))$ then by Theorem 5.1 we have $a_{\theta^*(F)}(\lambda) = \overline{a_F(S(\lambda))}$. This property is reminiscent of the classical Saito-Kurokawa subspace. Indeed if T is a primitive binary quadratic form and $F \in S_\ell(\mathrm{Sp}_4(\mathbf{Z}))$ is the theta lift of an anti-holomorphic form $\overline{f}$ on $\widetilde{\mathrm{SL}}_2$ then $a_F(T) = \overline{a_f(\det(T))} = a_{\overline{f}}(-\det(T))$.

As a corollary to Theorem 5.1 we deduce the following.

Corollary 5.6. *Suppose $\lambda_1, \lambda_2 \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ are strongly primitive elements satisfying $S(\lambda_1) = S(\lambda_2)$. If $\varphi \in \mathrm{SK}_\ell$ then $a_\varphi(\lambda_1) = a_\varphi(\lambda_2)$.*

Lemma 5.7. *Suppose $T = [a, b, c]$ is a half integral symmetric matrix with $a, b, c \in \mathbf{Z}$. Then there exists a strongly primitive element $\lambda = [T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ such that $T = S(\lambda)$.*

Proof. We let $T_1 = \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}$ so that if $\lambda = [T_1, T_2]$ then $S(\lambda) = T$. Suppose $r \in \mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})$ satisfies $\lambda r^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ and write $r^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ with $w, x, y, z \in \mathbf{Q}$. Then

$$\lambda r^{-1} = \left[\begin{pmatrix} w & y \\ * & * \end{pmatrix}, \begin{pmatrix} x & z \\ * & * \end{pmatrix} \right] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}.$$

So $x, y, z, w \in \mathbf{Z}$ which implies $\det(r^{-1}) = \det(r)^{-1} \in \mathbf{Z}$. So $r \in \mathrm{GL}_2(\mathbf{Z})$. $\square$

Definition 5.8. Define the *quaternionic Maass Spezialschar* MS_ℓ as the subspace of $\mathcal{A}_{0,\mathbf{Z}}(G, \ell)$ consisting of forms φ satisfying condition (i) and (ii) below.

(i) If $\lambda_1, \lambda_2 \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ are strongly primitive and $S(\lambda_1) = S(\lambda_2)$ then

$$a_\varphi(\lambda_1) = a_\varphi(\lambda_2).$$

Let $\lambda \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$. By Lemma 5.7 there exists a strongly primitive element $\breve{\lambda} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ such that $S(\lambda) = S(\breve{\lambda})$. If $\varphi \in \mathcal{A}_{0,\mathbf{Z}}(G, \ell)$ satisfies condition (i) then

$$a_\varphi^{\mathrm{prim}}(\lambda) := a_\varphi(\breve{\lambda})$$

is well defined independent of the choice of $\breve{\lambda}$.

(ii) If $\lambda = [T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus}$ satisfies $T_1, T_2 \succ 0$ (see (5.1)) then

$$a_\varphi(\lambda) = \sum_{r \in \mathrm{GL}_2(\mathbf{Z}) \backslash (\mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})) \text{ s.t. } \lambda r^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}} |\det(r)|^{\ell-1} a_\varphi^{\mathrm{prim}}(\lambda r^{-1}).$$

Remark 5.9. Note that the two conditions of Definition 5.8 can be replaced by the single condition

$$a_\varphi(\lambda) = \sum_{r \in \mathrm{GL}_2(\mathbf{Z}) \backslash (\mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})) \text{ s.t. } \lambda r^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}} |\det(r)|^{\ell-1} a_\varphi((\lambda \breve{r^{-1}})),$$

where if $\mu \in M_2(\mathbf{Z})^2$ has $S(\mu) = [a, b, c]$ then $\breve{\mu} = \left[\begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix} \right]$.

Lemma 5.10. *We have an inclusion $\mathrm{SK}_\ell \subseteq \mathrm{MS}_\ell$.*

Proof. Suppose $\varphi = \theta^*(F) \in \mathrm{SK}_\ell$. The fact that φ satisfies property (i) of Definition 5.8 is the content of Corollary 5.6. It remains to show that φ satisfies condition (ii). Suppose $\lambda = [T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ is such that $T_1, T_2 \succ 0$. Let $r \in \mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})$ be such that $\lambda r^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$. By Theorem 5.1, it suffices to show that

$$a_\varphi^{\mathrm{prim}}(\lambda r^{-1}) = \overline{a_F({}^t r^{-1} S(\lambda) r^{-1})}. \tag{5.6}$$

Since $S(\lambda r^{-1}) = {}^t r^{-1} S(\lambda) r^{-1}$, equality (5.6) follows from Theorem 5.1 (see Remark 5.5). $\square$

6. Dirichlet Series for L-functions of Quaternionic Modular Forms

In this section, we present a conjectural Dirichlet series for the L-functions of irreducible, cuspidal, quaternionic automorphic representations Π of split SO(8) and show that this conjecture is satisfied by the Saito-Kurokawa lifts. By a *Dirichlet series* for an L-function of an automorphic representation Π, we mean a sum of the form $\sum_n a_n n^{-s}$ where the a_n are given explicitly in terms of Fourier coefficients of an automorphic form in Π.

6.1. **Conjecture on Dirichlet series.** We recall the formula of Andrianov [And74] for the standard L-function of Siegel modular forms of full level on Sp_4. Let f be a level one cuspidal Siegel modular eigenform of weight ℓ with Fourier expansion $f(Z) = \sum a_f(T)\exp(2\pi i \mathrm{tr}(TZ))$, where the sum is taken over half-integral, positive definite, symmetric matrices T. Let π be the automorphic representation generated by the automorphic function corresponding to f. For a fixed index T, let S denote a finite set of primes containing those that divide $8\det(T)$ and let $\chi_T(\cdot) = (\cdot, -\det(T))_2$ be the automorphic character induced by the Hilbert symbol. Then one has

$$a(T)\frac{L^S(\pi, Std, s)}{\zeta^S(2s)L^S(\chi_T, s+1)} = \sum_{g\in \mathrm{M}_2^S(\mathbf{Z})/\,\mathrm{GL}_2(\mathbf{Z})} \frac{a(g^tTg)}{|\det(g)|^{s+\ell-1}}. \tag{6.1}$$

Here $L^S(\pi, Std, s)$ is the standard partial L-function of π, ζ^S is the partial Riemann zeta-function, $L^S(\chi_T, s)$ is the partial Dirichlet L-function associated to χ_T, and $\mathrm{M}_2^S(\mathbf{Z})$ denotes the 2×2 integer matrices with nonzero determinant, only divisible by primes away from S. See [Pol22b] for a modern reference and derivation.

To state the conjecture for Dirichlet series on split SO(8), let φ be a level one cuspidal QMF and Π the irreducible automorphic representation generated by φ with standard L-function $L(\Pi, Std, s)$. Let $[T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ be strongly primitive with $T = S([T_1, T_2])$, and suppose S is a finite set of primes containing all those that divide $8\det(T)$. Following Definition 4.7, we write $a_\varphi([T_1, T_2])$ for the $[T_1, T_2]$-th Fourier coefficient of φ. Finally, let $\zeta_E(s)$ denote the Dedekind zeta function of the quadratic field E determined by the discriminant of T.

Conjecture 6.1. *Let the notation be as above. Then*

$$a_\varphi([T_1, T_2])\frac{L^S(\Pi, Std, s)}{\zeta^S(2s)\zeta_E^S(s+1)} = \sum_{g\in \mathrm{M}_2^S(\mathbf{Z})/\,\mathrm{GL}_2(\mathbf{Z})} \frac{a_\varphi([T_1, T_2]\cdot g)}{|\det(g)|^{s+\ell-1}}.$$

We call the right hand side the Dirichlet series for φ at $[T_1, T_2]$ and denote it by $D_\varphi(T_1, T_2)$. The conjecture was found by analyzing a hypothetical Rankin-Selberg integral for cusp forms on G similar to that used in the proof of Andrianov's formula. We hope to prove that this integral does in fact represent the standard L-function in a future work. At this point, we are able to provide some evidence for this conjecture as a consequence of Lemma 5.10 by considering the behavior of the L-function of the theta lift and the Maass relations.

6.2. **Evidence from the Maass Spezialschar.** Let $\varphi = \theta^*(F)$ be the Saito-Kurokawa lift of a cuspidal level one Siegel modular eigenform of weight ℓ, let Π be the automorphic representation of SO(8) generated by φ, and let π be the automorphic representation of Sp(4) generated by the automorphic form corresponding to F. By properties of the theta lifting [Ral82], we have that $L(\Pi, Std, s) = L(\pi, Std, s)\zeta(s-1)\zeta(s)\zeta(s+1)$. Proposition 6.4 shows that the Dirichlet series associated to any element of Maass Spezialschar MS_ℓ factors in a similar way. Before we can prove such a proposition, we need a lemma about the action of $\mathrm{M}_2(\mathbf{Z})$ on $V_0 = \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$.

We use the following lemma that regards strong primitivity.

Lemma 6.2. *Suppose $\lambda \in \mathrm{M}_2(\mathbf{Z})^2$ is strongly primitive and $r \in \mathrm{GL}_2(\mathbf{Q})$ satisfies $\lambda \cdot r \in \mathrm{M}_2(\mathbf{Z})^2$. Then $r \in \mathrm{M}_2(\mathbf{Z})$.*

Proof. We can write $r = k_1 d k_2$ with $k_j \in \mathrm{GL}_2(\mathbf{Z})$ and $d = \mathrm{diag}(m_1/n_1, m_2/n_2)$ with $\gcd(m_1, n_1) = 1$, $\gcd(m_2, n_2) = 1$. Suppose $\lambda \cdot k_1 = (\mu_1, \mu_2)$, so that $\mu_j \in \mathrm{M}_2(\mathbf{Z})$. Then $\lambda \cdot r \in \mathrm{M}_2(\mathbf{Z})$ implies $m_1\mu_1/n_1 \in M_2(\mathbf{Z})$ and $m_2\mu_2/n_2 \in \mathrm{M}_2(\mathbf{Z})$. But from the fact that $\gcd(m_i, n_i) = 1$, we obtain $\mu_1/n_1 \in \mathrm{M}_2(\mathbf{Z})$ and $\mu_2/n_2 \in \mathrm{M}_2(\mathbf{Z})$.

Set $g = k_2^{-1}\mathrm{diag}(n_1, n_2)k_1^{-1}$. Then $\lambda \cdot g^{-1} \in \mathrm{M}_2(\mathbf{Z})^2$ by the previous line. Hence $g \in \mathrm{GL}_2(\mathbf{Z})$ so $n_1, n_2 = \pm 1$. Consequently, $r \in \mathrm{M}_2(\mathbf{Z})$ as desired. □

Lemma 6.3. *Let S be the finite set of primes and $\lambda \in \mathrm{M}_2(\mathbf{Z})^2$ strongly primitive. Suppose $g \in \mathrm{M}_2^S(\mathbf{Z})$. Then if $r \in \mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})$ with $\lambda \cdot gr^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ we have that $r, gr^{-1} \in \mathrm{M}_2^S(\mathbf{Z})$.*

Proof. By Lemma 6.2, $gr^{-1} \in M_2(\mathbf{Z})$. Then since $g = (gr^{-1})r$ and because $\det(g)$ is a unit at primes in S, so are $\det(gr^{-1})$ and $\det(r)$. □

Proposition 6.4. *Let $\varphi \in \mathcal{A}_{0,\mathbf{Z}}(G, \ell)$ be an element of the quaternionic Maass Spezialschar, MS_ℓ so that the Fourier coefficients a_φ satisfy the conditions of Definition 5.8. Then for a fixed $\lambda = [T_1, T_2] \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}$, the Dirichlet series factors as*

$$D_\varphi(T_1, T_2) = \sum_{r \in \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2^S(\mathbf{Z})} |\det(r)|^{-s} \sum_{g \in \mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z})} \frac{a_\varphi^{\mathrm{prim}}(\lambda \cdot g)}{|\det(g)|^{s+\ell-1}}. \tag{6.2}$$

Proof. To prove this proposition, we calculate

$$\begin{aligned}
D_\varphi(T_1, T_2) &= \sum_{g \in \mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z})} \frac{a_\varphi(\lambda \cdot g)}{|\det(g)|^{s+\ell-1}} \\
&= \sum_{g \in \mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z})} |\det(g)|^{1-s-\ell} \left(\sum_{\substack{r \in \mathrm{GL}_2(\mathbf{Z})\backslash(\mathrm{GL}_2(\mathbf{Q}) \cap \mathrm{M}_2(\mathbf{Z})) \\ \text{with } \lambda \cdot gr^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}}} |\det(r)|^{\ell-1} a_\varphi^{\mathrm{prim}}(\lambda \cdot gr^{-1}) \right) \\
&= \sum_{\substack{g \in \mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z}) \\ r \in \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2^S(\mathbf{Z}) \\ \text{with } \lambda \cdot gr^{-1} \in \mathrm{M}_2(\mathbf{Z})^{\oplus 2}}} |\det(r)|^{-s} \frac{a_\varphi^{\mathrm{prim}}(\lambda \cdot gr^{-1})}{|\det(gr^{-1})|^{s+\ell-1}} \\
&= \sum_{d \in \mathbf{Z}_{>0}} \left(\sum_{\substack{r \in \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2^S(\mathbf{Z}) \\ |\det(r)| = d}} |\det(r)|^{-s} \sum_{\substack{g \in \mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z}) \\ \lambda \cdot gr^{-1} \in \mathrm{M}_2^S(\mathbf{Z}))^{\oplus 2}}} \frac{a_\varphi^{\mathrm{prim}}(\lambda \cdot gr^{-1})}{|\det(gr^{-1})|^{s+\ell-1}} \right),
\end{aligned}$$

where the second line follows from the quaternionic Maass relations, the third is Lemma 6.3, and the last is grading by $|\det(r)|$.

Now let $R_d = \{r_1, \dots, r_{N_d}\}$ be a fixed set of coset representatives for

$$\{r \in \mathrm{GL}_2(\mathbf{Z})\backslash \mathrm{M}_2^S(\mathbf{Z})\colon |\det(r)| = d\}$$

and let $\{g_j\colon j \in \mathbf{Z}_{>0}\}$ be a fixed set of coset representatives of $\mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z})$. Then applying Lemma 6.3 again, the sum rearranges as

$$D_\varphi(T_1,T_2) = \sum_{\substack{d\in\mathbf{Z}_{>0}\\ h\in\mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z})}} d^{-s}\frac{a_\varphi^{\mathrm{prim}}(\lambda\cdot h)}{|\det(h)|^{s+\ell-1}}\left(\sum_{i\in\{1,\dots,N_d\},j\in\mathbf{Z}^+}\mathrm{char}_{h\,\mathrm{GL}_2(\mathbf{Z})}(g_jr_i^{-1})\right),$$

where char is the characteristic function.

Recall that $N_d = |R_d|$ is the number of coset representatives of $\mathrm{GL}_2(\mathbf{Z})\backslash\mathrm{M}_2^S(\mathbf{Z})$ of determinant d. We now show that for a given $h\in\mathrm{M}_2^S(\mathbf{Z})$,

$$N_d = |\{(i,j)\colon r_i\in R_d,\ g_jr_i^{-1}\in h\,\mathrm{GL}_2(\mathbf{Z})\}|,$$

and this will complete the proof. For a fixed element $r_i\in R_d$, let $S(d,i) = \{g_j | g_j \in h\,\mathrm{GL}_2(\mathbf{Z})r_i\}$ be the set of fixed coset representatives g_j in the double coset determined by r_i. As these cosets are disjoint, it remains to show that $N_d = \sum_{i=1}^{N_d}|S(d,i)| = |\bigcup_{i=1}^{N_d}S(d,i)|$. As R_d is a complete set of coset representatives, we have $\bigcup_{i=1}^{N_d}S(d,i) = h(\mathrm{M}_2^S(\mathbf{Z})^{|\det|=d})$, but this is a union of N_d $\mathrm{GL}_2(\mathbf{Z})$ cosets with $\mathrm{GL}_2(\mathbf{Z})$ now acting on the right. Thus, there are exactly N_d representatives g_j in the set, and the Dirichlet series factors as in (6.2). □

We will also need the following well-known fact.

Lemma 6.5. *We have an equality of meromorphic functions,*

$$\sum_{r\in\mathrm{GL}_2(\mathbf{Z})\backslash\mathrm{M}_2^S(\mathbf{Z})}|\det(r)|^{-s} = \zeta^S(s)\zeta^S(s-1).$$

Corollary 6.6. *Let F be denote a weight ℓ, level 1, degree 2, cuspidal Siegel eigenform, then Conjecture 6.1 holds for $\varphi = \theta^*(F)$.*

Proof. By Lemma 5.10, $\theta^*(F)$ is in the Mass Spezialschar and thus for any choice of strongly primitive $[T_1,T_2]\in\mathrm{M}_2(\mathbf{Z})^{\oplus 2}$ the Dirichlet series factors as (6.2) by Proposition 6.4. Now by Lemma 5.7 and Theorem 5.1 we have that $a^{\mathrm{prim}}_{\theta^*(F)}(\lambda\cdot g) = a_F(S(\lambda)\cdot g)$ for all $g\in\mathrm{M}_2^S(\mathbf{Z})$. Hence, by Lemma 6.5 and (6.1)

$$\begin{aligned}D_{\theta^*(F)}(T_1,T_2) &= \sum_{r\in\mathrm{GL}_2(\mathbf{Z})\backslash\mathrm{M}_2^S(\mathbf{Z})}|\det(r)|^{-s}\sum_{g\in\mathrm{M}_2^S(\mathbf{Z})/\mathrm{GL}_2(\mathbf{Z})}\frac{a_F(S(\lambda)\cdot g)}{|\det(g)|^{s+\ell-1}}\\ &= \zeta^S(s)\zeta^S(s-1)a(S(\lambda))\frac{L^S(\pi_F,Std,s)}{\zeta^S(2s)L^S(\chi_{S(\lambda)},s+1)}\\ &= a_{\theta^*(F)}(\lambda)\frac{L^S(\Pi_{\theta^*(F)},Std,s)}{\zeta^S(s+1)\zeta^S(2s)L^S(\chi_{S(\lambda)},s+1)}.\end{aligned}$$

The last equality follows from [Ral82], which implies that

$$L^S(\Pi_{\Theta^*(F)},Std,s) = \zeta^S(s)\zeta^S(s-1)\zeta^S(s+1)L^S(\pi_F,Std,s).$$

To complete the proof, we note that $\zeta_E^S(s) = \zeta^S(s)L^S(\chi_{S(\lambda)},s)$. □

7. The Fourier-Jacobi coefficient

In this section, we complete the proofs of some of the main theorems of the paper, namely Theorems 1.1, 1.2, and 1.3. The main technical result of this section is Proposition 7.3. This proposition, in conjunction with Corollary 7.2, is used to complete the proofs of these results.

7.1. **Fourier Coefficients along a Siegel parabolic.** Let $\varphi\colon G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{V}_\ell$ denote a vector valued automorphic form on $G(\mathbf{A})$ such that $\varphi(gk) = k^{-1}\cdot\varphi(g)$ for all $k \in K^0$ and $g \in G(\mathbf{A})$.

Recall the inclusion $V_{2,n} \subseteq V_{3,n+1}$ as well as the fixed positive definite two plane $V_2^+ \subseteq V_{2,n}$. Recall also the orthogonal basis for V_2^+ defined in Section 3 consisting of vectors y_0 and y_1 such that $(y_0, y_0) = (y_1, y_1) = 2$ and the character χ_{y_0} defined in (3.7). Let ξ^φ denote the Fourier coefficient of φ along N_Q corresponding to the character χ_{y_0} as in (4.4), and let H denote the stabilizer of y_0 in $\mathrm{SO}(V_{3,n+1}) \le M_Q$. Then ξ^φ defines an automorphic function

$$\xi^\varphi\colon H(\mathbf{A}) \to \mathbf{V}_\ell, \qquad \xi^\varphi(h) = \int_{V_{3,n+1}\backslash V_{3,n+1}(\mathbf{A})} \psi^{-1}((v, y_0))\varphi(\exp(b_1 \wedge v)h)dv.$$

Let $V_{2,n+1}$ denote the orthogonal complement of $\mathbf{Q}y_0$ in $V_{3,n+1}$.Through its action on $V_{3,n+1}/\mathbf{Q}y_0 \simeq V_{2,n+1}$ the group H is identified with $\mathrm{SO}(V_{2,n+1})$.

Write $R = M_R \cdot N_R$ for the parabolic subgroup in H stabilizing the line $\mathbf{Q}b_2$. The reader is referred to §4.1 for notation concerning the group H. The following proposition gives a "soft" formula for the Fourier coefficients of ξ^φ along the unipotent radical $N_R \simeq V_{1,n+1}$.

Lemma 7.1. *Let $T \in V_{1,n}$ and write ξ_T^φ to denote the Fourier coefficient of ξ^φ along N_R corresponding to the character*

$$\eta_T\colon N_R(\mathbf{A}) \to \mathbf{C}^1, \qquad \exp(b_2 \wedge v) \mapsto \psi((T, v)).$$

If $\varphi_{N_P,\varepsilon_{[y_0,T]}}(g) = \int_{N_P(\mathbf{Q})\backslash N_P(\mathbf{A})} \varphi(ng)\varepsilon_{[y_0,T]}(n)^{-1}dn$ and $h \in H(\mathbf{A})$ then

$$\xi_T^\varphi(h) = \int_{\mathbf{Q}\backslash\mathbf{A}} \varphi_{N_P,\varepsilon_{[y_0,T]}}(\exp(sb_1 \wedge b_{-2})h)ds$$

Proof. Throughout the proof we adopt the convention that if $\mathcal{G}$ is an algebraic group over $\mathbf{Q}$ then $[\mathcal{G}] := \mathcal{G}(\mathbf{Q})\backslash\mathcal{G}(\mathbf{A})$. Fix $T \in V_{1,n}$. Unraveling definitions and applying the identification (3.7) we obtain

$$\begin{aligned}
\xi_T^\varphi(h) &= \int_{[V_{1,n}]}\int_{[V_{3,n+1}]} \psi^{-1}((v', T) + (v, y_0))\varphi(\exp(b_1 \wedge v)\exp(b_2 \wedge v')h)dvdv' \\
&= \int_{[V_{1,n}]}\int_{[V_{2,n}\oplus\mathbf{Q}b_{-2}]}\int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}((v', T) + (v, y_0))\varphi(\exp(b_1 \wedge (v + tb_2))\exp(b_2 \wedge v')h)dtdvdv' \\
&= \int_{[V_{1,n}]}\int_{[V_{2,n}\oplus\mathbf{Q}b_{-2}]} \psi^{-1}((v', T) + (v, y_0))\varphi_Z(\exp(b_1 \wedge v)\exp(b_2 \wedge v')h)dvdv' \\
&= \int_{\mathbf{Q}\backslash\mathbf{A}}\int_{[V_{1,n}]}\int_{[V_{2,n}]} \psi^{-1}((v', T) + (v, y_0))\varphi_Z(\exp(sb_1 \wedge b_{-2})\exp(b_1 \wedge v + b_2 \wedge v')h)dvdv'ds
\end{aligned} \tag{7.1}$$

A short calculation reveals that if $w = \exp(sb_1 \wedge b_{-2})$ with $s \in \mathbf{A}$ then

$$w\exp(b_1 \wedge v + b_2 \wedge v')w^{-1} = \exp(b_1 \wedge (v + sv') + b_2 \wedge v'). \tag{7.2}$$

By substituting (7.2) into (4.5), the integrand in the last line of (7.1) may be rewritten as

$$\sum_{[T_1,T_2]\in V_{2,n}\oplus V_{2,n}} \varphi_{N_P,\varepsilon_{[T_1,T_2]}}(\exp(sb_1\wedge b_{-2})h)\psi((T_1,v+sv')+(T_2,v')-(v',T)-(v,y_0)) \tag{7.3}$$

If $s\in\mathbf{A}$ and $[T_1,T_2]\in V_{2,n}\oplus V_{2,n}$ then the integral

$$\int_{[V_{1,n}]}\int_{[V_{2,n}]}\psi((T_1,v+sv')+(T_2,v')-(v',T)-(v,y_0))dvdv' \tag{7.4}$$

vanishes unless $(v',T-T_2-sT_1)+(v,y_0-T_1)=0$ for all $(v,v')\in V_{2,n}(\mathbf{A})\oplus V_{1,n}(\mathbf{A})$. Equivalently, the integral (7.4) is non vanishing if and only if $T_1=y_0$ and there exists $t\in\mathbf{Q}$ such that $T_2=T+ty_0$. It follows that (7.1) can be simplified to the form

$$\xi_T^{\varphi}(h)=\int_{\mathbf{Q}\backslash\mathbf{A}}\sum_{t\in\mathbf{Q}}\varphi_{N_P,\varepsilon_{[y_0,T+ty_0]}}(\exp(sb_1\wedge b_{-2})h)ds \tag{7.5}$$

If $t\in\mathbf{Q}$ and $s\in\mathbf{A}$ then $\varphi_{N_P,\varepsilon_{[y_0,T+ty_0]}}(\exp(sb_1\wedge b_{-2})h)=\varphi_{N_P,\varepsilon_{[y_0,T]}}(\exp((t+s)b_1\wedge b_{-2})h)$. So (7.5) unfolds to the expression in the proposition. $\square$ $\square$

The following corollary is a consequence of Lemma 7.1, Corollary 4.10, and the fact that $y_0\in V_{1,n}^{\perp}$.

Corollary 7.2. *Suppose $\varphi\colon G(\mathbf{Q})\backslash G(\mathbf{A})\to\mathbf{V}_\ell$ is a weight ℓ cuspidal quaternionic modular form on G. Then $\xi_T^{\varphi}(h)\not\equiv 0$ only if $T\in V_{1,n}$ is positive definite. Moreover, with notation as in (4.9) the Fourier expansion of ξ^{φ} along R takes the form*

$$\xi^{\varphi}(h_fh_\infty)=\sum_{T\in V_{1,n}\colon (T,T)>0} a_T(\xi^{\varphi},h_f)\int_{\mathbf{R}}\mathcal{W}_{[2\pi y_0,2\pi T]}(\exp(s_\infty b_1\wedge b_{-2})h_\infty)ds_\infty \tag{7.6}$$

where $h_fh_\infty\in H(\mathbf{A})$ and $a_T(\xi^{\varphi},h_f)=\int_{\mathbf{A}_f}a_{[y_0,T]}(\varphi,\exp(s_fb_1\wedge b_{-2})h_f)ds_f$.

7.2. **The Archimedean Component of ξ_T^{φ}.** The summand appearing in (7.6) is the same as the summand appearing in the Fourier expansion of a holomorphic modular form on $H(\mathbf{A})$. In this subsection we further refine (7.6) and explain how our refinement can be used to obtain a genuine scalar valued holomorphic modular form on $H(\mathbf{R})$. First we recall that M_R is identified with the product $\mathbb{G}_m\times\mathrm{SO}(V_{1,n})$ via its action on the decomposition (4.1). We write elements $m\in M_R$ as pairs $m=(t,u)$ with the understanding that $t\in\mathbb{G}_m$ and $u\in\mathrm{SO}(V_{1,n})$. The coordinate $t\in\mathbb{G}_m$ is normalized so that $(t,1)\cdot b_2=tb_2$.

Proposition 7.3. *Fix $T\in V_{1,n}$ such that T is positive definite and $[y_0,T]\succeq 0$. If $h_\infty=(t,u)\in M_R(\mathbf{R})^0$ with $t\in\mathbf{R}_{>0}$ and $u\in\mathrm{SO}(V_{1,n})(\mathbf{R})^0$ then*

$$\int_{\mathbf{R}}\mathcal{W}_{[y_0,T]}(\exp(s_\infty b_1\wedge b_{-2})h_\infty)ds_\infty=\frac{\pi t^\ell e^{-(2-t(T,u\cdot y_1))}}{2}\sum_{-\ell\le v\le\ell} i^v\frac{x^{\ell+v}y^{\ell-v}}{(\ell-v)!(\ell+v)!}. \tag{7.7}$$

Proof. To begin, we apply (4.8) to explicate the integrand in the left hand side of (7.7). Let $s\in\mathbf{R}_{>0}$ and recall that $h_\infty=(t,u)$ with $t\in\mathbf{R}_{>0}$ and $u\in\mathrm{SO}(V_{1,n})(\mathbf{R})^0$. Then unraveling definitions one obtains

$$\beta_{[y_0,T]}(\exp(sb_1\wedge b_{-2})h_\infty)=-2st+i(2-t(T,u\cdot y_1)) \tag{7.8}$$

To simplify notation we write $w = t(T, u \cdot y_1)$. If $-\ell \leq v \leq \ell$ then (7.8) together with a manipulation in elementary calculus gives

$$\int_{-\infty}^{\infty} \left(\frac{\beta_{[y_0,T]}(\exp(sb_1 \wedge b_{-2})h_\infty)}{|\beta_{[y_0,T]}(\exp(sb_1 \wedge b_{-2})h_\infty)|} \right)^v K_v(|\beta_{[y_0,T]}(\exp(sb_1 \wedge b_{-2})h_\infty)|)ds$$
$$= \frac{1}{t} \sum_{k=0}^{\lfloor |v|/2 \rfloor} \binom{|v|}{2k} (i\,\mathrm{sgn}(v)(2-w))^{|v|-2k} \int_0^\infty \frac{s^{2k}}{\sqrt{(s^2+|2-w|^2)^{|v|}}} K_{|v|}\left(\sqrt{s^2+|2-w|^2}\right) ds.$$

Hence the formula [GR07, pg. 693, 6.596(3)] together with (4.8) implies

$$\int_{\mathbf{R}} \mathcal{W}_{[y_0,T]}(\exp(s_\infty b_1 \wedge b_{-2})h_\infty)ds_\infty$$
$$= t^\ell \sum_{-\ell \leq v \leq \ell} \left(\sum_{k=0}^{\lfloor |v|/2 \rfloor} \binom{|v|}{2k} \frac{(i\,\mathrm{sgn}(v)(2-w))^{|v|-2k} 2^{(2k-1)/2} \Gamma\left(\frac{2k+1}{2}\right)}{|2-w|^{|v|-(2k+1)/2}} \right.$$
$$\left. \times K_{|v|-(2k+1)/2}(|2-w|) \frac{x^{\ell+v} y^{\ell-v}}{(\ell-v)!(\ell+v)!} \right). \tag{7.9}$$

Claim 7.4. *If $h_\infty \in M_R(\mathbf{R})^0$ then $\mathrm{Im}(\beta_{[y_0,T]}(h_\infty)) = 2 - w$ is positive.*

To begin the proof of Claim (7.4) we note that if $h_\infty \in M_R(\mathbf{R})^0$ satisfies $\mathrm{Im}(\beta_{[y_0,T]}(h_\infty)) = 0$ then (7.8) implies that $\beta_{[y_0,T]}(h_\infty) = 0$ which contradicts our assumption that $[y_0, T] \succeq 0$. This means that $2 - t(T, u \cdot y_1) \neq 0$ for all $t \in \mathbf{R}_{>0}$ and $u \in \mathrm{SO}(V_{1,n})(\mathbf{R})^0$. Claim (7.4) now follows from the fact that $(T, u \cdot y_1) \neq 0$ for all $u \in \mathrm{SO}(V_{1,n})(\mathbf{R})^0$.

Given $X \in \mathbf{R}_{>0}$ and $-\ell \leq v \leq \ell$ define

$$S_v(X) := \sum_{k=0}^{\lfloor |v|/2 \rfloor} \binom{|v|}{2k} \frac{(i\mathrm{sgn}(v)X)^{|v|-2k} 2^{(2k-1)/2} \Gamma\left(\frac{2k+1}{2}\right)}{X^{|v|-(2k+1)/2}} K_{|v|-(2k+1)/2}(X). \tag{7.10}$$

Using Claim 7.4, the expression (7.9) simplifies to the form $t^\ell \sum_{-\ell \leq v \leq \ell} S_v(2-w) \frac{x^{\ell+v} y^{\ell-v}}{(\ell-v)!(\ell+v)!}$. Thus to complete the proof of Proposition (7.3) it remains to establish the following claim.

Claim 7.5. *If $-\ell \leq v \leq \ell$ then $S_v(X) = \frac{\pi e^{-X} i^v}{2}$.*

The formula [GR07, pg. 925, 8.468] implies that $S_0(X) = \pi e^{-X}/2$ as required. Moreover, by inspection of (7.10) we have $S_{-v}(X) = (-1)^v S_v(X)$. Hence we may assume that $1 \leq v \leq \ell$. It follows from [GR07, pg. 925, 8.468] that

$$S_v(X) = \frac{i^v \cdot \sqrt{\pi} \cdot e^{-X}}{\sqrt{2} X^{v-1}} \sum_{k=0}^{\lfloor v/2 \rfloor} \sum_{j=0}^{v-k-1} (-1)^k \binom{v}{2k} \frac{(v-k-1+j)! 2^{k-j-1/2} \Gamma(k+1/2)}{j!(v-k-1-j)!} X^{v-1-k-j}. \tag{7.11}$$

If $k \in \mathbf{Z}_{\geq 0}$ then $\Gamma(k+1/2) = (2k)!\sqrt{\pi} \cdot 4^{-k}/k!$. Therefore (7.11) is equal to the expression

$$\frac{i^v \pi e^{-X}}{2 \cdot X^{v-1}} \sum_{k=0}^{\lfloor v/2 \rfloor} \sum_{j=0}^{v-k-1} (-1)^k \cdot \frac{v!(v-k-1+j)! 2^{-k-j}}{(v-2k)!(v-k-1-j)! j! k!} X^{v-1-k-j} \tag{7.12}$$

Let $m = j + k$ so that $k = m - j$. Then (7.12) may be re-expressed in the form

$$\frac{i^v \pi e^{-X}}{2 \cdot X^{v-1}} \sum_{m=0}^{v-1} \sum_{j=0}^{m} (-1)^{m-j} \cdot \frac{v!(v-(m-j)-1+j)! 2^{-m}}{(v-2m+2j)!(v-1-m)!j!(m-j)!} X^{v-1-m}$$

$$= \frac{i^v \pi e^{-X}}{2} + \frac{i^v \pi e^{-X}}{2 \cdot X^{v-1}} \sum_{m=1}^{v-1} \sum_{j=0}^{m} (-1)^{m-j} \cdot \frac{v!(v-1-m+2j)! 2^{-m}}{(v-2m+2j)!(v-1-m)!j!(m-j)!} X^{v-1-m}$$

$$= \frac{i^v \pi e^{-X}}{2} + \frac{i^v \pi e^{-X}}{2 \cdot X^{v-1}} \sum_{m=1}^{v-1} (-1)^m \frac{v! 2^{-m}}{m!(v-1-m)!} \sum_{j=0}^{m} (-1)^j \cdot \frac{(v-1-m+2j)!}{(v-2m+2j)!} \binom{m}{j} X^{v-1-m}$$

$$= \frac{i^v \pi e^{-X}}{2}$$

The last equality above uses the fact that $\sum_{k=0}^{m} (-1)^k \binom{m}{k} F(k) = 0$ whenever $F(k)$ is polynomial in k of degree less than m. The proof of Claim 7.5 is now complete, which completes the proof of Proposition 7.3. $\square$

Let $\{\cdot, \cdot\}_{K^0} \colon \mathbf{V}_\ell \times \mathbf{V}_\ell \to \mathbf{C}$ denote the unique K^0-invariant symmetric bilinear form on $\mathbf{V}_\ell$ normalized so that

$$\{x^{\ell+v} y^{\ell-v}, x^{\ell-w} y^{\ell+w}\}_{K^0} = (-1)^{\ell+v} \delta_{v,w} (\ell+v)!(\ell-v)!$$

where $v, w \in \{-\ell, \cdots, \ell\}$ and $\delta_{v,w}$ is the Dirac delta symbol. Before proving the next corollary we introduce coordinates on the maximal compact subgroup $K_H := K \cap H$ and describe how the identity component K_H^0 acts on the representation $\mathbf{V}_\ell$. Let $W_2^+(\mathbf{R}) = \mathbf{R}\text{-span}\{u_2, v_2\}$ and let $W_{n+1}^-(\mathbf{R})$ denote the orthogonal complement $W_2^+(\mathbf{R})$ inside of $V_{2,n+1}(\mathbf{R})$. Then

$$V_{2,n+1}(\mathbf{R}) = W_2^+(\mathbf{R}) \oplus W_{n+1}^-(\mathbf{R}) \tag{7.13}$$

is a decomposition of $V_{2,n+1}(\mathbf{R})$ into definite subspaces. The maximal compact subgroup $K_H = K \cap H(\mathbf{R})$ is the stabilizer in $H(\mathbf{R})$ of the decomposition (7.13). So K_H has identity component $K_H^0 \simeq \mathrm{SO}(2) \times \mathrm{SO}(n+1)$. To describe the action of K_H^0 on $\mathbf{V}_\ell$ we first note that $\mathrm{SO}(n+1) \leq \mathrm{SO}(V^-)$ and thus $\mathrm{SO}(n+1)$ acts trivially on $\mathbf{V}_\ell$. It remains to describe the action of the subgroup

$$\mathrm{SO}(2) = \{\exp(\theta u_2 \wedge v_2) \colon \theta \in \mathbf{R}\}.$$

Since $u_2 \wedge v_2 = -\frac{1}{2}(e^+ - f^+) + \frac{1}{2}(e'^+ - f'^+)$ and $\{x, y\}$ is weight basis for the action of the $\mathfrak{sl}_2$-triple $\{e^+, h^+, f^+\}$ we have that

$$\exp(\theta u_2 \wedge v_2) \cdot (-ix + y)^{2\ell} = e^{-i\ell\theta} (-ix + y)^{2\ell}.$$

Theorem 1.1 is now proved with the following corollary.

Corollary 7.6. *Suppose $\varphi \colon G(\mathbf{Q}) \backslash G(\mathbf{A}) \to \mathbf{V}_\ell$ is a weight ℓ cuspidal quaternionic modular form on G. Let ξ^φ denote the $\mathbf{V}_\ell$ valued automorphic form obtain by restricting the χ_{y_0}-th Fourier coefficient of φ to the subgroup $H(\mathbf{A}) \leq M_Q(\mathbf{A})$ (see §7.1). Then the function*

$$\mathrm{FJ}_\varphi \colon H(\mathbf{A}) \to \mathbf{C}, \qquad h \mapsto \overline{\{\xi^\varphi(h), (-ix + y)^{2\ell}\}_{K^0}}$$

is the automorphic function corresponding to a weight ℓ holomorphic modular form on H. Moreover, the classical Fourier coefficients of FJ_φ are finite sums of the quaternionic Fourier coefficients of φ

Proof. For $h \in H(\mathbf{R})$, define $W_T^{FJ}(h)$ as

$$W_T^{FJ}(h) = \left\{ \int_{\mathbf{R}} \mathcal{W}_{[y_0,T]}(\exp(s_\infty b_1 \wedge b_{-2})h) ds_\infty, (-ix+y)^{2\ell} \right\}_{K^0}.$$

We claim that $\overline{W_T^{FJ}(hk)} = j(k, -iy_1)^{-\ell}\overline{W_T^{FJ}(h)}$ for all $h \in H(\mathbf{R})^0$ and $k \in K_H^0$, and that the function $Z \mapsto j(g, -iy_1)^{\ell}\overline{W_T^{FJ}(g)}$, where $Z = g \cdot i(-y_1)$, is a holomorphic function of Z. This will establish the corollary, and also give the form of the Fourier expansion of holomorphic modular form associated to ξ^{φ}.

Recall that $u_2 = \frac{1}{\sqrt{2}}(b_2 + b_{-2})$ and $v_2 = \frac{1}{\sqrt{2}}(b_4 + b_{-4}) = \frac{1}{\sqrt{2}}y_1$. Above it is calculated that

$$\exp(\theta u_2 \wedge v_2)(-ix+y)^{2\ell} = e^{-i\ell\theta}(-ix+y)^{2\ell}$$

and thus

$$W_T^{FJ}(he^{\theta u_2 \wedge v_2}) = e^{-i\ell\theta} W_T^{FJ}(h).$$

To prove the first claim, we must compute that, if $k = e^{\theta u_2 \wedge v_2}$, then $j(k, -iy_1) = e^{-i\theta}$. But an immediate check gives $u_2 \wedge v_2 \cdot (b_2 + iy_1 + b_{-2}) = i(b_2 + iy_1 + b_{-2})$, which implies that indeed $j(k, -iy_1) = e^{-i\theta}$.

For the second claim, suppose $g = \exp(x \wedge b_2)h$, where $h = \mathrm{diag}(t, u, t^{-1})$. Then one calculates $g \cdot (-iy_1) = Z$, with $Z = x + itu \cdot (-y_1)$ and $j(g, -iy_1) = t^{-1}$. Now we have immediately from the computation Proposition 7.3 of $W_T^{FJ}(h)$ that there is a nonzero constant η so that

$$j(g, -iy_1)^{\ell}\overline{W_T^{FJ}(g)} = t^{-\ell}e^{i(T,x)}\overline{W_T^{FJ}(h)} = \eta e^{i(T,x)}e^{t(T,u\cdot y_1)} = \eta e^{i(T,Z)}.$$

The corollary follows. □

7.3. **Fourier coefficients for** Sp_4**.** In this subsection we specialize to the case $n = 2$ so that $H = \mathrm{SO}(V_{2,3})$. Our first task is to describe a map $\mathrm{Sp}_4 \to H$ in explicit coordinates.

Recall $y_0 = b_3 + b_{-3}$, $y_1 = b_4 + b_{-4}$, $V_{1,2} = \mathrm{Span}(b_3 - b_{-3}, b_4, b_{-4})$, and $V_{2,3} = \mathbf{Q}b_2 + V_{1,2} + \mathbf{Q}b_{-2}$. Let W_4 be the standard representation of Sp_4, with symplectic basis e_1, e_2, f_1, f_2 and write V_5 for the kernel of the contraction map $\wedge^2 W_4 \to \mathbf{Q}$ given by the symplectic pairing. The Sp_4-representation $\wedge^2 W_4$ is identified with $\mathrm{Span}(b_2, b_3, b_4, b_{-4}, b_{-3}, b_{-2})$ via

- $b_2 = e_1 \wedge e_2$
- $b_3 = e_1 \wedge f_1$
- $b_4 = e_1 \wedge f_2$
- $b_{-4} = -e_2 \wedge f_1$
- $b_{-3} = e_2 \wedge f_2$
- $b_{-2} = -f_1 \wedge f_2$.

We put a blinear form on $\wedge^2 W_4$ as $u_1 \wedge u_2 = (u_1, u_2)e_1 \wedge f_1 \wedge e_2 \wedge f_2$ for $u_1, u_2 \in \wedge^2 W_4$. Then the above identification of bases respects the quadratic forms on both sides. Moreover, in this identification, $V_5 = \mathrm{Span}(b_2, b_3 - b_{-3}, b_4, b_{-4}, b_{-2})$, which is $V_{2,3}$.

Because Sp_4 acts on V_5 preserving the quadratic form, and because the identification of V_5 with $V_{2,3}$ respects the quadratic forms on each, we obtain a map $\pi \colon \mathrm{Sp}_4 \to \mathrm{SO}(V_5) \simeq \mathrm{SO}(V_{2,3})$. If φ is an automorphic function on $\mathrm{SO}(V_{2,3})$, define $\varphi^* := \varphi \circ \pi$.

Let

$$K_{\mathrm{Sp}_4} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : A + iB \in U(2) \right\}$$

denote the standard maximal compact subgroup of $\mathrm{Sp}_4(\mathbf{R})$. If $j_{\mathrm{Sp}_4}(\gamma, Z)$ is the standard factor of automorphy, then $j_{\mathrm{Sp}_4}(\bullet, i1_2)$ is a character $K_{\mathrm{Sp}_4} \to \mathbf{C}^\times$. We claim that if $k \in K_{\mathrm{Sp}_4}$

then $j_{\mathrm{Sp}_4}(k, i1_2) = j_{\mathrm{SO}(V_{2,3})}(\pi(k), -iy_1)$. Here $j_{\mathrm{SO}(V_{2,3})}$ is the factor of automorphy defined in Section 4.1. Indeed, to compute $j_{\mathrm{SO}(V_{2,3})}(\pi(k), -iy_1)$, we simply let $\pi(k)$ act on

$$b_2 - iy_1 + b_{-2} = e_1 \wedge e_2 - f_1 \wedge f_2 - i(e_1 \wedge f_2 - e_2 \wedge f_1) = (e_1 - if_1) \wedge (e_2 - if_2).$$

Now the claim is immediately verified.

If φ corresponds to a holomorphic modular form, then so does φ^*, and their Fourier coefficients are related as follows. Suppose $s = \left(\begin{smallmatrix} u & v \\ v & w \end{smallmatrix}\right)$ and set $n(s) = \left(\begin{smallmatrix} 1 & s \\ & 1 \end{smallmatrix}\right) \in \mathrm{Sp}_4$. Then one computes, under our identification of bases of V_5 with $V_{2,3}$, that

$$n(s) \cdot b_{-2} = \exp(b_2 \wedge (v(b_3 - b_{-3}) - ub_4 - wb_{-4})) \cdot b_{-2}. \tag{7.14}$$

Moreover, if $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ then

$$(T, s) = au + bv + cw = (v(b_3 - b_{-3}) - ub_4 - wb_{-4}, -(cb_4 + bb_3 + ab_{-4})). \tag{7.15}$$

From combining (7.14) and (7.15) one sees that the T-th Fourier coefficient of φ^* is the $-(cb_4 + bb_3 + ab_{-4})$ Fourier coefficient of φ.

7.4. **Application of triality to Fourier coefficients.** Recall the Lie algebra isomorphism $\Phi\colon \wedge^2 \Theta \xrightarrow{\sim} \mathfrak{g}_E$ (see (A.1)). Applying the identification (3.4), Φ^{-1} maps the element

$$\alpha' E_{12} + v_1 \otimes (\beta_1', \beta_2', \beta_3') + \delta_3 \otimes (\gamma_1', \gamma_2', \gamma_3') + \delta' E_{23} \in \mathfrak{g}_E$$

to the element

$$b_1 \wedge y_1' + b_2 \wedge y_2' = b_1 \wedge (\beta_3' b_3 - \alpha' b_4 + \gamma_2' b_{-4} + \beta_1' b_{-3}) + b_2 \wedge (\gamma_1' b_3 - \beta_2' b_4 + \delta b_{-4} + \gamma_3' b_{-3}) \in \wedge^2 V.$$

Consequently, if $(T_1, T_2) \in \mathrm{Span}(b_3, b_4, b_{-4}, b_{-3})$ and L is the linear map on the abelianization of the unipotent radical of the Heisenberg nilpotent given by

$$L(b_1 \wedge y_1' + b_2 \wedge y_2') = (T_1, y_1') + (T_2, y_2'),$$

then L corresponds to the element $w \in W_E$, $w = (\alpha, \beta, \gamma, \delta)$, where

$$\begin{aligned} T_1 &= \gamma_1 b_3 - \beta_2 b_4 + \delta b_{-4} + \gamma_3 b_{-3} \\ T_2 &= -\beta_3 b_3 + \alpha b_4 - \gamma_2 b_{-4} - \beta_1 b_{-3}. \end{aligned}$$

That is, with the above definitions of T_1, T_2, y_1', y_2', one has

$$\langle (\alpha, \beta, \gamma, \delta), (\alpha', \beta', \gamma', \delta') \rangle = (T_1, y_1') + (T_2, y_2').$$

We are now ready to prove Theorem 1.2 as the following corollary.

Corollary 7.7. *Suppose F is a quaternionic modular form of even weight ℓ on G and level one. Then there is a Siegel modular form $f(Z)$ on Sp_4 of weight ℓ and level one so that one has an identity of Fourier coefficients*

$$a_f\left(\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\right) = \overline{a_F(T_1, T_2)}$$

where $T_1 = -bb_4 + ab_3 + b_{-3}$ and $T_2 = -cb_4 - b_{-4}$

Proof. Observe that $(b_3 + b_{-3}, -(cb_4 + bb_3 + ab_{-4}))$ corresponds to the element

$$(-c, (0, 0, b), (1, a, 1), 0) \in W.$$

Applying triality in the form of Theorem A.1, we obtain $(-c, (0, b, 0), (a, 1, 1), 0) \in W$, which then corresponds to the pair $(-bb_4 + ab_3 + b_{-3}, -cb_4 - b_{-4})$. This completes the proof. $\square$

One sees that the pair $[T_1, T_2]$ is strongly primitive because of $T_1 = -bb_4 + ab_3 + b_{-3}, T_2 = -cb_4 - b_{-4}$, and $m \in M_2(\mathbf{Q})$ satisfies $[T_1, T_2]m$ is integral, then $m \in M_2(\mathbf{Z})$. Moreover, $S(T_1, T_2) = \left(\begin{array}{cc} a & b/2 \\ b/2 & c \end{array}\right)$. As a consequence, we obtain:

Corollary 7.8. *Suppose φ is a level one, cuspidal, quaternionic modular form of even weight $\ell \geq 16$ that is in the Maass Spezialschar. Then there is a holomorphic cuspidal modular form f on Sp_4 so that $\varphi = \theta^*(f)$. In particular, φ is in the Saito-Kurokawa subspace.*

Proof. Let f be the weight ℓ, level one holomorphic Siegel modular form for Sp_4 constructed in Corollary 7.7. By Theorem 5.1, the Fourier coefficients of $\theta^*(f)$ agree with those of φ for $[T_1, T_2]$ of the form $[-bb_4 + ab_3 + b_{-3}, -cb_4 - b_{-4}]$. Because both $\theta^*(f)$ and φ are in the quaternionic Maass Spezialschar, it follows from Definition 5.8 that the Fourier coefficients of φ and $\theta^*(f)$ agree for all $[T_1, T_2]$. Consequently, $\varphi = \theta^*(f)$ as desired. $\square$

Theorem 1.3 follows immediately from the corollary.

8. Hecke stability and theta lifts

Suppose φ is a cuspidal quaternionic eigenform on G. The purpose of this section is to give some equivalent conditions for φ to be in the Saito-Kurokawa subspace. In particular, we prove Theorem 8.10, which gives a mostly representation theoretic criterion for φ to be quaternionic Saito Kurokawa lift. One consequence of this theorem is Corollary 8.11, which is our starting point for the work of section 9 on periods.

The results we prove in this section are mostly derived from what is known about the theta correspondence between symplectic and orthogonal groups. Because we work with quaternionic modular forms on a special orthogonal group, instead of an orthogonal group, we spend some effort relating representations and automorphic forms on these two groups.

8.1. Restriction from $O(V)$.

Recall that V is our split quadratic space of dimension 8 with integral lattice $V(\mathbf{Z})$ spanned by the b_i, and $G = \mathrm{SO}(V)$. Set $G_1 = O(V)$, the orthogonal group of V. We write $K_{1,\ell} = G_1(\mathbf{Z}_\ell)$ for the subgroup of $G_1(\mathbf{Q}_\ell)$ that stabilizes $V(\mathbf{Z}_\ell)$ and $G_1(\widehat{\mathbf{Z}}) = \prod_\ell G_1(\mathbf{Z}_\ell)$. Set $k_0 \in G_1$ to be the element that fixes b_i for $i \in \{1, 2, 3, -1, -2, -3\}$ and exchanges b_4 with b_{-4}. We will at various times consider k_0 in $G_1(\mathbf{Q}_v)$ or $G_1(\mathbf{Q})$ or $K_{1,\ell}$.

We begin with a simple lemma.

Lemma 8.1. *The natural map*

$$G(\mathbf{Q})\backslash G(\mathbf{A})/G(\widehat{\mathbf{Z}}) \to G_1(\mathbf{Q})\backslash G_1(\mathbf{A})/G_1(\widehat{\mathbf{Z}})$$

is a bijection.

Proof. For the surjectivity, if $g_1 \in G_1(\mathbf{A})$, there exists $\gamma_1 \in G_1(\mathbf{Q})$ so that $(\gamma_1)_\infty g_{1,\infty} \in G(\mathbf{R})$, and thus there exists $k \in G_1(\widehat{\mathbf{Z}})$ so that $\gamma_1 g_1 k \in G(\mathbf{A})$. For the injectivity, if $g, g' \in G(\mathbf{A})$, and $g' = \gamma g k$ for some $\gamma \in G_1(\mathbf{Q})$ and $k \in G_1(\widehat{\mathbf{Z}})$, then comparing determinants at the archimedean place proves $\gamma \in G(\mathbf{Q})$, and at all the finite places proves $k \in G(\widehat{\mathbf{Z}})$. $\square$

Recall from Definition 4.2 that $\mathcal{A}_{0,\mathbf{Z}}(G, \ell)$ denotes the weight ℓ quaternionic cuspidal modular forms that are right invariant under $G(\widehat{\mathbf{Z}})$. Similarly denote $\mathcal{A}_{0,\mathbf{Z}}(G_1, \ell)$ the cuspidal quaternionic modular forms on G_1 that are right invariant under $G_1(\widehat{\mathbf{Z}})$, where in the definition of $\mathcal{A}(G_1, \ell)$ we assume the functions are valued in $\mathbf{V}_\ell$ (instead of a larger space) and are equivariant only for the identity component K^0 of a fixed maximal compact subgroup of $G_1(\mathbf{R})$.

We have a restriction map $Res : \mathcal{A}_{0,\mathbf{Z}}(G_1,\ell) \to \mathcal{A}_{0,\mathbf{Z}}(G,\ell)$. We use the lemma to define a lifting map $L : \mathcal{A}_{0,\mathbf{Z}}(G,\ell) \to \mathcal{A}_{0,\mathbf{Z}}(G_1,\ell)$.

Proposition 8.2. *Regarding the maps L and Res, we have the following facts:*

1. *The maps L and Res are inverse isomorphisms.*
2. *If φ is a Hecke eigenform on G, then $L(\varphi)$ is a Hecke eigenform on G_1.*

Proof. Part (1) of the proposition is immediate. For part (2), suppose $T \in \mathcal{H}_p(G_1)$, the algebra of smooth, compactly supported functions on $G_1(\mathbf{Q}_p)$ that are bi-invariant under $G_1(\mathbf{Z}_p)$. Let T_1 be the restriction of T to $G(\mathbf{Q}_p)$, so that $T_1 \in \mathcal{H}_p(G)$, the Hecke algebra of $G(\mathbf{Q}_p)$. Then $T(g) = T_1(g)$ if $g \in G(\mathbf{Q}_p)$ and $T(g) = T_1(gk_0)$ if $\det(g) = -1$. Now, because $L(\varphi)$ is invariant by k_0, we have $TL(\varphi) = T_1L(\varphi)$ (normalizing measures appropriately) so $Res(TL(\varphi)) = T_1\varphi = \lambda_1\varphi$. Applying L gives the desired conclusion. $\square$

Let T denote the diagonal torus of G or G_1. Let χ be an unramified character of $T(\mathbf{Q}_p)$, π_χ the irreducible unramified subquotient of $Ind_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(\delta_B^{1/2}\chi)$, and $\pi_{1,\chi}$ the irreducible unramified subquotient of $Ind_{B_1(\mathbf{Q}_p)}^{G_1(\mathbf{Q}_p)}(\delta_B^{1/2}\chi)$, where δ_B is the modulus character for the Borel subgroup of G. Here $B_1 \simeq B$ is the upper triangular subgroup of G_1. Let the normalized spherical vectors in the inductions be ϕ and ϕ_1. Observe that the restriction of ϕ_1 to G is ϕ.

Lemma 8.3. *Let the notation be as above. Suppose $\chi(\mathrm{diag}(1,1,1,p,p^{-1},1,1,1)) = 1$. Then the restriction to $G(\mathbf{Q}_p)$ of $\pi_{1,\chi}$ is π_χ, i.e., $\pi_{1,\chi}$ restricts irreducibly to $G(\mathbf{Q}_p)$.*

Proof. Let V_1 be the space of $\pi_{1,\chi}$ and V the space of π_χ. We write I_1 for the induction $Ind_{B_1(\mathbf{Q}_p)}^{G_1(\mathbf{Q}_p)}(\delta_B^{1/2}\chi)$ and I for $Ind_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(\delta_B^{1/2}\chi)$. Thus V_1 is the unique unramified subquotient of I_1 and V is the unique unramified subquotient of I.

We have a restriction map $I_1 \to I$, which is $G(\mathbf{Q}_p)$ equivariant. We let I_1' be the subspace of I_1 given by those functions f that satisfy $f(k_0g) = f(g)$ for all $g \in G_1(\mathbf{Q}_p)$. Note that the property of χ assumed in the lemma implies that I_1' contains the spherical vector ϕ_1. We claim that the restriction map defines an isomorphism of $G(\mathbf{Q}_p)$ representations $I_1' \to I$.

To see surjectivity, suppose $f \in I$. Define a function f_1 on $G_1(\mathbf{Q}_p)$ as $f_1(g) = f(g)$ if $g \in G(\mathbf{Q}_p)$, and $f_1(k_0g) = f(g)$ if again $g \in G(\mathbf{Q}_p)$. One checks that $f_1 \in I_1'$, using the property of χ assumed for this lemma. For injectivity, if $f_1(g) = 0$ for all $g \in G(\mathbf{Q}_p)$ and some $f_1 \in I_1'$, then $f_1(k_0g) = 0$ so $f = 0$. Consequently, $I_1' \to I_1$ is an isomorphism.

Let $U \subseteq I$ be the submodule generated by the spherical vector ϕ, and let $U_1 \subseteq I_1'$ be the submodule generated by the spherical vector ϕ_1. Let $\mathrm{vol}(K_p)$ denote the volume of K_p and $\mathrm{vol}(K_{1,p})$ denote the volume of $K_{1,p}$ for a fixed choice of Haar measure. Define

$$p_{K_p}(v) = \frac{1}{\mathrm{vol}(K_p)}\int_{K_p} k \cdot v\, dk$$

and similarly for $p_{K_{1,p}}$, so that these are projections. Now set $M = \{u \in U : p_{K_p}(gu) = 0\, \forall g \in G(\mathbf{Q}_p)\}$ and $M_1 = \{u \in U_1 : p_{K_{1,p}}(gu) = 0\, \forall g \in G_1(\mathbf{Q}_p)\}$. One has that M is the maximal proper submodule of U and M_1 is the maximal proper submodule of U_1. We have $V_1 = U_1/M_1$ and $V = U/M$.

If $f \in I$ or I_1', then $p_{K_{1,p}}(g \cdot f)(h) = \mathrm{vol}(K_{1,p})^{-1}\int_{K_{1,p}} f(hk_1g)\, dk_1$ and $p_{K_p}(g \cdot f)(h) = \mathrm{vol}(K_p)^{-1}\int_{K_p} f(hkg)\, dk$. Now observe that if $f \in I_1'$, then $Res(p_{K_{1,p}}(f)) = p_{K_p}(Res(f))$.

Indeed, if $h = bk$, with $k \in K_p$ and $b \in B$, then

$$\begin{aligned}\mathrm{vol}(K_{p,1})Res(p_{K_{1,p}}(f))(bk) &= \int_{K_{1,p}} f(bkk_1)\,dk_1 = \chi(b)\int_{K_{1,p}} f(k_1)\,dk_1 \\ &= 2\chi(b)\int_{K_p} f(k)\,dk = \mathrm{vol}(K_{1,p})p_{K_p}(Res(f))(bk).\end{aligned}$$

It follows that the restriction map induces a well-defined injection $V_1 = U_1/M_1 \to V = U/M$. As V is irreducible, this map is an isomorphism. □

Proposition 8.4. *For $\ell \geq 16$ even, the Saito-Kurokawa subspace of $\mathcal{A}_{0,\mathbf{Z}}(G,\ell)$ is Hecke stable.*

Proof. Let $f_1,\ldots,f_N$ be a basis of the weight ℓ, level one, holomorphic cuspidal Siegel modular forms on Sp_4. Let $F_j = L(\theta^*(f_j))$ be the theta lift of f_j to G_1, so that $\theta^*(f_j) = Res(F_j)$. Then each F_j is nonzero, and so the automorphic theta module $\Pi_{F_j} = \Theta_{G_1}(\pi_{f_j}) \neq 0$. By Howe duality [GT16] it is thus irreducible. Consequently, F_j is a Hecke eigenform. If the Satake parameters at a prime p of f_j are $\mathrm{diag}(\alpha_j,\beta_j,1,\beta_j^{-1},\alpha_j^{-1})$, it follows from [Ral82], as explained in [Gan23], that the Satake parameters of F_j at p are $\mathrm{diag}(\alpha_j,\beta_j,p,1,1,p^{-1},\beta_j^{-1},\alpha_j^{-1})$. For these particular Satake parameters, i.e., with two 1's, the restriction from G_1 to G remains irreducible by Lemma 8.3. Consequently, the finite part $\Pi_{F_j,f}$ restricts irreducibly to G, so $\theta^*(f_j) = Res(F_j)$ is again an eigenform. Thus SK_ℓ is spanned by eigenforms, so is Hecke stable. □

We will need Lemma 8.6 (below) soon. We setup this lemma now. Recall from Definition 4.3 that for an integer $\ell \geq 4$, we let π_ℓ^0 be the quaternionic discrete series representation of $G(\mathbf{R})^0$ with minimal K^0-type $S^{2\ell}(\mathbf{C}^2)\otimes\mathbf{1}\otimes\mathbf{1}$, as a representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(4)$. It has the same infinitesimal character as the highest weight submodule of $\mathrm{Sym}^{\ell-4}(\wedge^2 V)$.

We set π_ℓ to be the induction from $G(\mathbf{R})^0$ to $G(\mathbf{R})$ of π_ℓ^0. Let W_ℓ denote the underlying vector space for the representation π_ℓ^0. As a $(\mathfrak{g},K)$-module, the underlying vector space of π_ℓ is $W_\ell' := W_\ell \oplus W_\ell$. Fix $\mu \in K \setminus K^0$. If $k \in K^0$, then k acts on W_ℓ' as $k(w_1,w_2) = (kw_1, \mu k\mu^{-1}w_2)$. The element μ acts on W_ℓ' as $\mu(w_1,w_2) = (w_2,\mu^2 w_1)$. If $X \in \mathfrak{g}$, then X acts on W_ℓ' as $X(w_1,w_2) = (Xw_1, Ad(\mu)(X)w_2)$.

The representation π_ℓ extends to $G_1(\mathbf{R})$, because it is isomorphic to its k_0-conjugate. There are two possible extensions, which we denote by Π_ℓ and $\Pi_\ell\otimes\det$. Let $O(4)^-$ be the subgroup of $G_1(\mathbf{R})$ that acts as the identity on V^+. We fix the notation so that $O(4)^-$ acts as the identity on the minimal K-types of Π_ℓ (and via the determinant map on the minimal K-types of $\Pi_\ell\otimes\det$.)

Definition 8.5. We define $\mathcal{A}_{0,\mathbf{Z}}^{rep}(G,\ell)$ to be the space of $(\mathfrak{g},K^0)$ homomorphisms from π_ℓ^0 to $\mathcal{A}_{0,\mathbf{Z}}(G)$. We define $\mathcal{A}_{0,\mathbf{Z}}^{rep}(G_1,\ell)$ to be the space of $(\mathfrak{g},K)$ homomorphisms from Π_ℓ to $\mathcal{A}_{0,\mathbf{Z}}(G_1)$.

If $\varphi \in \mathcal{A}(G)$, set $\iota(\varphi) \in \mathcal{A}(G)$ as $\iota(\varphi)(g) = \varphi(k_0 g k_0^{-1})$. We say that φ is in the *plus subspace* if $\iota(\varphi) = \varphi$. We remark that it follows immediately from the definition of the theta lift θ^* that any $\varphi = \theta^*(f)$ is in the plus subspace.

Lemma 8.6. *If $\varphi \in \mathcal{A}_{0,\mathbf{Z}}^{rep}(G,\ell)$ is in the plus subspace, then $L(\varphi)$ is in $\mathcal{A}_{0,\mathbf{Z}}^{rep}(G_1,\ell)$.*

Proof. Suppose $F : W_\ell \to \mathcal{A}_{0,\mathbf{Z}}(G)$. Let $\phi \in \pi_\ell$, so that $\phi = (w_1,w_2)$ in the notation above. Define $F' : W_\ell' \to \mathcal{A}(G)$ as $F'(\phi) = F(w_1) + \mu^{-1}F(w_2)$. One checks that F' is a $(\mathfrak{g},K)$ homomorphism.

Define now $\Psi : \Pi_\ell \to \mathcal{A}(G_1)$ as $\Psi(\phi) = L(F'(\phi))$. This is $(\mathfrak{g}, K)$ equivariant. Observe that $k_0^{-1} \circ \Psi \circ k_0$ and Ψ are both $(\mathfrak{g}, K)$-equivariant maps from Π_ℓ to $\mathcal{A}(G_1)$. If τ denotes the minimal K-type of π_ℓ^0 and $F(\tau)$ lands in the plus space, then one checks that $k_0^{-1} \circ \Psi \circ k_0$ and Ψ agree on τ. By irreducibility of Π_ℓ, they then agree on Π_ℓ. This proves the lemma. $\square$

8.2. **Lifts back to** Sp_4**.** We begin with a preliminary result.

Lemma 8.7. *Suppose $\Pi = \Pi_f \otimes \Pi_\infty$ is a cuspidal automorphic representation on G_1, with Π_∞ a quaternionic discrete series Π_ℓ. Let φ be a cusp form in the space of Π, and χ a degenerate character of U_P, the unipotent radical of the Heisenberg parabolic. Then the Fourier coefficient $\varphi_\chi(g) \equiv 0$.*

Proof. Let $\alpha : \Pi \to \mathcal{A}_0(G_1)$ be the given embedding. Let L_χ denote the $\mathbf{C}$-valued linear form on $\mathcal{A}_0(G_1)$ given by $L_\chi(\xi) = \xi_\chi(1)$. Suppose $\varphi = \alpha(v_f \otimes v_\infty)$. Let $\{v_j\}$ be a basis of the $\mathbf{V}_\ell \subseteq \Pi_\infty$, and $\{v_j^\vee\}$ the dual basis. Consider the function $F_\chi : G_1(\mathbf{R}) \to \mathbf{V}_\ell$ defined as $F_\chi(g) = \sum_j L_\chi(g\alpha(v_f \otimes v_j))v_j^\vee$. Then $F_\chi(g)$ is a generalized Whittaker function of type χ, and is a *bounded* function of g because α lands in cusp forms. Consequently, $F_\chi(g) \equiv 0$ on $G_1(\mathbf{R})^0$.

Let $\iota = \mathrm{diag}(\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right))$ and k_0 be as usual. Then the $1, \iota, k_0, \iota k_0$ are in the standard Levi of the Heisenberg parbolic, and meet every connected component of $G_1(\mathbf{R})$. If x is one of these representatives, then note that $F_\chi(xg) : G_1(\mathbf{R})^0 \to \mathbf{V}_\ell$ is again a bounded generalized Whittaker function for a degenerate character (namely, $\chi \cdot x$). Thus $F_\chi(xg) \equiv 0$. Consequently, $F_\chi(g)$ is 0 on all of $G_1(\mathbf{R})$, so $\varphi_\chi(g)$ must also be identically 0. $\square$

From Lemma 8.7, we now immediately obtain the following proposition. Recall that we write $V = U + V_{2,n} + U^\vee$.

Proposition 8.8. *Let $\Pi = \Pi_f \otimes \Pi_\infty$ be a cuspidal automorphic representation on G_1 with Π_∞ a quaternionic discrete series. For a vector $v \in V_{2,n}$, let $N_v = \exp(U \wedge (U^\perp \cap v^\perp))$ be a certain subgroup of the unipotent radical of the Heisenberg parabolic. One has the following facts.*

1. *Suppose $\varphi \in \Pi$. Then the constant term of φ over N_v is identically 0.*
2. *The theta lift of Π to SL_2 is 0.*
3. *The theta lift of Π to Sp_4 is cuspidal.*

Proof. For the first part, the constant term of φ over N_v can be expanded in term of degenerate Fourier coefficients of φ, so is 0 by Lemma 8.7. For the second part, standard calculation with the definition of the Schrodinger model of the Weil representation immediately shows that the Fourier coefficients of the theta lifts $\theta_\phi(\varphi)$ to SL_2 factor through periods of φ over subgroups $S_v = \{g \in G_1 : gv = v\}$. These subgroups contain the N_v's, so the periods vanish. The third part follows from the second by the Rallis tower property, or directly from the first part by a similar calculation with theta lifts and periods. $\square$

We will shortly use the following direct corollary of results of Yamana [Yam14].

Theorem 8.9. *Suppose Π_1, Π_2 are automorphic cuspidal representations of G_1, which are isomorphic (but not necessarily equal in the space of automorphic forms.)*

1. *One has that $\Theta_{\mathrm{Sp}_4}(\Pi_1) \neq 0$ if and only if $\Theta_{\mathrm{Sp}_4}(\Pi_2) \neq 0$.*
2. *Let $\mathcal{A}_0(G_1)[\Pi_1]$ denote the Π_1-isotypic subspace of the cusp forms on G_1. If there exists $f \in \mathcal{A}_0(\mathrm{Sp}_4)$ and test data ϕ so that $\theta_\phi(f) \in \mathcal{A}_0(G_1)[\Pi_1]$ is nonzero, then every element of this space is in the image of the theta lift from Sp_4.*

Proof. For the first statement, note that Yamana proves that the these theta lifts are nonzero precisely if the local theta lifts $\Theta(\Pi_{i,v}) \neq 0$ for every place v of $\mathbf{Q}$, and the L-function $L(\Pi_i, Std, s)$ has a pole at $s = 2$. These latter conditions only depend on the isomorphism type of the Π_i.

For the second statement, let $B \subseteq \mathcal{A}_0(G_1)[\Pi_1]$ be the space of the theta lifts from Sp_4, let C be its orthogonal complement in $\mathcal{A}_0(G_1)[\Pi_1]$. Then B is a $G_1(\mathbf{A})$-representation, and thus so is C. Suppose for the sake of contradiction that C is nonzero. Let $\Pi_2 \subseteq C$. By the first part, $\Theta_{\mathrm{Sp}_4}(\Pi_2) \neq 0$, so there exists f' a cusp form on Sp_4, test data ϕ and $\varphi_2 \in \Pi_2$ so that $\theta_\phi(f') \in \mathcal{A}_0(G_1)[\Pi_1]$ and

$$\langle \varphi_2, \theta_\phi(f')\rangle = \langle \theta_\phi(\varphi_2), f'\rangle \neq 0.$$

(We are using that if $f' \in \Theta_{\mathrm{Sp}_4}(\Pi_2)$, then $\theta_\phi(f') \in \mathcal{A}_0(G_1)[\Pi_1]$, which follows from local global compatibility of the theta correspondence.) This contradicts that Π_2 is orthogonal to B, proving the claim. $\square$

We now have the following result, which follows from Theorem 8.9.

Theorem 8.10. *Suppose $\varphi \in \mathcal{A}_{0,\mathbf{Z}}^{rep}(G,\ell)$ is a Hecke eigenform in the plus subspace. The following conditions are equivalent:*

(1) $\varphi \in SK_\ell$, i.e., there exists a level one holomorphic Siegel modular form f of weight ℓ so that $\varphi = \theta^*(f)$ in $\mathcal{A}_{0,\mathbf{Z}}(G,\ell)$;
(2) $\Theta_{\mathrm{Sp}_4}(\Pi_{L(\varphi)}) \neq 0$;

Proof. It is clear that (1) implies (2): Indeed, if $\varphi = \theta^*(f)$, then there is test data ϕ so that $L(\varphi) = \theta_\phi(f) \in \mathcal{A}_0(G_1)$ is nonzero. Since $\langle \theta_\phi(f), \theta_\phi(f)\rangle \neq 0$, we see that

$$\int_{[G_1]\times[\mathrm{Sp}_4]} \overline{\theta_\phi(f)(g)}\theta_\phi(g,h)\overline{f(h)}\, dg\, dh \neq 0,$$

so $\Theta_{\mathrm{Sp}_4}(\Pi_{L(\varphi)}) \neq 0$.

We now prove that (2) implies (1). Let π' be the irreducible cuspidal automorphic representation of Sp_4 which is $\Theta_{\mathrm{Sp}_4}(\Pi_{L(\varphi)})$. It is irreducible by Howe duality and is cuspidal by Proposition 8.8. By local-global compatibility (see [Gan23]), π' is generated by a level one holomorphic Siegel modular form f' of weight ℓ. It follows that $\Theta_{G_1}(\pi')$ has nonzero inner product with some element of $\Pi_{L(\varphi)}$, and that $\Theta_{G_1}(\pi')$ is isomorphic to $\Pi_{L(\varphi)}$. By Theorem 8.9, there exists a level one cuspidal automorphic representation π on Sp_4 for which $\Theta_{G_1}(\pi) = \Pi_{L(\varphi)}$. Indeed, there exists a cuspidal automorphic representation π_1 on Sp_4 and $f_1 \in \pi_1$ so that $\theta_\phi(f_1) \in \Pi_{L(\varphi)}$. Because $\Pi_{L(\varphi)}$ is irreducible, $\Theta_{G_1}(\pi_1) \supseteq \Pi_{L(\varphi)}$. But $\Theta_{G_1}(\pi_1)$ itself is irreducible, so $\Theta_{G_1}(\pi_1) = \Pi_{L(\varphi)}$. One sees that π_1 must be unramified at every finite place and holomorphic discrete series at infinity, so we can take $\pi = \pi_1$.

Finally, let $f \in \pi$ be the level one Siegel modular form. Then $\theta^*(f)$ is nonzero, but also in Π_φ, so we must have $\theta^*(f) = \varphi$ as desired. $\square$

For a half-integral 2×2 symmetric matrix S, let $P_{S,\ell}$ be the weight ℓ Poincare series on Sp_4 associated to S, and let $Q_{S,\ell}$ be the associated Poincare lift; see Proposition 5.3.

Corollary 8.11. *Suppose $\varphi \in \mathcal{A}_{0,\mathbf{Z}}^{rep}(G,\ell)$ is an eigenform in the plus subspace, for $\ell \geq 16$ even. Then $\varphi \in SK_\ell$ if and only if $\langle \varphi, Q_{S,\ell}\rangle \neq 0$ for some S.*

Proof. If $\varphi \in SK_\ell$, then clearly such an S exists: We have $\varphi = \theta^*(f)$, and $f = \sum_j \alpha_j P_{S_j,\ell}$, so that $\varphi = \sum_j \beta_j Q_{S_j,\ell}$, and hence $\langle \varphi, Q_{S_j,\ell}\rangle \neq 0$ for some j.

Conversely, suppose $\langle \varphi, Q_{S,\ell}\rangle \neq 0$ for some S. This inner product is equal to $\langle L(\varphi), L(Q_{S,\ell})\rangle_{G_1}$, and $L(Q_{S,\ell})$ is a theta lift from Sp_4. Thus $\Theta_{\mathrm{Sp}_4}(\Pi_{L(\varphi)}) \neq 0$, so the corollary follows from Theorem 8.10. □

In the next section, we will study the inner products $\langle \varphi, Q_{S,\ell}\rangle \neq 0$ in terms of periods of φ.

9. Periods

The purpose of this section is to prove Theorem 1.4, restated below as Corollary 9.8. By Corollary 8.11, we can characterize the elements of the Saito-Kurokawa subspace in terms of inner products with the Q_S. Thus to prove Theorem 1.4 it remains to relate the inner products $\langle \varphi, Q_S\rangle$ to periods of φ. That is what we do in this section.

Suppose $v_1, v_2 \in V$. We denote H_{v_1,v_2} the stabilizer of v_1, v_2 in G. If $v_1, v_2 \in V(\mathbf{Q})$ or $V(\mathbf{R})$, then H_{v_1,v_2} is an algebraic group over $\mathbf{Q}$ or $\mathbf{R}$. If $v_1, v_2 \in L = V(\mathbf{Z})$, then we write $H_{v_1,v_2}(\mathbf{Z})$ for the subgroup of $G(\mathbf{Z})$ that fixes v_1, v_2.

Given $v_1, v_2 \in V(\mathbf{R})$ spanning a positive-definite two-plane, recall we define

$$B_{[v_1,v_2]}(g) = \frac{pr_{\mathfrak{su}_2}(g^{-1}(v_1\wedge v_2))^{\ell}}{||pr_{\mathfrak{su}_2}(g^{-1}(v_1\wedge v_2))||^{2\ell+1}}.$$

Then $B_{[v_1,v_2]}(g)$ is a function $G(\mathbf{R}) \to \mathbf{V}_\ell$. That it is defined (i.e., that the denominator is nonzero) follows from the following lemma, and the fact that if v_1, v_2 span a positive definite two-plane, then the projection of v_1, v_2 to V_4 still span a two-plane.

Lemma 9.1. *Suppose $u_1, u_2 \in V_4$ span a two-plane. Then $pr_{\mathfrak{su}_2}(u_1 \wedge u_2)$ is nonzero.*

Proof. This is clear for the basis elements u_1, u_2 of section 3. It now follows in general by $\mathrm{SO}(V_4)$ equivariance of the projection map. □

Suppose now that T is a fixed half-integral, positive definite, symmetric matrix. Set $X_T = \{v_1, v_2 \in L : S(v_1, v_2) = T\}$. If $\ell \geq 16$ is even, recall we set

$$Q_{T;\ell}(g) = \sum_{(v_1,v_2)\in X_T} B_{[v_1,v_2],\ell}(g).$$

Similarly, if $v_1, v_2 \in L$ span a positive-definite two plane, and $\ell \geq 16$ is even, we set

$$Q_{v_1,v_2;\ell}(g) = \sum_{\gamma \in H_{v_1,v_2}(\mathbf{Z})\backslash G(\mathbf{Z})} B_{[v_1,v_2];\ell}(\gamma g).$$

Suppose $D = -4\det(T)$ is odd and square-free. By [Sah13], the Poincare series P_T with these T span the space of cusp forms of weight ℓ and level one on Sp_4. By Theorem B.1, for such T, $Q_{T;\ell} = Q_{v_1,v_2;\ell}$ for any $v_1, v_2 \in L$ with $S(v_1, v_2) = T$. Thus we have:

Corollary 9.2. *Suppose $\ell \geq 16$ is even, and φ is a level one cuspidal quaternionic modular form on G of weight ℓ. Suppose moreover that φ is an eigenform in the plus subspace. Then $\varphi \in SK_\ell$ if and only if $\langle \varphi, Q_{v_1,v_2;\ell}\rangle \neq 0$ for some $v_1, v_2 \in L$ with $-4\det(S(v_1, v_2))$ odd and square-free.*

Our objective for the rest of this section is to reinterpret the inner product $\langle \varphi, Q_{v_1,v_2;\ell}\rangle$ as a period of φ over H_{v_1,v_2}. The inner product $\langle \varphi, Q_{v_1,v_2;\ell}\rangle$ is defined adelically. It can be interpreted as an integral over the real points of G, by the following lemma.

Lemma 9.3. *The canonical map*

$$G(\mathbf{Z})\backslash G(\mathbf{R}) \to G(\mathbf{Q})\backslash G(\mathbf{A})/G(\widehat{\mathbf{Z}})$$

is a bijection.

Proof. The map is clearly an injection. For the surjectivity, we must prove that $G(\mathbf{A}_f) = G(\mathbf{Q})G(\widehat{\mathbf{Z}})$. Thus suppose $g \in G(\mathbf{A}_f)$. By the Iwasawa decomposition for the Siegel parabolic NM of G, we can write $g = umk$ with $u \in N(\mathbf{A}_f)$, $m \in M(\mathbf{A}_f) \simeq \mathrm{GL}_4(\mathbf{A}_f)$ and $k \in K_f = G(\widehat{\mathbf{Z}})$. We can write $m = \gamma k_1$ for some $\gamma \in GL_4(\mathbf{Q}) \subseteq G(\mathbf{Q})$ and $k_1 \in K_f$. Thus $g = \gamma(\gamma^{-1}u\gamma)k_1 k$. But now $\gamma^{-1}u\gamma = \mu k_2$ for some $\mu \in N(\mathbf{Q})$ and $k_2 \in K_f$. The lemma follows. $\square$

To setup our result on the inner product $\langle \varphi, Q_{v_1,v_2;\ell}\rangle$, we need an additional lemma.

Lemma 9.4. *Suppose $v_1, v_2 \in V(\mathbf{R})$ span a positive-definite two-plane. Then the image of $K \cap H_{v_1,v_2}(\mathbf{R})$ in the long root SU_2/μ_2 is a nontrivial torus.*

Proof. Suppose u_1, u_2 span the orthogonal complement in V_4 of the projection of v_1, v_2 to V_4. Then the projection of $K \cap H_{v_1,v_2}(\mathbf{R})$ in the long root SU_2/μ_2 is the projection of the $\exp(tu_1 \wedge u_2)$ with $t \in \mathbf{R}$. Now the lemma follows by Lemma 9.1. $\square$

We write $K_{v_1,v_2} = K \cap H_{v_1,v_2}(\mathbf{R})$. It follows from Lemma 9.4 that K_{v_1,v_2} stabilizes a unique line in $\mathbf{V}_\ell$. Suppose $v' \in \mathbf{V}_\ell$ spans this line. Observe that $B_{[v_1,v_2]}(1)$ is in the same line, so that $\langle v', B_{[v_1,v_2]}(1)\rangle \neq 0$.

If $v_1, v_2 \in L$, and φ is a level one cuspidal quaternionic modular form on G, we define

$$P_{v_1,v_2}(\varphi) = \frac{1}{\langle v', B_{[v_1,v_2]}(1)\rangle} \int_{H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})} \langle v', \varphi(h)\rangle \, dh.$$

In terms of the vector-valued period $\mathcal{P}$ defined in the introduction, note that $P_{v_1,v_2}(\varphi) = \frac{\langle v', \mathcal{P}_{v_1,v_2}(\varphi)\rangle}{\langle v', B_{[v_1,v_2]}(1)\rangle}$. Observe that $\mathcal{P}_{v_1,v_2}(\varphi)$ is invariant by K_{v_1,v_2}, so by Lemma 9.4, $P_{v_1,v_2}(\varphi) \neq 0$ if and only if $\mathcal{P}_{v_1,v_2}(\varphi) \neq 0$.

Theorem 9.5. *Suppose $\ell \geq 22$. There is a nonzero constant C_{v_1,v_2} so that $\langle \varphi, Q_{v_1,v_2}\rangle = C_{v_1,v_2} P_{v_1,v_2}(\varphi)$ for all level one, weight ℓ cuspidal quaternionic modular forms φ.*

The proof is easy, once we have one more lemma. Say that $v \in \mathbf{V}_\ell$ is *completely degenerate* if v is in the $\mathrm{SL}_2(\mathbf{C})$ orbit of a highest weight line of $\mathbf{V}_\ell$.

Lemma 9.6. *Suppose $F : G(\mathbf{R}) \to \mathbf{V}_\ell$ is a quaterionic function so that $F(g)$ is never completely degenerate. Suppose also that $a : G(\mathbf{R}) \to \mathbf{C}$ is a smooth function with aF quaternionic. Then a is constant.*

Remark 9.7. Suppose $v \in \mathbf{V}_\ell$. Consider the condition:

- $\langle v, u\rangle = 0$ with $u \in V_2$ implies $u = 0$, where $\langle v, u\rangle \in S^{2\ell-1}(V_2)$ is the contraction.

The element v satisfies this condition precisely if v is not completely degenerate. Indeed, by acting by $\mathrm{SL}_2(\mathbf{C})$, it suffices to consider the case $u = x$. Then $\langle v, x\rangle = 0$ implies v is in $\mathbf{C}x^{2\ell}$. We also remark that if $\langle v, v\rangle \neq 0$, then v is not completley degenerate.

Proof of Lemma 9.6. First observe that, because both aF and F are K-equivariant, and F is never 0, we obtain that a is K-invariant. Now, let $\mathfrak{p} = V_2 \otimes W$, $\{w_\alpha\}$ be a basis of W, and $\{w_\alpha^\vee\}$ the dual basis of $W^\vee$. Then

$$D(aF) = aD(F) + \sum_\alpha (x \otimes w_\alpha)(a)\langle F, y\rangle w_\alpha^\vee - (y \otimes w_\alpha)(a)\langle F, x\rangle w_\alpha^\vee.$$

We have $D(F) = 0$, and the $w_\alpha^\vee$ are linearly independent. Thus we obtain

$$(x \otimes w_\alpha)(a)\langle F, y\rangle w_\alpha^\vee - (y \otimes w_\alpha)(a)\langle F, x\rangle w_\alpha^\vee = 0$$

for all α. But by Remark 9.7, $\langle F, y\rangle$ and $\langle F, x\rangle$ are linearly independent in $S^{2\ell-1}(V_2)$. Consequently $(x \otimes w_\alpha)(a) = 0$ and $(y \otimes w_\alpha)(a) = 0$ for all α. In other words, $Xa = 0$ for all $X \in \mathfrak{p}$. One concludes that α is constant. □

Proof of Theorem 9.5. Set

$$B_\varphi(g) = \int_{H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})} \varphi(hg)\, dh.$$

Then $B_\varphi(g)$ is quaternionic and left $H_{v_1,v_2}(\mathbf{R})$-invariant. Indeed, to see that $B_\varphi(g)$ is quaternionic, we simply need to justify differentiation under the integral $\int_{H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})}$. To do this, fix $X \in \mathfrak{p}$. Then we are intersted in proving the equality

$$\frac{d}{dt}\int_{H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})} \varphi(hge^{tX})\, dh = \int_{H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})} \frac{d}{dt}\varphi(hge^{tX})\, dh$$

at $t = 0$. One now justifies the exchange simply by the boundedness of cusp forms.

Now, we claim that there is a constant D_1 so that $B_\varphi(g) = D_1 B_{[v_1,v_2]}(g)$. Indeed, if $k \in K \cap g^{-1}H_{v_1,v_2}(\mathbf{R})g = K_{g^{-1}v_1, g^{-1}v_2}$, then

$$k^{-1} \cdot B_\varphi(g) = B_\varphi(gk) = B_\varphi(g)$$

because B_φ is left $H_{v_1,v_2}(\mathbf{R})$-invariant. Thus $B_\varphi(g)$ and $B_{[v_1,v_2]}(g)$ lie on the same line in $\mathbf{V}_\ell$ for every $g \in G(\mathbf{R})$. Because $\langle B_{[v_1,v_2]}(g), B_{[v_1,v_2]}(g)\rangle \neq 0$ for all g, we can apply Lemma 9.6 to deduce that $B_\varphi(g) = D_1 B_{[v_1,v_2]}(g)$ for some constant D_1.

Now, the constant D_1 is determined by evaluating both sides at $g = 1$ and pairing with v', so we obtain $D_1 = P_{v_1,v_2}(\varphi)$.

Now one has

$$\begin{aligned}\langle \varphi, Q_{v_1,v_2}\rangle &= \int_{H_{v_1,v_2}(\mathbf{Z})\backslash G(\mathbf{R})} \langle B_{[v_1,v_2]}(g), \varphi(g)\rangle\, dg \\ &= \int_{H_{v_1,v_2}(\mathbf{R})\backslash G(\mathbf{R})} \langle B_{[v_1,v_2]}(g), B_\varphi(g)\rangle\, dg \\ &= P_{v_1,v_2}(\varphi)\int_{H_{v_1,v_2}(\mathbf{R})\backslash G(\mathbf{R})} \langle B_{[v_1,v_2]}(g), B_{[v_1,v_2]}(g)\rangle\, dg.\end{aligned}$$

To justify the unfolding of the integral, we must prove that the first integral converges absolutely. We have

$$|\langle B_{[v_1,v_2]}(g), \varphi(g)| \leq ||B_{[v_1,v_2]}(g)|| \cdot ||\varphi(g)|| \leq C||B_{[v_1,v_2]}(g)||.$$

Thus

$$\int_{H_{v_1,v_2}(\mathbf{Z})\backslash G(\mathbf{R})} |\langle B_{[v_1,v_2]}(g), \varphi(g)|\rangle|\, dg \leq C \int_{H_{v_1,v_2}(\mathbf{Z})\backslash G(\mathbf{R})} ||B_{[v_1,v_2]}(g)||\, dg$$
$$= C' \int_{H_{v_1,v_2}(\mathbf{R})\backslash G(\mathbf{R})} ||B_{[v_1,v_2]}(g)||\, dg$$

because $H_{v_1,v_2}(\mathbf{Z})\backslash H_{v_1,v_2}(\mathbf{R})$ has finite volume and $B_{[v_1,v_2]}(g)$ is left-invariant by $H_{v_1,v_2}(\mathbf{R})$. Now the result, for $\ell \geq 22$, follows from Theorem C.1. □

Putting everything together, we have obtained the following result.

Corollary 9.8. *Suppose $\ell \geq 22$ is even. Suppose φ is a level one, weight ℓ, cuspidal quaternionic eigenform on G in the plus subspace. Then φ is a Saito-Kurakawa lift if and only if there exists $v_1, v_2 \in L$ with $S(v_1, v_2) > 0$ and $D := -4\det(S(v_1, v_2))$ odd and square free, so that the period $P_{v_1,v_2}(\varphi) \neq 0$.*

Appendix A. Triality

The purpose of this section is to work out facts regarding triality on D_4, and how it interacts with the notion of quaternionic modular forms. This is an important ingredient in the proof of Theorem 1.2.

To define triality on D_4, recall the trilinear form $(x, y, z) = tr_{\mathbb{O}}(x(yz))$ on the octonions $\mathbb{O}$. See subsection 3.2 for notation regarding the octonions. The group $G' = \mathrm{Spin}(\mathbb{O})$ is defined as the set of triples $(g_1, g_2, g_3) \in \mathrm{SO}(\mathbb{O})^3$ that satisfy

$$(g_1 x, g_2 y, g_3 z) = (x, y, z) \text{ for all } x, y, z \in \mathbb{O}.$$

The association $g \mapsto g_1$ gives a map of groups $G' \to G$, which induces an isomorphism on Lie algebras.

Permutation of the g_j induces an S_3 action on G' as follows. For $g \in \mathrm{SO}(\mathbb{O})$, let $g^* = c \circ g \circ c$, where $c : \mathbb{O} \to \mathbb{O}$ is the octonionic conjugation. If $\sigma \in S_3$ has $\mathrm{sgn}(\sigma) = 1$, we define $\sigma(g_1, g_2, g_3) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, g_{\sigma^{-1}(3)})$. For $\sigma \in S_3$ with $\mathrm{sgn}(\sigma) = -1$, we define $\sigma(g_1, g_2, g_3) = (g^*_{\sigma^{-1}(1)}, g^*_{\sigma^{-1}(2)}, g^*_{\sigma^{-1}(3)})$. One verifies easily that this maps $G' \to G'$ and is an S_3 action.

In this section, we understand how this triality interacts with the notion of quaternionic modular forms on G'. Specifically, we prove the following theorem.

Theorem A.1. *Suppose $\sigma \in S_3$. Let P' be the Heisenberg parabolic of G', defined as the inverse image of P in G', and similarly define M', N', K'.*

1. *One has $\sigma(M') = M'$, $\sigma(N') = N'$, $\sigma(K') = K'$, $\sigma(\mathfrak{p}) = \mathfrak{p}$. Moreover, if $v \in \mathbf{V}_\ell$, and $k \in K'$, then $\sigma(k) \cdot v' = k \cdot v'$.*
2. *The action of S_3 on N' induces an action of S_3 on the space of Bhargava cubes $W = N^{ab} = (N')^{ab}$.*
3. *If φ is a weight ℓ quaternionic modular form on G', then $\varphi_\sigma(g) := \varphi(\sigma^{-1}(g))$ is a weight ℓ quaternionic modular form on G', and the Fourier coefficients a_{φ_σ} satisfy $a_{\varphi_\sigma}(w)(g_f) = a_\varphi(\sigma^{-1}(w))(\sigma^{-1}(g_f))$.*

To prove the theorem, we will make explicit the action of triality on $\mathfrak{g}' = Lie(G') \simeq Lie(G) \simeq \wedge^2 V$.

The Lie algebra $\mathfrak{g}' = Lie(G')$ of G' is the set of triples $(X_1, X_2, X_3) \in Lie(\mathrm{SO}(\mathbb{O}))^3$ that satisfy

$$(X_1 x, y, z) + (x, X_2 y, z) + (x, y, X_3 z) = 0$$

for all $x, y, z \in \mathbb{O}$. We denote $\mathfrak{g} = Lie(\mathrm{SO}(\mathbb{O}))$ the Lie algebra of $G = \mathrm{SO}(\mathbb{O})$.

We have the following well-known lemma.

Lemma A.2. *For $j = 1, 2, 3$, the map $g \mapsto g_j$ induces an isomorphism of Lie algebras $\mathfrak{g}' \to \mathfrak{g}$.*

For $x \in \mathbb{O}$, denote $\ell_x : \mathbb{O} \to \mathbb{O}$ left multiplication by x and $r_x : \mathbb{O} \to \mathbb{O}$ right multiplication by x.

Proposition A.3. *If $u, v \in \mathbb{O}$, then*

$$(u \wedge v, \frac{1}{2}(\ell_{u^*}\ell_v - \ell_{v^*}\ell_u), \frac{1}{2}(r_{u^*}r_v - r_{v^*}r_u))$$

is a triality triple, i.e., in the Lie algebra of $\mathrm{Spin}(\mathbb{O})$.

Proof. This follows from [SV00, Theorem 3.5.5]. □

We would like to explicitly calculate the triality triples for a basis of elements of $\wedge^2\Theta$. This can be done using the following lemmas. Let $V_7 \subseteq \mathbb{O}$ denote the space of trace 0 elements.

Lemma A.4. *Suppose $u, v \in V_7$ and $Im(uv) = 0$, so that $uv = vu$. Then $(u \wedge v, u \wedge v, u \wedge v)$ is a triality triple.*

Proof. The proof uses the *alternative identity*: For arbitrary $a, b, c \in \Theta$, one has $c(ab) + (ab)c = (ca)b + a(bc)$. □

Lemma A.5. *Suppose W_1, W_2, W_3 are two-dimensional isotropic subspaces of $\mathbb{O}$ and $W_i \cdot W_{i+1} = 0$. Let u_j, v_j be a basis of W_j. Then there are nonzero $\alpha_1, \alpha_2, \alpha_3$ so that $(\alpha_1 u_1 \wedge v_1, \alpha_2 u_2 \wedge v_2, \alpha_3 u_3 \wedge v_3)$ is in $Lie(\mathrm{Spin}(\mathbb{O}))$.*

Proof. First note that if $(X, Y, Z) \in Lie(\mathrm{Spin}(\mathbb{O}))$ and $(g_1, g_2, g_3) \in \mathrm{Spin}(\mathbb{O})$ then

$$(g_1 X, g_2 Y, g_3 Z) \in Lie(\mathrm{Spin}(\mathbb{O})).$$

This follows immediately from the definitions. Let u, v be as in Lemma A.4; then we know $(u \wedge v, u \wedge v, u \wedge v)$ is a triality triple. Now, if W_1, W_2, W_3 are as in the statement of this lemma, there exists $(g_1, g_2, g_3) \in \mathrm{Spin}(\mathbb{O})$ so that $g_i \,\mathrm{Span}\{u, v\} = W_i$; this follows from, for example, Lemma 2.2(3) and Lemma 2.3(3) of [PWZ19]. The lemma follows. □

Observe that if (X, Y, Z) is a triality triple, then $X + Y + Z \in Lie(G_2)$. Because we already know how the Lie algebra of G_2 embeds in $\wedge^2\mathbb{O}$ (it is the kernel of the map $\wedge^2 V_7 \to V_7$ given by $u \wedge v \mapsto Im(uv)$), we can frequently use this fact together with Lemma A.5 to compute many triality triples. We do this now.

Proposition A.6. *We have the following triality triples:*

(1) $(\epsilon_1 \wedge e_j, e^*_{j+1} \wedge e^*_{j-1}, -\epsilon_2 \wedge e_j)$ *for* $j \in \mathbf{Z}/3\mathbf{Z}$;

(2) $(\epsilon_1 \wedge e^*_j, -\epsilon_2 \wedge e^*_j, e_{j+1} \wedge e_{j-1})$ *for* $j \in \mathbf{Z}/3\mathbf{Z}$.

Proof. One computes that $W_1 = \mathrm{Span}\{\epsilon_1, e_1\}, W_2 = \mathrm{Span}\{e^*_2, e^*_3\}$, $W_3 = \mathrm{Span}\{\epsilon_2, e_1\}$ satisfy $W_i \cdot W_{i+1} = 0$. Because $(\epsilon_1 - \epsilon_2) \wedge e_1 + e^*_2 \wedge e^*_3$ is in $Lie(G_2)$, the first point follows in the case $j = 1$. The other cases follow from the $j = 1$ case by applying the $\mathrm{SL}_3 \subseteq G_2$ action.

The second point is similar. □

We now setup the main theorem of this section. Let F be a field of characteristic 0, and set $E = F \times F \times F$ with its usual S_3-action, given by permuting the factors. We think of E as a cubic norm structure. So, the norm on E is $N_E(z_1, z_2, z_3) = z_1 z_2 z_3$ and the adjoint on E is $(z_1, z_2, z_3)^\# = (z_2 z_3, z_3 z_1, z_1 z_2)$. For $z, z' \in E$ one sets $z \times z' = (z + z')^\# - z^\# - (z')^\#$.

Let $\mathfrak{g}_E$ be the associated Lie algebra, as in [Pol20, section 4.2]. We have

$$\mathfrak{g}_E = (\mathfrak{sl}_3 \oplus E^0) \oplus V_3 \otimes E \oplus V_3^\vee \otimes E^\vee.$$

Here V_3 is the standard representation of $\mathfrak{sl}_3$ and $V_3^\vee$ is the dual representation. Moreover, E^0 is the trace 0 subspace of E, and E^0 acts on E via multiplication $\lambda \cdot x = \lambda x$, while E^0 acts on $E^\vee = E$ as $\lambda \cdot \gamma = -\lambda\gamma$. The S_3 action on E induces an S_3 action on $\mathfrak{g}_E$.

We will recall the Lie bracket on $\mathfrak{g}_E$. Before doing so, for $u \in E^0$, let $\Psi_u \in E^0 \subseteq \mathfrak{g}_E$ be the map $E \to E$ defined as $\Psi_u(z) = uz$.

For $X \in E$ and $\gamma \in E^\vee$, there is a map $\Phi_{\gamma,X} : E \to E$ defined as

$$\Phi_{\gamma,X}(Z) = -\gamma \times (X \times Z) + (\gamma, Z)X + (\gamma, X)Z.$$

Here (γ, Z) and (γ, X) denotes the canonical pairing between E and the dual $E^\vee$. One sets $\Phi_{\gamma,X} - \frac{2}{3}(X, \gamma)1_E =: \Phi'_{\gamma,X}$. The definition of the Lie bracket on $\mathfrak{g}_E$ from [Pol20] uses the $\Phi'_{\gamma,X}$.

We require the following lemma, which relates the $\Phi'_{\gamma,X}$ with the Ψ_u.

Lemma A.7. *Suppose $x \in E$, $\gamma \in E^\vee$. Then $\Phi'_{\gamma,x} = \Psi_{2u}$ with $u = x\gamma - \frac{1}{3}(x, \gamma)1_E$.*

Proof. This is a direct computation:

$$\Phi'_{\gamma,x}(z) = -\gamma \times (x \times z) + (\gamma, z)x + \frac{1}{3}(x, \gamma)z.$$

Now $(\gamma, z)x = (x_1(\gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3), \ldots)$, $(x, \gamma)z = (z_1(x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3), \ldots)$, and one has

$$\begin{aligned}\gamma \times (x \times z) &= \gamma \times (x_2 z_3 + x_3 z_2, x_1 z_3 + x_3 z_1, x_1 z_2 + x_2 z_1)\\ &= ((x_1 z_3 + x_3 z_1)\gamma_3 + (x_1 z_2 + x_2 z_1)\gamma_3, \ldots).\end{aligned}$$

Combining the above and using the symmetry between the indices gives the claim. □

We now explicate the Lie bracket on $\mathfrak{g}_E$: Take $\phi_3 \in \mathfrak{sl}_3$, $\phi_J \in E^0$, $v, v' \in V_3$, $\delta, \delta' \in V_3^\vee$, $X, X' \in J$ and $\gamma, \gamma' \in E^\vee$. Then

$$\begin{aligned}
[\phi_3, v \otimes X + \delta \otimes \gamma] &= \phi_3(v) \otimes X + \phi_3(\delta) \otimes \gamma.\\
[\phi_J, v \otimes X + \delta \otimes \gamma] &= v \otimes \phi_J(X) + \delta \otimes \phi_J(\gamma)\\
[v \otimes X, v' \otimes X'] &= (v \wedge v') \otimes (X \times X')\\
[\delta \otimes \gamma, \delta' \otimes \gamma'] &= (\delta \wedge \delta') \otimes (\gamma \times \gamma')\\
[\delta \otimes \gamma, v \otimes X] &= (X, \gamma) v \otimes \delta + \delta(v)\Phi_{\gamma,X} - \delta(v)(X, \gamma)\\
&= (X, \gamma)\left(v \otimes \delta - \frac{1}{3}\delta(v)\right) + \delta(v)\left(\Phi_{\gamma,X} - \frac{2}{3}(X, \gamma)\right).
\end{aligned}$$

Note that $v \otimes \delta - \frac{1}{3}\delta(v) \in \mathfrak{sl}_3$ and $\Phi_{\gamma,X} - \frac{2}{3}(X, \gamma) = \Phi'_{\gamma,X} \in E^0$.

Furthermore, the action of $\mathfrak{sl}_3$ and E^0 on $V_3^\vee$ and $E^\vee$ is determined by the equalities $(\phi_3(v), \delta) + (v, \phi_3(\delta)) = 0$ and $(\phi_J(X), \gamma) + (X, \phi_J(\gamma)) = 0$.

For $j \neq k$, let $E_{jk} \in \mathfrak{sl}_3$ be the matrix with a 1 in the (j,k) position and 0's elsewhere. Let v_1, v_2, v_3 be the standard basis of V_3 and $\delta_1, \delta_2, \delta_3$ the dual basis of $V_3^\vee$. Denote by

$$\Phi \colon \mathfrak{g}_E \to \wedge^2 \mathbb{O} \tag{A.1}$$

the linear isomorphism given as follows:

(1) $\Phi(E_{jk}) = e_k^* \wedge e_j$;
(2) $\Phi(v_j \otimes (x_1, x_2, x_3)) = x_1 \epsilon_1 \wedge e_j + x_2 e_{j+1}^* \wedge e_{j-1}^* + x_3(-\epsilon_2 \wedge e_j)$;
(3) $\Phi(\delta_j \otimes (\gamma_1, \gamma_2, \gamma_3)) = \gamma_1(-\epsilon_2 \wedge e_j^*) + \gamma_2 e_{j+1} \wedge e_{j-1} + \gamma_3 \epsilon_1 \wedge e_j^*$;
(4) on E^0, Φ is: $\Phi(\Psi_{2u}) = (u_1 - u_3)\epsilon_1 \wedge \epsilon_2 + u_2(e_1 \wedge e_1^* + e_2 \wedge e_2^* + e_3 \wedge e_3^*)$.

Theorem A.8. *The linear map Φ is a Lie algebra isomorphism, respecting the S_3 actions.*

Proof of Theorem A.8. There are lots of brackets to check.

(1) Suppose $\ell \neq k$. Then $[\Phi(E_{\ell,k}), \Phi(v_j \otimes x)] = [e_k^* \wedge e_\ell, x_1 \epsilon_1 \wedge e_j + x_2 e_{j+1}^* \wedge e_{j-1}^* + x_3(-\epsilon_2 \wedge e_j)] = 0$ if $j \neq k$ and is equal to $\Phi(v_\ell \otimes x)$ if $j = k$.
(2) Suppose $\ell \neq k$. Then $[\Phi(E_{\ell,k}), \Phi(\delta_j \otimes \gamma)] = [e_k^* \wedge e_\ell, \gamma_1(-\epsilon_2 \wedge e_j^*) + \gamma_2 e_{j+1} \wedge e_{j-1} + \gamma_3 \epsilon_1 \wedge e_j^*] = 0$ if $\ell \neq j$ and is equal to $\Phi(-\delta_k \otimes \gamma)$ if $\ell = j$.
(3) Suppose $\alpha_1 + \alpha_2 + \alpha_3 = 0$, and $h = \alpha_1 E_{11} + \alpha_2 E_{22} + \alpha_3 E_{33} \in \mathfrak{h}_{\mathrm{SL}_3}$, the diagonal Cartan of $\mathfrak{sl}_3$. Then $[\Phi(h), \epsilon_1 \wedge e_j] = \alpha_j \epsilon_1 \wedge e_j$. Using the S_3 action on $\wedge^2\mathbb{O}$, and that we already know $h \in Lie(G_2) \subseteq (\wedge^2\mathbb{O})^{S_3}$, we obtain $[\Phi(h), \Phi(v_j \otimes x)] = \Phi(\alpha_j v_j \otimes x)$. Similarly, we obtain $[\Phi(h), \Phi(\delta_j \otimes x)] = \Phi(-\alpha_j \delta_j \otimes x)$.
(4) Note that $\epsilon_1 \wedge e_j, e_{j+1}^* \wedge e_{j-1}^*$, and $-\epsilon_2 \wedge e_j$ all commute. Thus $[\Phi(v_j \otimes x), \Phi(v_j \otimes x')] = 0$. Similarly, $[\Phi(\delta_j \otimes \gamma), \Phi(\delta_j \otimes \gamma')] = 0$.
(5) We compute $[\Phi(v_j \otimes x), \Phi(v_{j+1} \otimes x')]$. One has
   (a) $[\epsilon_1 \wedge e_j, \epsilon_1 \wedge e_{j+1}] = 0$;
   (b) $[\epsilon_1 \wedge e_j, e_{j-1}^* \wedge e_j^*] = \epsilon_1 \wedge e_{j-1}^*$;
   (c) $[\epsilon_1 \wedge e_j, -\epsilon_2 \wedge e_{j+1}] = e_j \wedge e_{j+1}$;
   Thus $[\Phi(v_j \otimes (x_1, 0, 0)), \Phi(v_{j+1} \otimes x')] = \Phi(\delta_{j-1} \otimes (0, x_1 x_3', x_1 x_2'))$. Using the S_3 action on $\wedge^2\mathbb{O}$, one deduces that $[\Phi(v_j \otimes x), \Phi(v_{j+1} \otimes x')] = \Phi(\delta_{j-1} \otimes (x \times x'))$.
(6) We compute $[\Phi(\delta_j \otimes \gamma), \Phi(\delta_{j+1} \otimes \gamma')]$. One has
   (a) $[-\epsilon_2 \wedge e_j^*, -\epsilon_2 \wedge e_{j+1}^*] = 0$;
   (b) $[-\epsilon_2 \wedge e_j^*, e_{j-1} \wedge e_j] = -\epsilon_2 \wedge e_{j-1}$;
   (c) $[-\epsilon_2 \wedge e_j^*, \epsilon_1 \wedge e_{j+1}^*] = e_j^* \wedge e_{j+1}^*$;
   Thus $[\Phi(\delta_j \otimes (\gamma_1, 0, 0)), \Phi(\delta_{j+1} \otimes \gamma')] = \Phi(v_{j-1} \otimes (0, \gamma_1 \gamma_3', \gamma_1 \gamma_2'))$. Using the S_3 action on $\wedge^2\mathbb{O}$, one deduces that $[\Phi(\delta_j \otimes \gamma), \Phi(\delta_{j+1} \otimes \gamma')] = \Phi(v_{j-1} \otimes (\gamma \times \gamma'))$.
(7) $[\Phi(\Psi_{2u}), \Phi(v_j \otimes x)] = \Phi(v_j \otimes (2ux))$. Indeed, we have

$$\begin{aligned}
[\Phi(\Psi_{2u}), \Phi(v_j \otimes x)] &= [(u_1 - u_3)\epsilon_1 \wedge \epsilon_2 + u_2(e_1 \wedge e_1^* + e_2 \wedge e_2^* + e_3 \wedge e_3^*), \\
&\qquad x_1 \epsilon_1 \wedge e_j + x_2 e_{j+1}^* \wedge e_{j-1}^* + x_3(-\epsilon_2 \wedge e_j)] \\
&= (u_1 - u_3)(x_1 \epsilon_1 \wedge e_j + x_3 \epsilon_2 \wedge e_j) \\
&\qquad + u_2(-x_1 \epsilon_1 \wedge e_j + 2x_2 e_{j+1}^* \wedge e_{j-1}^* + x_3 \epsilon_2 \wedge e_j) \\
&= 2\Phi(v_j \otimes ux).
\end{aligned}$$

(8) Similarly, one computes $[\Phi(\Psi_{2u}), \Phi(\delta_j \otimes \gamma)] = \Phi(\delta_j \otimes (-2u\gamma))$.
(9) It is immediately checked that $[\Phi(\phi), \Phi(\Psi_u)] = 0$ if $\phi \in \mathfrak{sl}_3$ and $[\Phi_{\Psi_u}, \Phi(\Psi_{u'})] = 0$.

(10) We are left to compute $[\Phi(\delta \otimes \gamma), \Phi(v \otimes x)]$. First observe that, by explicit computation,

$$[e_{j+1} \wedge e_{j-1}, e_{k+1}^* \wedge e_{k-1}^*] = -e_k \wedge e_j^* + \delta_{jk}(e_1 \wedge e_1^* + e_2 \wedge e_2^* + e_3 \wedge e_e^*).$$

Using this, one obtains

$$\begin{aligned}[\Phi(\delta_j \otimes \gamma), \Phi(v_k \otimes x)] &= [\gamma_1(-\epsilon_2 \wedge e_j^*) + \gamma_2 e_{j+1} \wedge e_{j-1} + \gamma_3 \epsilon_1 \wedge e_j^*, \\ &\qquad x_1 \epsilon_1 \wedge e_k + x_2 e_{k+1}^* \wedge e_{k-1}^* + x_3(-\epsilon_2 \wedge e_k)] \\ &= \gamma_1 x_1 (e_j^* \wedge e_k + \delta_{jk} \epsilon_1 \wedge \epsilon_2) \\ &\quad + \gamma_2 x_2(-e_k \wedge e_j^* + \delta_{jk}(e_1 \wedge e_1^* + e_2 \wedge e_2^* + e_3 \wedge e_e^*)) \\ &\quad + \gamma_3 x_3 (e_j^* \wedge e_k - \delta_{jk} \epsilon_1 \wedge \epsilon_2) \\ &= (\gamma, x) e_j^* \wedge e_k + \delta_{jk}((\gamma_1 x_1 - \gamma_3 x_3)\epsilon_1 \wedge \epsilon_2 \\ &\quad + \gamma_2 x_2 (e_1 \wedge e_1^* + e_2 \wedge e_2^* + e_3 \wedge e_3^*))\end{aligned}$$

Set $I = e_1^* \wedge e_1 + e_2^* \wedge e_2 + e_3^* \wedge e_3$. Then the above is

$$(\gamma, x)(e_j^* \wedge e_k - \delta_{jk} I/3) + \delta_{jk}(\gamma_1 x_1 - \gamma_3 x_3)\epsilon_1 \wedge \epsilon_2 + \delta_{jk}(\gamma_2 x_2 - (\gamma, x)/3)(-I).$$

But

$$[\delta_j \otimes \gamma, v_k \otimes x] = (\gamma, x)(E_{kj} - \delta_{jk} 1_3/3) + \Phi'_{\gamma, X} = (\gamma, x)(E_{kj} - \delta_{jk} 1_3/3) + \Psi_{2u}$$

with $u = x\gamma - \frac{1}{3}(x, \gamma) 1_E$. This finishes the proof that Φ is a Lie algebra homomorphism.

To see that Φ is S_3 invariant: we have already checked this on everything but the $\Phi(\Psi_{2u})$. But if $\sigma \in S_3$ and $u \in E^0$, we have

$$\begin{aligned}\sigma(\Phi(\Psi_{2u})) &= \sigma(\Phi([\delta_1 \otimes u, v_1 \otimes 1_E])) = \sigma([\Phi(\delta_1 \otimes u), \Phi(v_1 \otimes 1_E)]) \\ &= [\sigma(\Phi(\delta_1 \otimes u)), \sigma(\Phi(v_1 \otimes 1_E))] = [\Phi(\delta_1 \otimes \sigma(u)), \Phi(v_1 \otimes 1_E)] \\ &= \Phi(\Psi_{2\sigma(u)}).\end{aligned}$$

This finishes the proof of the theorem. □

From [Pol20], there is given a Cartan involution on $\mathfrak{g}_E$. On $\mathfrak{g}_E$, this involution Θ is given as:

(1) on $\mathfrak{sl}_3$, it is $\Theta(X) = -X^t$
(2) on E^0, it is given as $\Theta(\Phi'_{\gamma, X}) = -\Phi_{\iota(X), \iota(\gamma)}$. Thus $\Theta(\Psi_{2u}) = -\Psi_{2u}$.
(3) on $V_3 \otimes E$ it is $\Theta(v \otimes x) = \iota(v) \otimes \iota(x)$, and
(4) on $V_3^\vee \otimes E^\vee$ it is $\Theta(\delta \otimes \gamma) = \iota(\delta) \otimes \iota(\gamma)$.

Note that this Cartan involution commutes with the S_3 action on $\mathfrak{g}_E$.

We also have a Cartan involution on $\wedge^2 \mathbb{O}$ given as $\Theta(u \wedge v) = \iota(u) \wedge \iota(v)$, where $\iota(b_j) = b_{-j}$.

Corollary A.9. *The map $\Phi : \mathfrak{g}_E \simeq \wedge^2 \mathbb{O}$ respects the Cartan involution on each side. In particular, the Cartan involution on $\wedge^2 \mathbb{O}$ commutes with the S_3 action.*

Proof. The proof is an immediate check from the above formulas. □

We need one additional lemma.

Lemma A.10. *Suppose $\sigma \in S_3$. The generalize Whittaker function W_χ on G' satisfies $W_{\sigma^{-1}(\chi)}(\sigma^{-1}(g)) = W_\chi(g)$.*

Proof. The two functions agree on G_2. Moreover, they both satisfy the same equivariance properties and moderate growth that make the generalized Whittaker functions unique. Thus, they are equal. $\square$

Proof of Theorem A.1. Everything is now straightforward, using Theorem A.8, Corollary A.9, and Lemma A.10. $\square$

Appendix B. An integral orbit problem

In this section, we study an integral orbit problem that is needed for the work on periods in section 9. More specifically, we prove Theorem B.1 below which is then used in the proof of Theorem 1.4.

We setup this theorem now. Suppose $L = \mathbf{Z}^{2n}$ is the standard split lattice inside of the split $2n$-dimensional quadratic space V. Let $G(\mathbf{Z})$ denote the stabilizer of L inside of $G(\mathbf{Q})$, where G is the special orthogonal group of V. Set $(v_1, v_2) = q(v_1+v_2) - q(v_1) - q(v_2)$ the split bilinear form on V. Recall that if $T_1, T_2 \in V$, we set

$$S(T_1,T_2) = \frac{1}{2}\left(\begin{array}{cc} (T_1,T_1) & (T_1,T_2) \\ (T_2,T_1) & (T_2,T_2) \end{array}\right).$$

Note that if $T_1, T_2 \in L$, then $S(T_1,T_2)$ is a half-integral symmetric matrix. For a general half-integral symmetric matrix T, set

$$X_T = \{(T_1,T_2) \in L^2 : S(T_1,T_2) = T\}.$$

Theorem B.1. *Suppose $n \geq 4$. Let T be a half-integral symmetric matrix, and assume $D = -4\det(T)$ is odd and square-free. Then $G(\mathbf{Z})$ acts transitively on X_T.*

Define a blinear form on $\wedge^2 V$ as

$$(x_1 \wedge x_2, y_1 \wedge y_2) = (x_1,y_2)(x_2,y_1) - (x_1,y_1)(x_2,y_2).$$

Note that $(T_1 \wedge T_2, T_1 \wedge T_2) = -4\det(S(T_1,T_2))$. To prove Theorem B.1 we will use the following lemmas.

Lemma B.2. *Suppose $n \geq 3$, and suppose $v \in L$ is primitive, with $q(v) = a$. Then there exists $g \in G(\mathbf{Z})$ so that $gv = ab_1 + b_{-1}$.*

Proof. Throughout the proof, we set $X = \mathrm{Span}(b_1, \ldots, b_n)$ and $Y = \mathrm{Span}(b_{-1}, \ldots, b_{-n})$.

Suppose $v = u + w$, with $u \in X$ and $w \in Y$. Using the $\mathrm{GL}_n(\mathbf{Z})$ action from the Levi of the Siegel parabolic, we can assume $u = a'b_1$ for an integer a'. If $a' = 0$, the lemma follows by using the $\mathrm{GL}_n(\mathbf{Z})$ action on w. Suppose then that $a' \neq 0$. We use the $\mathrm{GL}_{n-1}(\mathbf{Z}) \subseteq \mathrm{GL}_n(\mathbf{Z})$ that fixes b_1, b_{-1} to move w to an element of the form $w = w_1 b_{-1} + w_2 b_{-2}$.

Observe that $gcd(w_1, w_2, a') = 1$. Now, we can apply a unipotent element in the unipotent group opposite to the Siegel parabolic to move w to $w' = w_1 b_{-1} + w_2 b_{-2} + a' b_{-3}$. Thus w' is primitive, so we can use the $\mathrm{GL}_n(\mathbf{Z})$ action to move w' to b_{-1}. Hence we have found a $g' \in G(\mathbf{Z})$ so that $gv = u' + b_{-1}$ with $u' \in X$. We can now finish the proof by using elements in the unipotent radical of the Siegel parabolic to move u' to an element of the form ab_1. $\square$

Lemma B.3. *Suppose $n \geq 4$, and suppose $v_1, v_2 \in L$ with $D := -4\det(S(v_1,v_2))$ odd and squarefree. Then there exists $u_1, u_2 \in L$ with $\mathrm{Span}(u_1,u_2)$ isotropic and $(v_1 \wedge v_2, u_1 \wedge u_2) = 1$.*

Proof. One first verifies that there exists $g \in G(\mathbf{Z})$ so that $gv_1 = ab_1 + b_{-1}$ and $gv_2 = rb_1 + sb_{-1} + m(\beta b_2 + b_{-2})$. Indeed, we first apply Lemma B.2 to move v_1 into the specified form. We then apply Lemma B.2 again for the subgroup of $G(\mathbf{Z})$ stablizing b_1, b_{-1} to move v_2 into the specified form.

Note that if we set $\alpha = as - r$, then we have $D = \alpha^2 - 4m^2 a\beta$.

Now, if v_1, v_2 are in the above specialized form, then there exists u_1, u_2 spanning an isotropic 2-plane so that $(v_1 \wedge v_2, u_1 \wedge u_2) = 1$. Indeed, because $D = \alpha^2 - 4m^2 a\beta$, the integers α and m are relatively prime, so there exists integers x, y so that $\alpha x - my = 1$. Now one verifies $(v_1 \wedge v_2, u_1 \wedge u_2) = 1$, where $u_1 = b_1 + b_3$ and $u_2 = xb_{-1} + yb_2 - xb_{-3}$. $\square$

Lemma B.4. *Suppose $n \geq 3$, u_1, u_2 are istropic and $u_1 \wedge u_2$ is primitive in $\wedge^2 L$. Then there exist $g \in G(\mathbf{Z})$ so that $gu_1 = b_1$ and $gu_2 = b_2$.*

Proof. Recall that P is the parabolic subgroup stabilizing $\mathrm{Span}(b_1, b_2)$. One can leverage the Iwasawa decomposition to check that $G(\mathbf{Q}) = P(\mathbf{Q})G(\mathbf{Z})$. Now, by Witt's theorem, there exists $g = pk \in G(\mathbf{Q})$ so that $gu_1 = b_1$ and $gu_2 = b_2$. Here $p \in P(\mathbf{Q})$ and $k \in G(\mathbf{Z})$. Thus $ku_1, ku_2 \in \mathrm{Span}(b_1, b_2)$ and $(ku_1) \wedge (ku_2)$ is primitive. One now finishes the proof by using an element in $\mathrm{GL}_2(\mathbf{Z})$ in the Levi subgroup of the Heisenberg parabolic. $\square$

Proof of Theorem B.1. By Lemma B.3 and Lemma B.4, we can assume $(T_1 \wedge T_2, b_1 \wedge b_2) = 1$. Say $T_j = x_j + y_j$ with $x_j \in X$, $y_j \in Y$.

Observe that $1 = (b_1 \wedge b_2, y_1 \wedge y_2)$. Thus $y_1 \wedge y_2$ is primitive in $\wedge^2 Y$, so we may use the action of $\mathrm{GL}_n(\mathbf{Z})$ from the Levi of the Siegel parabolic to assume $y_1 = b_{-1}$, $y_2 = b_{-2}$.

Suppose now

$$S(T_1, T_2) = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

Let N denote the unipotent radical of the Siegel parabolic of G. Then there exists $n \in N(\mathbf{Z})$ so that $nT_1 = ab_1 + b_{-1}$ and $nT_2 = bb_1 + cb_2 + b_{-2}$. This claim follows easily by writing down the action of $N(\mathbf{Z})$.

The theorem is proved. $\square$

Appendix C. The finiteness of an integral

In this section, we prove the finiteness of an integral that arises in the work on periods in section 9. For ease of notation, set $H = H_{v_1, v_2}$. We prove the following theorem. This theorem is used in the proof of Theorem 9.5, which is then used in proof of Theorem 1.4.

Theorem C.1. *Suppose $\ell \geq 22$. Then the integral*

$$\int_{H(\mathbf{R})\backslash G(\mathbf{R})} ||B_{[v_1,v_2]}(g)||\, dg$$

is finite.

Observe that one can identify the quotient space $H(\mathbf{R})\backslash G(\mathbf{R})$ with pairs of vectors $(v_1', v_2') \in V^2$ for which $S(v_1', v_2') = S(v_1, v_2)$. To prove the finiteness of the integral, we work to put the invariant measure dg in these coordinates.

Notation: Throughout this section, we use the notation $A \approx B$ to mean that there is a nonzero constant β so that $A = \beta B$.

We begin by understanding some differential forms on a vector space. Suppose $V = \mathbf{R}^n$, with coordinates $(z_1, z_2, \ldots, z_n)$. We will later take $n = 8$, but for now, we work more generally.

Proposition C.2. *We have the following facts concerning forms on V.*

1. *$\omega_{[1]} := d(z_1^2 + \cdots + z_n^2) = \sum_j 2z_j dz_j$ is $\mathrm{SO}(n)$-invariant.*
2. *Set $\omega_{[n]} = dz_1 \wedge \cdots \wedge dz_n$ and $\eta_j = (-1)^{j-1} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$, so that $dz_j \wedge \eta_j = \omega_{[n]}$. Let $\omega_{[n-1]} := \sum_j z_j \eta_j$. Then $\omega_{[n-1]}$ is $\mathrm{SL}_n(\mathbf{R})$ invariant. In fact, if $g \in \mathrm{GL}_n(\mathbf{R})$, then $g^* \omega_{[n-1]} = \det(g) \omega_{[n-1]}$.*
3. *$\omega_{[1]} \wedge \omega_{[n-1]} = 2(\sum_j z_j^2) \omega_{[n]}$.*

Proof. One interprets $\eta_j \in \wedge^{n-1}(V) \simeq \det(V) V^\vee$. Then, if δ_j is dual to z_j, we have that $\sum_j z_j \otimes \delta_j$ is $\mathrm{GL}_n(\mathbf{R})$-invariant. The proposition follows. □

We construct invariant differential forms on V^2. We use the variables $(w_1, \ldots, w_n)$ for the coordinates on the first copy of V, and the z_j's as coordinates on the second copy of V. Set $q_{11} = \sum_j w_j^2$, $q_{12} = \sum_j w_j z_j$, and $q_{22} = \sum_j z_j^2$. To distinguish between differential forms defined using the coordinates w from forms defined using z, we use superscripts. So, we let $\omega^z_{[n-1]}$ denote the form $\omega_{[n-1]}$ defined in Proposition C.2, and let $\omega^w_{[n-1]} = \sum_j w_j \eta^w_j$, where $\eta^w_j = (-1)^{j-1} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_n$.

Now, if $j < k$, set $\eta^z_{j,k} = (-1)^{j+k-1} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_n$, so that $dz_j \wedge dz_k \wedge \eta^z_{j,k} = \omega^z_{[n]}$. Similarly define $\eta^w_{j,k}$. Now let

$$\omega^{w,z}_{[n-2]} = \sum_{1 \le j < k \le n} (w_j z_k - w_k z_j) \eta^w_{j,k}$$

and $\omega^{w,z}_{[2n-3]} = \omega^{w,z}_{[n-2]} \wedge \omega^z_{[n-1]}$.

Proposition C.3. *We have the following facts concerning differential forms on V^2.*

1. *The q_{ij} are $\mathrm{SO}(n)$-invariant.*
2. *The $(n-2)$-form $\omega^{w,z}_{[n-2]}$ is $\mathrm{SL}_n(\mathbf{R})$-invariant.*

Proof. We leave this to the reader. □

Set $\omega^{w,z'}_{[n-1]} = \sum_j z_j \eta^w_j$. This is again an $\mathrm{SL}_n(\mathbf{R})$-invariant differential form. Let $\det(Q) = q_{11} q_{22} - q_{12}^2$.

Proposition C.4. *One has the following equalities:*

1. *$2q_{22} \det(Q) \omega^w_{[n-1]} \wedge \omega^z_{[n-1]} = (-q_{11} q_{22} dq_{11} + 2q_{11} q_{22} dq_{12} - q_{11} q_{12} dq_{22}) \wedge \omega^{w,z}_{[2n-3]}$.*
2. *$2q_{22} \det(Q) \omega^{w,z'}_{[n-1]} \wedge \omega^z_{[n-1]} = (-q_{22}^2 dq_{11} + 2q_{12} q_{22} dq_{12} - q_{12}^2 dq_{22}) \wedge \omega^{w,z}_{[2n-3]}$.*
3. *$2q_{22} \det(Q) \omega^{w,z}_{[n-2]} \wedge \omega^z_{[n]} = (-1)^n \det(Q) dq_{22} \wedge \omega^{w,z}_{[2n-3]}$.*

Proof. One computes $dq_{ij} \wedge \omega^{w,z}_{[2n-3]}$ in terms of $\omega^w_{[n-1]} \wedge \omega^z_{[n-1]}, \omega^{w,z'}_{[n-1]} \wedge \omega^z_{[n-1]}, \omega^{w,z}_{[n-2]} \wedge \omega^z_{[n]}$ then inverts this expression. □

Lemma C.5. *One has*

1. *$4q_{22} \det(Q) \omega^w_{[n]} \wedge \omega^z_{[n]} = (-1)^n dq_{11} \wedge dq_{12} \wedge dq_{22} \wedge \omega^{w,z}_{[2n-3]}$.*

Proof. From the previous proposition, $(-1)^n dq_{22} \wedge \omega^{w,z}_{[2n-3]} = 2q_{22}\omega^{w,z}_{[n-2]} \wedge \omega^z_{[n]}$. Now, wedging with $dq_{11} \wedge dq_{12}$, we get

$$dq_{11} \wedge dq_{12} \wedge \omega^{w,z}_{[n-2]} \wedge \omega^z_{[n]} = 2\left(\sum_{j',k'} w_{j'}z_{k'}dw_{j'} \wedge dw_{k'}\right) \wedge \left(\sum_{j,k}(w_jz_k - w_kz_j)\eta^w_{j,k}\right) \wedge \omega^z_{[n]}.$$

It is now a straightforward computation. □

Suppose $V = V_a^+ \oplus V_b^-$ is a quadratic space of signature (a,b), so that $a+b=n$. In our application, $a=b=4$. We let $u_1,\ldots,u_a$, $v_1,\ldots,v_a$ be the coordinates on the two copies of V_a^+ in V^2, and $x_1,\ldots,x_b$, $y_1,\ldots,y_b$ be the coordinates on the two copies of V_b^- in V^2. Let $r_{11} = \sum_j x_j^2$, $r_{12} = \sum_j x_jy_j$ and $r_{22} = \sum_j y_j^2$. Similarly, let $t_{11} = \sum_j u_j^2$, $t_{12} = \sum_j u_jv_j$, and $t_{22} = \sum_j v_j^2$. Finally, set $s_{ij} = t_{ij} - r_{ij}$. Let T, R, S be the two-by-two matrices with entries t_{ij}, r_{ij}, s_{ij}, so that $S = T - R$. We set

$$V_S^2 = \{(v_1', v_2') \in V^2 : ((v_i', v_j')) = S\}$$

to be the collection of vectors with Gram matrix S.

Theorem C.6. *Suppose S is positive-definite. Then there is a nonzero cubic homogeneous polynomial $P(R,T)$ in the variables r_{ij} and t_{ij} so that $4r_{22}t_{22}\det(R)\det(T)\omega^{w,z}_{[2n-3]}$ has the same restriction to V_S^2 as $P(R,T)dr_{11} \wedge dr_{12} \wedge dr_{22} \wedge \omega^{u,v}_{[2a-3]} \wedge \omega^{x,y}_{[2b-3]}$.*

Proof. We have $\omega^{w,z}_{[2n-3]} = \omega^{w,z}_{[n-2]} \wedge \omega^z_{[n-1]}$. Now,

$$\omega^z_{[n-1]} = \omega^v_{[a-1]} \wedge \omega^y_{[b]} + (-1)^a \omega^v_{[a]} \wedge \omega^y_{[b-1]}.$$

Also

$$\omega^{w,z}_{[n-2]} = \omega^{u,v}_{[a-2]} \wedge \omega^x_{[b]} + \omega^u_{[a]} \wedge \omega^{x,y}_{[b-2]} + (-1)^{a-1}\sum_{1\le j\le a, 1\le k\le b}(u_jy_k - v_jx_k)\eta^u_j \wedge \eta^x_k.$$

Observe that $dt_{ij} = dr_{ij}$ when restricted to V_S^2. We can use formulas above to put our $2n-3$ form in the desired shape. For example,

$$\begin{aligned}\omega^{u,v}_{[a-2]} \wedge \omega^v_{[a]} \wedge \omega^x_{[b]} \wedge \omega^y_{[b-1]} &\approx \frac{1}{t_{22}}dt_{22} \wedge \omega^{u,v}_{[2a-3]} \wedge \omega^x_{[b]} \wedge \omega^y_{[b-1]}\\ &\approx \frac{1}{t_{22}}\omega^{u,v}_{[2a-3]} \wedge \omega^x_{[b]} \wedge (r_{22}\omega^y_{[b]})\\ &\approx \frac{r_{22}\det(T)}{r_{22}t_{22}\det(R)\det(T)}dr_{11} \wedge dr_{12} \wedge dr_{22} \wedge \omega^{u,v}_{[2a-3]} \wedge \omega^{x,y}_{[2b-3]}.\end{aligned}$$

As another example,

$$\begin{aligned}&\sum_{j,k} u_jv_k\eta^u_j \wedge \eta^x_k \wedge \omega^v_{[a-1]} \wedge \omega^y_{[b]}\\ &\approx \omega^u_{[a-1]} \wedge \omega^v_{[a-1]} \wedge \omega^{x,y'}_{[b-1]} \wedge \omega^y_{[b]}.\end{aligned}$$

Then $\omega^u_{[a-1]} \wedge \omega^v_{[a-1]} = \frac{1}{t_{22}\det(T)}\alpha(t) \wedge \omega^{u,v}_{[2a-3]}$, where $\alpha(t)$ is a linear combination of the dt_{ij} with quadratic coefficients. Additionally, one has that

$$\omega^{x,y'}_{[b-1]} \wedge \omega^y_{[b]} \approx \frac{1}{r_{22}}dr_{22} \wedge \omega^{x,y'}_{[b-1]} \wedge \omega^y_{[b-1]} \approx \frac{1}{r_{22}\det(R)}\beta(r) \wedge \omega^{x,y}_{[2b-3]}$$

where $\beta(r)$ is a 2-form in the dr_{ij} with linear coefficients.

The other terms are similar to one of the above two examples. This completes the proof, except for the non-vanishing of $P(R,T)$. However, it is easy to see using Lemma C.5 that $\omega^{w,z}_{[2n-3]}$ has nonzero restriction to V_S^2, so the theorem follows. □

To go further, we parametrize V_S^2 explicitly in terms of the Borel subgroup $B(\mathbf{R})$ of $\mathrm{GL}_2(\mathbf{R})$ and compact sets. First, let $B'(\mathbf{R})$ be the subgroup of $B(\mathbf{R})$ with positive diagonal entries. Next, denote $A_1 = \{(a_1,a_2) \in V_a^2 : T(a_1,a_2) = 1_2\}$. Here $T(a_1,a_2)$ is the 2×2 matrix with entries the inner products (a_i,a_j) . Similarly, denote $B_1 = \{(b_1,b_2) \in V_b^2 : R(b_1,b_2) = 1_2\}$. Finally, set $V_{S,o}^2$ to be the open subset of V_S^2 consisting of pairs $((u,x),(v,y))$ with $\det(R(x,y)) \neq 0$. We define a diffeomorphism $\Psi : B'(\mathbf{R}) \times A_1 \times B_1 \to V_{S,o}^2$ as follows.

Given $b \in B'(\mathbf{R})$, let $r(b) = bb^t$, $t(b) = S + bb^t$, $r_2(b) = r(b)^{1/2}$, and $t_2(b) = t(b)^{1/2}$. Here the square roots are the unique symmetric positive definite ones. Then

$$\Psi(b,(a_1,a_2),(b_1,b_2)) = (r_2(b)(a_1,a_2)^t, t_2(b)(b_1,b_2)^t).$$

We require the following lemma.

Lemma C.7. *Suppose $g \in \mathrm{GL}_2(\mathbf{R})$. Let $T = gg^t$, and let σ_A be the pullback of $\omega^{u,v}_{[2a-3]}$ to A_1. Then the pullback of $g^*\omega^{u,v}_{[2a-3]} = \omega^{g(u,v)^t}_{[2a-3]}$ to A_1 is $t_{22}\det(T)^{a-1/2}\sigma_A$.*

Proof. The action of $g = (g_{ij})$ on the variables is $u \mapsto g_{11}u + g_{12}v$ and $v \mapsto g_{21}u + g_{22}v$.

We write $g = bk$. First observe that $k^*\omega^{u,v}_{[2a-3]}$ pulls back to σ_A. Indeed, the action of $\mathrm{SO}(2)$ preserves A_1, and so $k^*\sigma_A$ is another $\mathrm{SO}(a)$-invariant $2a-3$ form on A_1 (which has dimension $2a-3$), so $k^*\sigma_A$ is proportional to σ_A. This defines a continuous homomorphism $\mathrm{SO}(2) \to \mathbf{R}^\times$, which therefore must be trivial. Thus $k^*\sigma_A = \sigma_A$.

For the pull back of $b^*\omega^{u,v}_{[2a-3]}$, it is easy to compute explicitly. If $b = (b_{ij})$, then $u \mapsto b_{11}u + b_{12}v$ and $v \mapsto b_{22}v$. Thus $u_j \mapsto b_{11}u_j + b_{12}v_j$ and $v_j \mapsto b_{22}v_j$. We obtain that $b^*\omega^{u,v}_{[2a-3]} = b_{22}^2\det(b)^{a-1}\omega^{u,v}_{[2a-3]} + \epsilon$, where ϵ is divisible by $\omega^v_{[a]}$. But, because $v_1^2 + \cdots + v_a^2 = 1$ is fixed on A_1, $\omega^v_{[a]}$ pulls back to 0.

Combining the above, and the fact that the k action commutes with pullback, we obtain the lemma. □

One computes that if $R = bb^t$, then

$$dr_{11}\wedge dr_{12}\wedge dr_{22} = 4b_{22}\det(b)db_{11}\wedge db_{12}\wedge db_{22}.$$

We can now compute $\Psi^*\omega^{w,z}_{[2n-3]}$.

Theorem C.8. *One has $\Psi^*\omega^{w,z}_{[2n-3]} = \frac{1}{4}P(R,T)\det(T)^{a-3/2}\det(R)^{b-3/2}dr_{11}\wedge dr_{12}\wedge dr_{22}\wedge \sigma_A\wedge\sigma_B$, where $P(R,T)$ is a cubic homogeneous polynomial in the variables r_{ij}, t_{ij}.*

Proof. One has $\omega^{t_2(b)(u,v)^t}_{[2a-3]}$ is $t_2(b)^*\omega_{[2a-3]}$ plus terms involving the db_{ij}. Similarly for $\omega^{r_2(b)(x,y)^t}_{[2b-3]}$. Thus the theorem follows from Lemma C.7 and Theorem C.6. □

Lemma C.9. *There is a positive constant D, that only depends upon S, so that $|r_{ij}| \leq D\det(T)$ and $|t_{ij}| \leq D\det(T)$. Consequently, there is a positive constant C so that $|P(R,T)| \leq C\det(T)^3$.*

Proof. We have $r_{11}, r_{22} \leq \mathrm{tr}(R) \leq \mathrm{tr}(T)$, and $t_{11}, t_{22} \leq \mathrm{tr}(T)$. Additionally, $r_{12}^2 \leq r_{11}r_{22}$, so $|r_{12}| \leq \mathrm{tr}(R)/2 \leq \mathrm{tr}(T)$ (by AM-GM) and similarly $|t_{12}| \leq \mathrm{tr}(T)$. So it suffices to prove that $\mathrm{tr}(T) \leq D\det(T)$ for some $D > 0$. To do this, let $\epsilon > 0$ be the minimum of the quadratic form v^tSv on the unit circle in the plane. Let λ_1, λ_2 be the eigenvalues of T, with corresponding unit eigenvectors e_1, e_2. Then since $T \geq S$, $\lambda_j = e_j^tTe_j \geq e_j^tSe_j \geq \epsilon$. Consequently,

$$\mathrm{tr}(T) = \lambda_1 + \lambda_2 = \lambda_1\lambda_2(\lambda_1^{-1} + \lambda_2^{-1}) \leq 2\epsilon^{-1}\det(T).$$

This proves the lemma. $\square$

Proof of Theorem C.1. We will prove that if $\ell > 2n+5$, then the integral of $||B_{[v_1,v_2]}(g)||$ over $H(\mathbf{R})\backslash G(\mathbf{R})/K$ is finite.

We have $\langle B_{[v_1,v_2]}(g), B_{[v_1,v_2]}(g)\rangle \approx \det(T)^{-(\ell+1)}$, so $||B_{[v_1,v_2]}(g)|| \approx \det(T)^{-(\ell+1)/2}$. By Theorem C.8 and Lemma C.9, the measure dg on $H(\mathbf{R})\backslash G(\mathbf{R})/K$ is bounded by a positive constant times $\det(T)^n dt_{11} \wedge dt_{12} \wedge dt_{22}$. We therefore must bound

$$\int_{T \geq S} \det(T)^{n-(1+\ell)/2} dt_{11} \wedge dt_{12} \wedge dt_{22}.$$

Now $d^*T := \det(T)^{-3/2}dT$, where $dT = |dt_{11} \wedge dt_{12} \wedge dt_{22}|$, is a $\mathrm{GL}_2(\mathbf{R})$ invariant measure on the positive definite cone. We therefore must bound $\int_{T\geq S} \det(T)^{n+1-\ell/2} d^*T$. Making a change of variables, it suffices to bound $\int_{T\geq 1} \det(T)^{n+1-\ell/2}d^*T = \int_{T\geq 1} \det(T)^{n-(1+\ell)/2}dT$.

We write $T = 1 + bb^t$, so that

$$\det(T) = 1 + \mathrm{tr}(bb^t) + \det(b)^2 = 1 + b_{12}^2 + b_{11}^2 + b_{22}^2 + b_{11}^2b_{22}^2 \geq 1 + b_{11}^2 + b_{12}^2 + b_{22}^2.$$

We thus must bound

$$\int_{b_{11}, b_{22} \geq 0, b_{12} \in \mathbf{R}} (1 + b_{11}^2 + b_{22}^2 + b_{12}^2)^{n-(1+\ell)/2} b_{11} b_{22}^2 db_{11} \wedge db_{12} \wedge db_{22}.$$

Let $r = (b_{11}^2 + b_{22}^2 + b_{12}^2)^{1/2}$. As $r \to \infty$, the integrand decays as $r^{-\ell+2n+2}$. Hence if $\ell - 2n - 2 > 3$, the integral converges. This proves the theorem. $\square$

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