Last time:

Lemma: $F$, a field, $p(x) \in F[x]$ irreducible polynomial. Let $a$ be a zero of $p(x)$ in some extension field of $F$. If $\phi: F \to F'$ is a field isomorphism, and $b$ is a zero of $\phi(p(x))$ in some extension field of $F'$, then there is an isomorphism $F(a) \to F'(b)$ taking $a$ to $b$ and agrees with $\phi$ on $F$.

**Pf:**

$F(a) \cong F[x]/\langle p(x) \rangle \cong F'[x]/\langle \phi(p(x)) \rangle = F'(b)$

**Thm:** Let $\phi: F \to F'$ be an isomorphism of fields, $f(x) \in F[x]$ a polynomial. If $E$ is a splitting field for $f(x)$ over $F$, and $E'$ is a splitting field for $\phi(f(x))$ over $F'$, then there is an isomorphism from $E$ to $E'$ that agrees with $\phi$ on $F$.

**Corollary:** Let $F$ be a field, $f(x) \in F[x]$. Then any
Corollary: Let \( E \) be a field, \( f(x) \in F[x] \). Then any two splitting fields of \( f(x) \) over \( E \) are isomorphic.

**Pf of Cor:** Apply the Theorem with \( F = F' \), \( \phi \) the identity map.

**Pf of Thm:** We induct on the degree of \( f(x) \).

- \( \text{deg}(f) = 1 \). In this case, \( E = F \) and \( E' = F' \), \( \phi : E \to E' \) is the desired map.

- \( \text{deg}(f) > 1 \): Let \( p(x) \) be an irreducible factor of \( f(x) \), \( a \) a zero of \( p(x) \) in \( E \), and \( b \) a zero of \( \phi(p(x)) \) in \( E' \).

By the Lemma above, there is an isomorphism \[ \phi' : F(a) \rightarrow F'(b) \]
agreeing with \( \phi \) on \( F \) and taking \( a \) to \( b \).

Now: \( f(x) = (x-a)g(x) \), \( g(x) \in F(a)[x] \), i.e. \( g(x) \) has coefficients in \( F(a) \).

Moreover, \( E \) is the splitting field for \( g(x) \) over \( F \).

Indeed, clearly \( E \) splits \( g(x) \), and if \( a_2, \ldots, a_n \) are the
Indeed, clearly $E$ splits $g(x)$, and if $a_2, \ldots, a_n$ are the roots of $g$ in $E$, then

$$E = F(a, a_2, \ldots, a_n) = F(a) (a_2, \ldots, a_n)$$

$E$ is the splitting field for $g(x)$ over $F(a)$.

Similarly, $E'$ is a splitting field for $\sigma(g(x))$ over $F'(b)$.

Since $\deg(g) < \deg(f)$, $F$ an isomorphism

$$\beta : E \rightarrow E'$$

that agrees with $\sigma$ on $F(a)$,

and thus agrees with $\delta$ on $F$.

\[\square\]

**Example:** $a \in \mathbb{Q}$ is positive, $\omega = e^{2\pi i/n}$ is a primitive $n^{th}$ root of unity.

Consider $x^n - a \in \mathbb{Q}[x]$.

The roots of $x^n - a$ in $\mathbb{C}$ are $a^{1/n}, \omega a^{1/n}, \omega^2 a^{1/n}, \ldots, \omega^{n-1} a^{1/n}$

$$\Rightarrow \quad x^n - a = (x - a^{1/n})(x - \omega a^{1/n}) \cdots (x - \omega^{n-1} a^{1/n})$$

and the splitting field of $x^n - a$ over $\mathbb{Q}$ is

$$\mathbb{Q}(a^{1/n}, \omega).$$

Note: $\mathbb{Q}(a^{1/n}, \omega a^{1/n}) = \mathbb{Q}(a^{1/n}, \omega)$.
Note: \( \mathbb{Q}(a^n, \omega a^n) = \mathbb{Q}(a^n, \omega) \)

Also: \( \omega \) is not a root of \( x^n - 1 \)

E.g. If \( n = 3, \ a = 2, \) the splitting field of \( x^3 - 2 \) over \( \mathbb{Q} \) is \( \mathbb{Q}(\sqrt[3]{2}, \ e^{2\pi i/3}) \)

Algebraic Extensions

Def 1: Let \( E \) be an extension field of a field \( F, \) and \( a \in E. \) We say \( a \) is algebraic over \( F \) if \( a \) is the zero of some nonzero polynomial in \( F[x]. \)

If \( a \) is not algebraic over \( F, \) then \( a \) is said to be transcendental over \( F. \)

An extension \( E \) of \( F \) is called algebraic over \( F \) if every element of \( E \) is algebraic over \( F. \)

If \( E \) is not algebraic over \( F, \) it is called a transcendental extension of \( F. \)

Examples
Examples

(0) Every element $a \in F$ is algebraic over $F$: If $a \in F$, then $a$ is the zero of the polynomial $x - a \in F[x]$.

(1) $\sqrt{2}$ is algebraic over $\mathbb{Q}$, as $\sqrt{2}$ is the root of $x^2 - 2 \in \mathbb{Q}[x]$.

(2) $a \in \mathbb{Q}$ is positive, $n$ integer, $a^n$ is algebraic over $\mathbb{Q}$ as $a^n$ is a root of $x^n - a \in \mathbb{Q}[x]$.

(3) $i = \sqrt{-1}$ is algebraic over $\mathbb{Q}$.

(4) If it is known that $\pi$ and $e$ are transcendental over $\mathbb{Q}$, they are algebraic over $\mathbb{R}$. It is not known whether $\pi + e$ is transcendental over $\mathbb{Q}$.

(5) If $F$ is a field, $x \in F(x)$ the fraction field of $F[x]$, then $x$ is transcendental over $F$.

(5') More generally, suppose $g(x) \in F(x)$ is any nonconstant polynomial. Then $g$ is transcendental over $F$. Indeed, consider

$$g^n + a_{n-1}g^{n-1} + \ldots + a_0, \quad a_i \in F.$$

Then this cannot be $0$ in $F(x)$, because it is a
Then this cannot be $0$ in $F(x)$, because it is a non-constant polynomial. (It has degree $n \deg(g)$.)

**Theorem.** Let $E$ be an extension field of the field $F$, and let $a \in E$. If $a$ is transcendental over $F$, then $F(a) \cong F(x)$. If $a$ is algebraic over $F$, then $F(a) \cong F[x]/\langle p(x) \rangle$ where $p$ is a polynomial in $F[x]$ of minimum degree such that $p(a) = 0$.

Moreover, $p(x)$ is irreducible over $F$.  