A QUATERNIONIC SAITO-KUROKAWA LIFT AND CUSP FORMS ON G_2

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Abstract. We consider a special theta lift $\theta(f)$ from cuspidal Siegel modular forms f on Sp_4 to "modular forms" $\theta(f)$ on $\mathrm{SO}(4,4)$, in the sense of [Pol20a]. This lift can be considered an analogue of the Saito-Kurokawa lift, where now the image of the lift is representations of $\mathrm{SO}(4,4)$ that are quaternionic at infinity. We relate the Fourier coefficients of $\theta(f)$ to those of f, and in particular prove that $\theta(f)$ is nonzero and has algebraic Fourier coefficients if f does. Restricting the $\theta(f)$ to $G_2 \subseteq \mathrm{SO}(4,4)$, we obtain cuspidal modular forms on G_2 of arbitrarily large weight with all algebraic Fourier coefficients. In the case of level one, we obtain precise formulas for the Fourier coefficients of $\theta(f)$ in terms of those of f. In particular, we construct nonzero cuspidal modular forms on G_2 of level one with all integer Fourier coefficients.

1. Introduction

Recall the notion of "modular forms" on G_2 from [GGS02]. These are elements of the space $Hom_{G_2(\mathbf{R})}(\pi_n, \mathcal{A}(G_2))$ of $G_2(\mathbf{R})$-equivariant homomorphisms from the quaternionic discrete series π_n [GW94, GW96] to the space of automorphic forms on $G_2(\mathbf{A})$. Equivalently, see [Pol20a] or [Pol19], they are certain $Sym^{2n}(\mathbf{C}^2)$-valued automorphic functions on $G_2(\mathbf{A})$ that are annihilated by a special linear differential operator $\mathcal{D}_n$. In [GGS02], Gan-Gross-Savin developed the theory of the (non-degenerate) Fourier coefficients of modular forms on G_2 using as a key input a certain Archimedean multiplicity one result of Wallach [Wal03], (see also [Gan00, section 15].) The full Fourier expansion, including the degenerate terms, of modular forms on quaterionic exceptional groups was then developed in [Pol20a]. See also [Pol20b].

While the general theory of the Fourier expansion is now worked out for these modular forms on quaternionic exceptional groups, there is currently a short supply of concrete examples; these can be found in [GGS02], [Wei06] and [Pol20b]. In [GGS02] and in [Pol20b] it is proved that the examples given have most of their Fourier coefficients being algebraic numbers. However, the modular forms constructed in [GGS02] and [Pol20b] are not *cusp forms*, and in fact no examples have been given of cusp forms that have all algebraic Fourier coefficients–or, for that matter, any explicit examples of nonzero cusp forms. Thus, it is natural to ask if there exist cuspidal modular forms on G_2 all of whose Fourier coefficients are algebraic. Our first theorem settles this question in the affirmative.

Theorem 1.0.1. *Suppose $w \geq 16$ is even. Then there are nonzero cuspidal modular forms on G_2 of weight w, all of whose Fourier coefficients are algebraic numbers.*

To prove this theorem, we develop an analogue of the Saito-Kurokawa lift, or more generally, the Oda-Rallis-Schiffman lift [Oda78], [RS78, RS81], see also [Kud78]. Recall that the Saito-Kurokawa lift can be considered as a very special case of the θ-lift from holomorphic modular forms on $\widetilde{\mathrm{SL}_2}$ to holomorphic Siegel modular forms on $\mathrm{SO}(5) = \mathrm{PGSp}_4$, and the Fourier coefficients of the lift can be neatly described in terms of those of the input on $\widetilde{\mathrm{SL}_2}$. These types of special θ lifts, in turn, go back to Doi-Naganuma [DN70], Niwa [Niw75] and Shintani [Shi75].

The lifts we consider now start off with cuspidal Siegel modular forms f on Sp_4, and via very special test data for the Weil-representation, we lift them to automorphic forms on $\mathrm{SO}(4,4)$. The

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lifts are cuspidal by a general argument of Rallis [Ral84, Chapter I, section 3]. With our special test data, we are able to check that the lifts are nonzero (quaternionic) modular forms in the sense of [Wei06] and [Pol20a]. Moreover, we can prove that the lift to $\mathrm{SO}(4,4)$ preserves algebraicity in the sense that if f has Fourier coefficients in some field E containing the cyclotomic extension of $\mathbf{Q}$, then $\theta(f)$ also has Fourier coefficients in E. The archimedean theta correspondence has a long history, see especially [Mg89], [Li89], [Li90] and the references contained therein. By making explicit computations with special test data for the Weil representation, we are able to guarantee that these special lifts to $\mathrm{SO}(4,4)$ are nonvanishing quaternionic modular forms while simultaneously controlling the algebraicity of their Fourier expansion.

Except for the cuspidality on $\mathrm{SO}(4,4)$, these properties of the theta lift partially generalize to a special quaternionic θ-lift from cuspidal Siegel modular forms on Sp_4 to quaternionic modular forms on $\mathrm{SO}(4,8k+4)$. Let us remark that this special θ-lift from $\mathrm{Sp}(4)$ to $\mathrm{SO}(4,8k+4)$ is also inspired by [Nar08] (following unpublished work of Arakawa) who lifts cuspidal modular forms on SL_2 to quaternionic modular forms on $\mathrm{Sp}(1,q)$, which sits inside $\mathrm{SO}(4,4q)$.

Continuing with the dual pair $\mathrm{Sp}_4 \times \mathrm{SO}(4,4)$, one can restrict automorphic functions on $\mathrm{SO}(4,4)$ to G_2. Perhaps surprisingly, the restriction of cuspidal nonzero modular forms on $\mathrm{SO}(4,4)$ to G_2 remains a *cuspidal* nonzero modular form. Moreover, the algebraicity of the Fourier coefficients is preserved under restriction to G_2. Because the cuspidal Hecke eigenforms on Sp_4 always have Fourier coefficients in a finite extension of $\mathbf{Q}$, the above procedure produces cuspidal modular forms on G_2 of arbitrarily large weight, with all algebraic Fourier coefficients.

The method of θ-lifting to an orthogonal group and then restricting to G_2 comes from Rallis-Schiffmann [RS89]. There, the authors lift from $\widetilde{SL}_2$ to split $\mathrm{SO}(7)$, and then restrict to G_2. These Rallis-Schiffmann lifts to G_2 do not produce *modular forms* on G_2. Indeed, Li and Schwermer [LS93, section 3.8] compute that the discrete series on $G_2(\mathbf{R})$ that are in the image of the Rallis-Schiffmann lift are the ones that have minimal K-type nontrivial representations of the short-root $\mathrm{SU}(2)$, as opposed to the long-root $\mathrm{SU}(2)$. See also [YN12], where the authors construct special automorphic functions on $\mathrm{SO}(4,1)$ with algebraic Fourier coefficients, by restricting theta functions from $\mathrm{SO}(4,2)$.

In the case when the input $F(Z)$ is a *level one* cuspidal holomorphic modular form on $\mathrm{Sp}(4)$, we obtain precise formulas for the Fourier coefficients $\theta(F)$ on $\mathrm{SO}(V) = \mathrm{SO}(4,4)$ and $\theta(F)|_{G_2}$ on G_2. See Theorem 4.1.1 and Corollary 4.2.4. As a consequence of these results, one has the following.

Theorem 1.0.2. *Suppose $F(Z) = \sum_{T>0} a_F(T) q^T$ is a level one cuspidal holomorphic modular form on* $\mathrm{Sp}(4)$ *of even weight $w \geq 16$, with Fourier coefficients $a_F(T)$ in some ring $R \subseteq \mathbf{C}$. Assume moreover that the Fourier coefficient $a_F(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) \neq 0$. Then $\theta(F)|_{G_2}$ is a nonzero cuspidal modular form on G_2 with Fourier coefficients in R.*

The theorem produces cuspidal modular forms on G_2 with all integral Fourier coefficients. As there is a weight 20, level one cusp form F for $\mathrm{Sp}(4;\mathbf{Z})$, with Fourier coefficients in $\mathbf{Z}$ and $a_F(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = 4$ [Sko92], one obtains the following corollary.

Corollary 1.0.3. *There is a nonzero, weight* 20*, level one cuspidal modular form on G_2, with Fourier coefficients in $\mathbf{Z}$.*

The structure of the paper is as follows. In section 2, we define various notations, and discuss general properties of the theta lift from $\mathrm{Sp}(4)$ to $\mathrm{SO}(V)$. Section 3 is archimedean in nature, and contains the key results that lead to the fact that our special theta lifts are modular forms and have algebraic Fourier coefficients. In section 4 we put together the work of sections 2 and 3 to obtain the main results Theorem 4.1.1 and Corollary 4.2.4; the latter implies Theorem 1.0.2. Finally, in sections 5 and 6 we obtain results on the algebraicity and nonvanishing of the theta lift to $\mathrm{SO}(V)$ and G_2; in particular, we deduce Theorem 1.0.1.

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2. Generalities on the theta lift

In this section, we describe those results on the θ-lift from Sp_4 to $\mathrm{SO}(4,n)$ that do not depend on our specific test data or the notion of modular forms on quaternionic groups. We also give various definition and notations that will be used throughout the paper.

2.1. **Weil representation.**

We discuss various notations, definitions, and recall well-known facts for the Weil representation [Wei64] restricted to $\mathrm{SO}(V)\times \mathrm{Sp}(W)$ when $\dim(V)=m$ is even. See [Kud86],[Ral84, Ral82].

2.1.1. *Generalities.* Suppose k is a local or global field of characteristic 0. Let (V,q) be a non-degenerate quadratic space over k, and let $(x,y)=q(x+y)-q(x)-q(y)$ be the associated symmetric bilinear form, so that $q(x)=\frac{1}{2}(x,x)$. Set $\det(V)\in k^\times/(k^\times)^2$ the element $\det((v_i,v_j)_{i,j})$ where $v_1,\dots,v_m$ is a basis of V. Set $\mathrm{disc}(V)=(-1)^{m(m-1)/2}\det(V)\in k^\times/(k^\times)^2$ the discriminant of V. When k is a local field, let $(a,b)_2\in\mu_2$ be the Hilbert symbol of k and denote by $\chi_V: k^\times\to\mu_2$ the character $\chi_V(x)=(x,\mathrm{disc}(V))_2$. When k is a global field, write χ_V for the quadratic character of $\mathbf{A}_k$ whose local components are the $(\cdot,\mathrm{disc}(V))_2$ just defined.

Let $\psi:\mathbf{A}/k\to\mathbf{C}^\times$ be a fixed additive character. Below when $k=\mathbf{Q}$, we will take ψ to be the standard choice, so that $\psi_\infty(x)=e^{2\pi i x}$. For f a Schwartz-Bruhat function, we write

$$\widehat{f}(x)=\int_V \psi((y,x))f(y)\,dy$$

the Fourier transform of f. The integral is over $V(k)$ if k is a local field, or the adelic points $V(\mathbf{A})$ if k is a global field. The measure dy is normalized so that if $f_1(x)=\widehat{f}(x)$ then $\widehat{f_1}(x)=f(-x)$. Let $\gamma(\psi,q(\cdot))$ be the Weil index[1][Wei64].

Now suppose W is a symplectic space over k, and $W=X\oplus Y$ is a Langrangian decomposition. Moreover, assume given a symplectic basis $e_1,\dots,e_n,f_1,\dots,f_n$ for W, so that $X=\mathrm{Span}\{e_1,\dots,e_n\}$ and $Y=\mathrm{Span}\{f_1,\dots,f_n\}$. We write

$$\mathbf{X}=X\otimes V=V^n \text{ and } \mathbf{Y}=Y\otimes V=V^n.$$

When $m=\dim(V)$ is even, we have a Weil representation ω_ψ of $\mathrm{SO}(V)\times\mathrm{Sp}(W)$ on the Schwartz space $S(\mathbf{X})$ locally and globally.

We let $\mathrm{SO}(V)$ and $\mathrm{Sp}(W)$ act on the right of V, resp. W. Then if $g\in\mathrm{SO}(V)$ and $\phi\in S(\mathbf{X})$ one has $\omega_\psi(g,1)\phi(x)=\phi(xg)$. The Weil representation restricted to $\mathrm{Sp}(W)$ is the unique representation for which

$$\left(\begin{pmatrix}\alpha & \beta\\ & {}^t\alpha^{-1}\end{pmatrix}\phi\right)(x)=\chi_V(\det(\alpha))|\det(\alpha)|^{m/2}\psi(\langle x\alpha,x\beta\rangle/2)\phi(x\alpha) \tag{1}$$

and

$$\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\phi\right)(x)=\gamma(\psi,q)^n\widehat{\phi}(x)$$

where

$$\widehat{\phi}(x)=\int_{V^n}\psi(\mathrm{tr}((x_i,y_j)))\phi(e_1\otimes y_1+\cdots+e_n\otimes y_n)\,dy_1\cdots dy_n.$$

As above, the integral is over the k-points $\mathbf{X}(k)=V^n(k)$ if k is a local field and the adelic points $\mathbf{X}(\mathbf{A})=V^n(\mathbf{A})$ if k is a global field.

[1] In what follows, our quadratic spaces (V,q) will be such that $\gamma(\psi,q(\cdot))=1$.

2.1.2. *Partial Fourier transform.* We will require the use of a partial Fourier transform to go between different models of this Weil representation. We explain this now. Suppose

$$V = U \oplus V_0 \oplus U^\vee$$

with $U, U^\vee$ isotropic, paired nontrivially via the symmetric form, and V_0 non-degenerate, $V_0 = (U \oplus U^\vee)^\perp$. Note that $\operatorname{disc}(V_0) = \operatorname{disc}(V)$ so that $\chi_V = \chi_{V_0}$. With

$$\mathbf{X}_U := X \otimes V_0 \oplus W \otimes U \text{ and } \mathbf{Y}_U := Y \otimes V_0 \oplus W \otimes U^\vee$$

one has that $\mathbf{W} = W \otimes V = \mathbf{X}_U \oplus \mathbf{Y}_U$ is a Lagrangian decomposition.

There is a different model of ω_ψ, now on $S(\mathbf{X}_U)$. The intertwining operator between these two models is given by a partial Fourier transform. This transform is defined as follows. First, we have

$$\mathbf{X} = (U \oplus V_0 \oplus U^\vee) \otimes X = U \otimes X \oplus V_0 \otimes X \oplus U^\vee \otimes X$$

and

$$\mathbf{X}_U = U \otimes W \oplus V_0 \otimes X = U \otimes X \oplus V_0 \otimes X \oplus U \otimes Y.$$

Then if $\phi \in S(\mathbf{X})$,

$$F(\phi)(x, y, z) = \int_{(U^\vee \otimes X)(\mathbf{A})} \psi(\langle z', z\rangle)\phi(x, y, z')\, dz'$$

where $x \in U \otimes X$, $y \in V_0 \otimes X$, $z \in U \otimes Y$ and $z' \in U^\vee \otimes X$.

The partial Fourier transform defines an isomorphism $S(\mathbf{X}) \to S(\mathbf{X}_U)$. Thus by transport of structure, there is Weil representation $\omega_{\psi,\mathbf{Y}_U}$ of $\mathrm{SO}(V) \times \mathrm{Sp}(W)$ on $S(\mathbf{X}_U) = S(X \otimes V_0) \otimes S(W \otimes U)$. This representation has the following well-known and useful property. Denote by P_U the parabolic subgroup of $\mathrm{Sp}(\mathbf{W})$ that stabilizes $\mathbf{Y}_U$. For $p \in P_U$, let $\det_{P_U}(p) = \det(p : \mathbf{W}/\mathbf{Y}_U \to \mathbf{W}/\mathbf{Y}_U)$ be the determinant of the map induced by p on $\mathbf{W}/\mathbf{Y}_U$. Finally, set

$$p_{\mathbf{X}_U} : \mathbf{W} \to \mathbf{X}_U, \quad p_{\mathbf{Y}_U} : \mathbf{W} \to \mathbf{Y}_U$$

the projections corresponding to the decomposition $\mathbf{W} = \mathbf{X}_U \oplus \mathbf{Y}_U$.

Proposition 2.1.1. *Assume that* $\operatorname{disc}(V) = 1$. *Suppose* $p \in P_U \cap (\mathrm{SO}(V) \times \mathrm{Sp}(W))$ *and* $\phi \in S(\mathbf{X}_U)$. *Then*

$$\omega_{\psi,\mathbf{Y}_U}(p)\phi(x) = |\det_{P_U}(p)|^{1/2}\psi(\langle p_{\mathbf{X}_U}(xp), p_{\mathbf{Y}_U}(yp)\rangle/2)\phi(p_{\mathbf{X}_U}(xp)). \tag{2}$$

In particular, $\mathrm{Sp}(W)$ *acts linearly on* $S(W \otimes U)$.

Proof. As mentioned, this is well-known. See, for example, Kudla's notes "The local theta correspondence", Lemma 4.2 or [Ral84, pg 340-341]. Because one already knows that the Weil representation exists on $S(\mathbf{X})$, the proposition can be checked by assuming $\phi = F(\phi')$ for $\phi' \in S(\mathbf{X})$ and checking (2) for generators of $\mathrm{Sp}(W)$ and $\mathrm{SO}(V)$. □

If $\phi \in S(\mathbf{X}(\mathbf{A}))$, the θ-function associated to ϕ is

$$\theta(g, h; \phi) = \sum_{\xi \in \mathbf{X}(k)} \omega_\psi(g, h)\phi(\xi).$$

Here $g \in \mathrm{SO}(V)(\mathbf{A})$ and $h \in \mathrm{Sp}(W)(\mathbf{A})$. The function $\theta(g, h; \phi)$ is an automorphic form on $\mathrm{SO}(V)(\mathbf{A}) \times \mathrm{Sp}(W)(\mathbf{A})$.

If $\phi' = F(\phi) \in S(\mathbf{X}_U)$, then one can similarly define

$$\theta(g, h; \phi') = \sum_{\xi \in \mathbf{X}_U(k)} \omega_{\psi,\mathbf{Y}_U}(g, h)\phi'(\xi).$$

Proposition 2.1.2. *Let the notation be as above, so that* $\phi' = F(\phi)$. *Then* $\theta(g, h; \phi') = \theta(g, h; \phi)$.

Proof. This follows from Poisson summation, and is well-known. □

Finally, given a cuspidal automorphic form f on $\mathrm{Sp}(W)$ and $\phi \in S(\mathbf{X}(\mathbf{A}))$, the theta-lift of f is

$$\theta(f,\phi)(g) = \int_{\mathrm{Sp}(W)(k)\backslash \mathrm{Sp}(W)(\mathbf{A})} \theta(g,h;\phi) f(h)\, dh.$$

By Proposition 2.1.2,

$$\theta(f,\phi)(g) = \int_{\mathrm{Sp}(W)(k)\backslash \mathrm{Sp}(W)(\mathbf{A})} \theta(g,h;F(\phi)) f(h)\, dh.$$

2.2. **Definitions and notation.** We now give various definitions and specific notations that we will use below.

2.2.1. *The group* $\mathrm{SO}(V)$. Throughout the paper $n = 8k+4$ for a non-negative integer k. Moreover, V is an $8k+8$-dimensional $\mathbf{Q}$-vector space that has a quadratic form q of signature $(4,n) = (4, 8k+4)$. Inside of V, $\Lambda \subseteq V$ is an even unimodular lattice for the quadratic form q. Thus we assume that the discriminant of V is trivial. The symmetric bilinear form associated to q is $(x,y) = q(x+y) - q(x) - q(y)$ for $x, y \in V$.

We fix a decomposition

$$V = U \oplus V_0 \oplus U^\vee$$

as above where $U, U^\vee$ are isotropic and two-dimensional, and duals to each other under the symmetric pairing on V. Here $V_0 = (U \oplus U^\vee)^\perp$ is an orthogonal space of signature $(2, n-2)$, and we denote by q_0 the restriction of q to V_0. We write b_1, b_2 for a fixed basis of U, and b_{-1}, b_{-2} for the dual basis of $U^\vee$.

The Lie algebra $\mathfrak{so}(V)$ is identified with $\wedge^2 V$. More precisely, we let $\mathrm{SO}(V)$ and $\mathfrak{so}(V)$ act on the right of V, and the identification $\mathfrak{so}(V) \simeq \wedge^2 V$ is taken so that $x \cdot v_1 \wedge v_2 = (x, v_1)v_2 - (x, v_2)v_1$ for $x, v_1, v_2 \in V$. The Lie bracket on $\mathfrak{so}(V) \simeq \wedge^2 V$ is $[u_1 \wedge u_2, X] = u_1 \wedge (u_2 X) + (u_1 X) \wedge u_2$ if $X \in \mathfrak{so}(V)$ and $u_1, u_2 \in V$. In particular

$$[u_1 \wedge u_2, v_1 \wedge v_2] = u_1 \wedge ((u_2, v_1)v_2 - (u_2, v_2)v_1) + ((u_1, v_1)v_2 - (u_1, v_2)v_1)) \wedge u_2.$$

2.2.2. *The Cartan involution and modular forms.* Over $\mathbf{R}$, we fix an orthogonal decomposition $V_0 \otimes \mathbf{R} = \mathbf{C} \oplus V_+$ so that $q_0(z,h) = |z|^2 - q_+(h)$, where (V_+, q_+) is a positive-definite orthogonal space and $|\cdot|$ is the usual norm on $\mathbf{C}$. With this decomposition fixed, we have $q((x,(z,h),\delta)) = \delta(x) + |z|^2 - q_+(h)$, where $x \in U$, $\delta \in U^\vee$ and z, h as above.

We obtain a majorant of q as follows. Define $\iota : U \to U^\vee$ via $\iota(b_j) = b_{-j}$ for $j = 1, 2$. Extend ι to V by defining $\iota(z,h) = (z,-h)$ for $(z,h) \in V_0$. Then

$$((x,z,h,\delta), \iota(x,z,h,\delta)) = (x, \iota(x)) + (\iota(\delta), \delta) + 2|z|^2 + 2q_+(h) \geq 0.$$

We set $r(v) = ||v||^2 = \frac{1}{2}(v, \iota(v))$ the majorant.

The involution $\iota : V \to V$ gives rise to a Cartan involution θ_ι on $\mathrm{SO}(V)(\mathbf{R})$ and $\mathfrak{so}(V)$. Specifically, on $\mathrm{SO}(V)(\mathbf{R})$ the Cartan involution is given by conjugation by $\iota \in O(V)$, and on $\mathfrak{so}(V)$ by $v_1 \wedge v_2 \mapsto \iota(v_1) \wedge \iota(v_2)$. The maximal compact subgroup K_V of $\mathrm{SO}(V)(\mathbf{R})$ is the subgroup of $\mathrm{SO}(V)$ that fixes the majorant $||\cdot||^2$.

We now fix an $\mathfrak{sl}_2$ triple (e_ℓ, h_ℓ, f_ℓ) of $\mathfrak{so}(V) \otimes \mathbf{C}$, as follows. First, set $u_1 = (b_1 + b_{-1})/\sqrt{2}$, $u_2 = (b_2 + b_{-2})/\sqrt{2}$, so that u_1, u_2 are orthonormal elements of V. Denote by $V_0^+ = \mathbf{C}$ the subspace of V_0 where ι acts by $+1$, and let v_1^+, v_2^+ be a fixed orthonormal basis of V_0^+. Set

$$\begin{aligned}
e_\ell &= \frac{1}{2}(u_1 - iu_2) \wedge (v_1 - iv_2) \\
h_\ell &= i\,(u_1 \wedge u_2 + v_1 \wedge v_2) \\
f_\ell &= -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 + iv_2).
\end{aligned}$$

Then $h_\ell = [e_\ell, f_\ell]$, $[h_\ell, e_\ell] = 2e_\ell$, and $[h_\ell, f_\ell] = -2f_\ell$. Moreover, the Cartan involution θ_ι acts as the identity on this $\mathfrak{sl}_2(\mathbf{C})$ spanned by e_ℓ, h_ℓ, f_ℓ.

Denote by $V_2 \cong \mathbf{C}^2$ the defining representation of SU(2), and $\mathbb{V}_w = Sym^{2w}V_2$, which is a representation of $\mathrm{SU}(2)/\mu_2$. We fix a K_V^0-equivariant projection

$$p_K : \wedge^2 V \otimes \mathbf{C} \to \mathfrak{su}_2 \otimes \mathbf{C} \simeq \mathfrak{sl}_2(\mathbf{C}) \simeq \mathbb{V}_1 = Sym^2(V_2)$$

by projecting to the $\mathfrak{sl}_2(\mathbf{C}) = \mathfrak{su}_2 \otimes \mathbf{C}$ spanned by e_ℓ, h_ℓ, f_ℓ. This projection induces a map $K_V^0 \to \mathrm{SU}(2)/\mu_2$.

We briefly recall the definition of *modular forms* on $\mathrm{SO}(V)$ and G_2 from [Pol20a], building on [GGS02]; see [Pol20a, Definition 1.1.1]. Specifically, if $w \geq 1$ is an integer, and $G = \mathrm{SO}(V)$, then a modular form of weight w on G is an automorphic function $F : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbb{V}_w$ satisfying

- $F(gk) = k^{-1} \cdot F(g)$ for all $k \in K_V^0 \subseteq G(\mathbf{R})$ and $g \in G(\mathbf{A})$;
- F is annihilated by a certain first-order linear differential operator D_w.

Modular forms of weight w on G_2 are defined analogously. See, for example, *loc cit*, or also [Pol19], [Pol20b].

2.2.3. *The dual pair.* Denote by W the defining four-dimensional representation of $\mathrm{Sp}(4) = \mathrm{Sp}(W)$. We fix a polarization $W = X \oplus Y$ of Lagrangian subspaces. Write e_1, e_2, f_1, f_2 for a fixed symplectic basis of W, so that X is spanned by e_1, e_2 and Y by f_1, f_2.

The vector space $\mathbf{W} = V \otimes W$ comes equipped with the symplectic form $(\ ,\ ) \otimes \langle\ ,\ \rangle$. As in subsection 2.1, we set $\mathbf{X} = V \otimes X$, $\mathbf{Y} = V \otimes Y$, $\mathbf{X}_U = U \otimes W \oplus V_0 \otimes X$, $\mathbf{Y}_U = U^\vee \otimes W \oplus V_0 \otimes Y$, so that $\mathbf{W} = \mathbf{X} \oplus \mathbf{Y} = \mathbf{X}_U \oplus \mathbf{Y}_U$ are isotropic decompositions.

As mentioned, we let our groups act on the right of the spaces that define them, i.e., we let $\mathrm{SO}(V)$ act on the right of V and $\mathrm{Sp}(W)$ act on the right of W. Denote by $P_Y = M_Y N_Y$ the Siegel parabolic subgroup of $\mathrm{Sp}(W)$ that is the stabilizer of Y and M_Y the Levi subgroup fixing the decomposition $X \oplus Y$. Thus $M_Y \simeq \mathrm{GL}(X) \simeq \mathrm{GL}_2$ and we identify N_Y with the symmetric 2×2 matrices. Denote by $P_U = M_U N_U$ the Heisenberg parabolic subgroup of $\mathrm{SO}(V)$ that stabilizes the isotropic subspace $U^\vee$ and M_U the Levi subgroup that fixes the decomposition $V = U \oplus V_0 \oplus U^\vee$. Thus $M_U \simeq \mathrm{GL}(U) \times \mathrm{SO}(V_0) \simeq \mathrm{GL}_2 \times \mathrm{SO}(V_0)$. Finally, we denote $Z = [N_U, N_U]$ the center of N_U. Thus, the unipotent group Z is one-dimensional and spanned by the root space for the highest root of $\mathrm{SO}(V)$.

2.3. **The Fourier coefficients of the lift.** Suppose that $\chi : N_U(\mathbf{Q})\backslash N_U(\mathbf{A}) \to \mathbf{C}^\times$ is a character of the unipotent radical of the Heisenberg parabolic of $\mathrm{SO}(V)$. In this subsection, we relate the χ-Fourier coefficient of a θ-lift $\theta(f, \phi)_\chi$ of f with the Fourier coefficients of f along the unipotent radical of the Siegel parabolic of $\mathrm{Sp}(W)$.

Denote by $[R]$ the quotient space $R(k)\backslash R(\mathbf{A})$, if R is a linear algebraic group over k. In more detail,

$$\begin{aligned}
\theta(f, \phi)_\chi(g) &= \int_{[N_U]} \chi^{-1}(n)\theta(f, \phi)(ng)\, dn \\
&= \int_{[N_U]\times[\mathrm{Sp}(W)]} \chi^{-1}(n)\theta(ng, h; \phi)f(h)\, dh\, dn \\
&= \int_{[Sp(W)]} \theta_\chi(g, h; \phi)f(h)\, dh
\end{aligned}$$

where

$$\theta_\chi(g, h; \phi) = \int_{[N_U]} \chi^{-1}(n)\theta(ng, h; \phi)\, dn.$$

If $\phi \in \mathbf{X}_U(\mathbf{A})$ we can relate $\theta(f, \phi)_\chi(g)$ to the Fourier coefficients of f along N_Y, as follows.

Suppose $v_1, v_2 \in V_0$. We can associate to the pair v_1, v_2 a character of N_U and N_Y, as follows. First, one identifies the abelianized unipotent radical N_U/Z with $V_0 \otimes U^\vee$ via the exponential map on the Lie algebra $\mathfrak{so}(V) \simeq \wedge^2 V$. More precisely, if $\delta \in U^\vee$, $v \in V_0$, and $x \in V$, then

$$\begin{aligned} x \exp(v \otimes \delta) &= x + x(v \otimes \delta) + \frac{1}{2} x(v \otimes \delta)^2 \\ &= x + (v,x)\delta - (\delta, x)v - \frac{1}{2}(\delta,x)(v,v)\delta. \end{aligned}$$

This defines an element of N_U, and we denote by $n(v \otimes \delta)$ the associated element of N_U/Z. We record now the following formulas:

$$\begin{aligned} b_1 \exp(x_1 b_{-1} + x_2 b_{-2}) &= b_1 - x_1 - \frac{1}{2}((x_1,x_1)b_{-1} + (x_2,x_1)b_{-2}) \\ b_2 \exp(x_1 b_{-1} + x_2 b_{-2}) &= b_2 - x_2 - \frac{1}{2}((x_1,x_2)b_{-1} + (x_2,x_2)b_{-2}) \\ y \exp(x_1 b_{-1} + x_2 b_{-2}) &= y + (x_1,y)b_{-1} + (x_2,y)b_{-2}. \end{aligned}$$

Associated to v_1, v_2 the element $v_1 \otimes b_1 + v_2 \otimes b_2 \in V_0 \otimes U$ gives a linear map $L_{v_1,v_2} : N_U/Z \simeq V_0 \otimes U^\vee \to \mathbf{Q}$ defined as

$$L_{v_1,v_2}(n(x_1 \otimes b_{-1} + x_2 \otimes b_{-2})) = (v_1, x_1) + (v_2, x_2).$$

We then obtain a character $\chi_{v_1,v_2} : N_U(\mathbf{Q}) \backslash N_U(\mathbf{A}) \to \mathbf{C}^\times$ as $\chi_{v_1,v_2}(n(x \otimes \delta)) = \psi(L_{v_1,v_2}(x \otimes \delta.))$.

Associated to v_1, v_2 we can also define a character on N_Y, as follows. First, using the basis e_1, e_2, f_1, f_2, one has an identification $N_Y \simeq Sym^2(X) \simeq Sym^2(k^2)$. If $x \in Sym^2(k^2)$ is a 2×2 symmetric matrix, let $n_Y(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Sp}(4)$ be the associated unipotent element. Suppose T is a 2×2 rational symmetric matrix. Associated to T, we have a character $N_Y(k) \backslash N_Y(\mathbf{A}) \to \mathbf{C}^\times$ as $\chi_T(n_Y(x)) = \psi(\mathrm{tr}(T,x))$. For an automorphic function f on $\mathrm{Sp}(W)(\mathbf{A})$, we write

$$f_T(g) = f_{\chi_T}(g) = \int_{[N_Y]} \chi_T^{-1}(n) f(ng)\, dn$$

the χ_T-Fourier coefficient of f. Finally, if $v_1, v_2 \in V$, set

$$S(v_1, v_2) = \frac{1}{2} \begin{pmatrix} (v_1,v_1) & (v_1,v_2) \\ (v_1,v_2) & (v_2,v_2) \end{pmatrix}$$

a 2×2 symmetric matrix. Thus $\chi_{S(v_1,v_2)}$ is a character of $N_Y(k) \backslash N_Y(\mathbf{A})$.

The next proposition is familiar from work of Piatetski-Shapiro [PS83] and Rallis [Ral84]. For ϕ in $S(\mathbf{X}_U(\mathbf{A}))$, it will relate the χ_{v_1,v_2} Fourier coefficient of $\theta(f,\phi)$ to the $\chi_{S(v_1,v_2)}$ Fourier coefficient of f.

Proposition 2.3.1. *Suppose f is a cuspidal automorphic function on $\mathrm{Sp}(W)(\mathbf{A})$, $\phi \in S(\mathbf{X}_U(\mathbf{A}))$, and $\theta(f,\phi)$ the automorphic function on $\mathrm{SO}(V)$ that is the theta-lift of f. Suppose $v_1, v_2 \in V_0$ and that the pair is non-degenerate in the sense that $S(v_1,v_2)$ has nonzero determinant. Set*

$$z_{v_1,v_2} = b_1 \otimes f_1 + b_2 \otimes f_2 + v_1 \otimes e_1 + v_2 \otimes e_2 \in \mathbf{X}_U = U \otimes W \oplus V_0 \otimes X.$$

Then

$$\theta(f,\phi)_{\chi_{-v_1,-v_2}}(g) = \int_{N_Y(\mathbf{A}) \backslash \mathrm{Sp}(W)(\mathbf{A})} \omega_{\psi,\mathbf{Y}_U}(g,h)\phi(z_{v_1,v_2}) f_{-S(v_1,v_2)}(h)\, dh. \tag{3}$$

In the proof of this proposition and frequently below, we drop the $\otimes$ sign and write vw for $v \otimes w$ if no confusion seems likely.

Proof of Proposition 2.3.1. Following the arguments of [PS83, section 5], first suppose that $g \in \mathrm{SO}(V_0)(\mathbf{A})$ and $\phi = \phi_U \otimes \phi_{V_0}$, with $\phi_U \in S(U \otimes W)$ and $\phi_{V_0} \in S(V_0 \otimes X)$. Then

$$\begin{aligned}\theta(g,h;\phi) &= \sum_{w_1,w_2 \in W} \sum_{y_1,y_2 \in V_0} \omega_{\psi,U^\vee \otimes W}(1,h)\phi_U(b_1w_1+b_2w_2)\omega_{\psi,V_0\otimes Y}(g,h)\phi_{V_0}(y_1e_1+y_2e_2)\\ &= \theta_{V_0\otimes Y}(g,h;\phi_{V_0})\left(\sum_{w_1,w_2\in W}\omega_{\psi,U^\vee\otimes W}(1,h)\phi_U(b_1w_1+b_2w_2)\right)\end{aligned}$$

in obvious notation.

Taking the constant term along $Z \subseteq N_U$, one obtains that

$$\begin{aligned}\theta_Z(g,h) &= \int_{[Z]} \theta(zg,h)\,dz\\ &= \theta_{V_0\otimes Y}(g,h;\phi_{V_0})\left(\sum_{w_1,w_2\in W,\langle w_1,w_2\rangle=0}\omega_{\psi,U^\vee\otimes W}(1,h)\phi_U(b_1w_1+b_2w_2)\right). \qquad (4)\end{aligned}$$

This follows from the fact that

$$\omega_\psi(\exp(zb_{-1}\wedge b_{-2}))\phi(b_1w_1+b_2w_2+y_1e_1+y_2e_2) = \psi(-z\langle w_1,w_2\rangle)\phi(b_1w_1+b_2w_2+y_1e_1+y_2e_2),$$

as is immediately computed from (1).

Now, the inner sum in (4) can be written as

$$\theta_U^0(1,h;\phi_U)+\theta_U^1(1,h;\phi_U)+\theta_U^2(1,h;\phi_U),$$

where $\theta_U^j(1,h;\phi_U)$ consists of the sum of the terms in (4) for which $\dim_{\mathbf{Q}} Span(w_1,w_2) = j$. Moreover, we have

$$\theta_U^2(1,h;\phi_U) = \sum_{\gamma\in N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{Q})} \omega(1,\gamma h)\phi_U(b_1f_1+b_2f_2).$$

Thus we have

$$\theta_Z(g,h) = \theta_{V_0\otimes Y}(g,h)(\theta_U^0(1,h;\phi_U)+\theta_U^1(1,h;\phi_U)) + \theta_{V_0\otimes Y}(g,h)\theta_U^2(1,h;\phi_U).$$

Consider an element $x_1b_{-1}+x_2b_{-2} \in V_0\otimes U^\vee$. Set

$$z := (b_1w_1+b_2w_2+y_1e_1+y_2e_2)\exp(x_1b_{-1}+x_2b_{-2})$$

Then

Lemma 2.3.2. *Suppose $w_1,w_2 \in Y$. Then*

$$\frac{1}{2}\langle pr_{\mathbf{X}_U}(z), pr_{\mathbf{Y}_U}(z)\rangle = (x_1,y_1)\langle w_1,e_1\rangle + (x_1,y_2)\langle w_1,e_2\rangle + (x_2,y_1)\langle w_2,e_1\rangle + (x_2,y_2)\langle w_2,e_2\rangle.$$

Proof. We have

$$\begin{aligned} z = {}&(b_1 - x_1 - \frac{1}{2}((x_1,x_1)b_{-1}+(x_2,x_1)b_{-2}))w_1 + (b_2 - x_2 - \frac{1}{2}((x_1,x_2)b_{-1}+(x_2,x_2)b_{-2}))w_2\\ &+ (y_1+(x_1,y_1)b_{-1}+(x_2,y_1)b_{-2})e_1 + (y_2+(x_1,y_2)b_{-1}+(x_2,y_2)b_{-2})e_2.\end{aligned}$$

Thus if $w_1,w_2\in Y$,

$$pr_{\mathbf{X}_U}(z) = b_1w_1+b_2w_2+y_1e_1+y_2e_2$$

and

$$\begin{aligned} pr_{\mathbf{Y}_U}(z) = {}&-(x_1+\frac{1}{2}((x_1,x_1)b_{-1}+(x_2,x_1)b_{-2}))w_1 - (x_2+\frac{1}{2}((x_1,x_2)b_{-1}+(x_2,x_2)b_{-2}))w_2\\ &+((x_1,y_1)b_{-1}+(x_2,y_1)b_{-2})e_1+((x_1,y_2)b_{-1}+(x_2,y_2)b_{-2})e_2.\end{aligned}$$

The lemma follows easily. $\square$

From the fact that the pair v_1, v_2 is non-degenerate, applying Lemma 2.3.2 one sees that the terms with θ_U^0 and θ_U^1 vanish upon taking the χ_{v_1,v_2}-Fourier coefficient. Thus we obtain

$$\theta(f;\phi)_{\chi_{-v_1,-v_2}}(g) = \int_{N_Y(k)\backslash \mathrm{Sp}(W)(\mathbf{A})} \omega_\psi(g,h)\phi(z_{v_1,v_2})f(h)\,dh.$$

Let $s = \left(\begin{smallmatrix} s_1 & s_2 \\ s_2 & s_3 \end{smallmatrix}\right)$. Then

$$z_{v_1,v_2}n_Y(s) = b_1f_1 + b_2f_2 + v_1(e_1 + s_1f_1 + s_2f_2) + v_2(e_2 + s_2f_1 + s_3f_2)$$

and thus

$$\frac{1}{2}\langle pr_{\mathbf{X}_U}(z_{v_1,v_2}n_Y(s)), pr_{\mathbf{Y}_U}(z_{v_1,v_2}n_Y(s))\rangle = \mathrm{tr}(S(v_1,v_2)s).$$

The statement of the proposition now follows in the restricted setting $\phi = \phi_U \otimes \phi_{V_0}$ and $g \in \mathrm{SO}(V_0)$.

For the general case, one reduces to the above special case. Indeed, we have proved (3) when $g = 1$ (or more generally, $g \in \mathrm{SO}(V_0)$) and $\phi = \phi_U \otimes \phi_{V_0}$. By linearity, the proposition follows for $g = 1$ and all ϕ. Finally, defining $\phi' = \omega_{\psi,\mathbf{Y}_U}(g,1)\phi$, (3) for ϕ' and $g = 1$ gives (3) for ϕ and g. This completes the proof of the proposition. $\square$

Let us now consider the integral (3) when $\phi = F(\phi')$ for $\phi' \in S(\mathbf{X}(\mathbf{A}))$. More precisely, in section 3, we will need an expression for $\omega_{\psi,\mathbf{Y}_U}(1,h)F(\phi')(z_{v_1,v_2})$ when $h \in M_Y(\mathbf{A})$. We compute this now. For $h \in M_Y$, we abuse notation and let $\det(h)$ denote the determinant of h acting on X.

Lemma 2.3.3. *Suppose $\phi' \in S(\mathbf{X})$, $\phi = F(\phi')$ is in $S(\mathbf{X}_U)$ and $h \in M_Y$. Then*

$$\omega_{\psi,\mathbf{Y}_U}(1,h)\phi(z_{v_1,v_2}) =$$
$$|\det(h)|^{(\dim(V_0)+4)/2}\int_{X^2}\psi(\langle z_1,f_1\rangle + \langle z_2,f_2\rangle)\phi'((v_1e_1+v_2e_2+b_{-1}z_1+b_{-2}z_2)h)\,dz_1\,dz_2. \tag{5}$$

In (5) and in the proof of Lemma 2.3.3, the domains of integration are the k points if k is a local field, or the adelic points if k is a global field.

Proof. We have

$$\omega_{\psi,\mathbf{Y}_U}(1,h)\phi(z_{v_1,v_2}) = |\det(h)|^{\dim V_0/2}\phi(b_1(f_1h) + b_2(f_2h) + v_1(e_1h) + v_2(e_2h)).$$

Thus

$$\begin{aligned}\omega_{\psi,\mathbf{Y}_U}(1,h)\phi(z_{v_1,v_2}) &= |\det(h)|^{\dim V_0/2}\int_{U^\vee\otimes X}\psi(\langle z', b_1(f_1h)+b_2(f_2h)\rangle)\phi'(v_1(e_1h)+v_2(e_2h)+z')\,dz' \\ &= |\det(h)|^{\dim V_0/2}\int_{X^2}\psi(\langle b_{-1}z_1+b_{-2}z_2, b_1(f_1h)+b_2(f_2h)\rangle) \\ &\qquad \times \phi'(v_1(e_1h)+v_2(e_2h)+b_{-1}z_1+b_{-2}z_2)\,dz_1\,dz_2 \\ &= |\det(h)|^{(\dim(V_0)+4)/2}\int_{X^2}\psi(\langle z_1,f_1\rangle+\langle z_2,f_2\rangle) \\ &\qquad \times \phi'((v_1e_1+v_2e_2+b_{-1}z_1+b_{-2}z_2)h)\,dz_1\,dz_2.\end{aligned}$$

In the last line we have made the variable change $z_j \mapsto z_jh$ for $j = 1, 2$. This gives the lemma. $\square$

2.4. **Theta lift of Poincare series.** Finally, we discuss the theta lift of certain Poincare series, in an abstract, formal setting. For the rest of the paper we specialize to the case $k = \mathbf{Q}$. Suppose we have a non-degenerate symmetric 2×2 rational matrix T, and χ_T denotes the associated character of $N_Y(\mathbf{Q})\backslash N_Y(\mathbf{A})$. Suppose that $\mu_T : \mathrm{Sp}(W)(\mathbf{A}) \rightarrow \mathbf{C}$ is a function satisfying $\mu_T(n_Y(s)h) = \psi((T,s))\mu_T(h)$ for all $h \in \mathrm{Sp}(W)(\mathbf{A})$. The Poincare series associated to μ_T is the automorphic function

$$P(h;\mu_T) = \sum_{\gamma \in N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{Q})} \mu_T(\gamma h),$$

if the sum converges absolutely.

In section 3, we will prove that for a particular special choice of archimedean test data $\phi_\infty \in S(\mathbf{X}(\mathbf{R}))$, the theta lift of holomorphic Siegel modular forms is a quaternionic modular form on $\mathrm{SO}(V)$. The proof follows the method of Oda [Oda78] and Niwa [Niw75], whereby one proves that the lifts of certain Poincare series on $\mathrm{Sp}(W)$ are quaternionic on $\mathrm{SO}(V)$, and deduces the general case from the fact that the Poincare series span the cuspidal Siegel modular forms [Kli90, Chapter 3]. We now write out the formal calculation.

Lemma 2.4.1. *Suppose the sum defining $P(h,\mu_T)$ converges absolutely to a cuspidal automorphic form on $\mathrm{Sp}(W)(\mathbf{A})$. Let $\phi \in S(\mathbf{X}(\mathbf{A}))$ and suppose moreover that*

$$\sum_{y_1,y_2 \in V} \int_{N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{A})} |\mu_T(h)||(\omega_\psi(g,h)\phi)(y_1e_1+y_2e_2)|\,dh \tag{6}$$

is finite. Then

$$\theta(P(\cdot;\mu_T);\phi)(g) = \sum_{y_1,y_2 \in V, S(y_1,y_2)=-T} \left(\int_{N_Y(\mathbf{A})\backslash \mathrm{Sp}(W)(\mathbf{A})} \mu_T(h)(\omega_{\psi,\mathbf{Y}}(g,h)\phi)(y_1e_1+y_2e_2)\,dh \right).$$

Proof. Because $P(h;\mu_T)$ is assumed cuspidal, the integral defining the theta lift converges absolutely. Computing formally,

$$\begin{aligned} \theta(P(\cdot;\mu_T);\phi)(g) &= \int_{[Sp(W)]} \theta(g,h;\phi)P(h;\mu_T)\,dh \\ &= \int_{N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{A})} \theta(g,h;\phi)\mu_T(h)\,dh \\ &= \int_{N_Y(\mathbf{A})\backslash \mathrm{Sp}(W)(\mathbf{A})} \mu_T(h) \left(\int_{[N_Y]} \psi((T,s))\theta(g,n_Y(s)h;\phi)\,ds \right) dh. \end{aligned}$$

The finiteness assumption of the lemma proves that the above formal manipulations are justified. Now

$$\theta(g,h;\phi) = \sum_{y_1,y_2 \in V} \omega_{\psi,\mathbf{Y}}(g,h)\phi(y_1e_1+y_2e_2)$$

and $\omega_{\psi,\mathbf{Y}}(1,n_Y(s))\phi_0(y_1e_1+y_2e_2) = \psi(S(y_1,y_2),s)\phi_0(y_1e_1+y_2e_2)$ for any $\phi_0 \in S(\mathbf{X})$. One obtains

$$\int_{[N_Y]} \psi((T,s))\theta(g,n_Y(s)h;\phi)\,ds = \sum_{y_1,y_2 \in V, S(y_1,y_2)=-T} \omega_{\psi,\mathbf{Y}}(g,h)\phi(y_1e_1+y_2e_2).$$

The lemma follows. $\square$

3. Special archimedean test data

This section is almost entirely archimedean. We define the special test data in $S(\mathbf{X}(\mathbf{R}))$ and prove that the theta lift of weight $N := w + 2 - n/2$ Siegel modular forms on $\mathrm{Sp}(W)$ are weight w modular forms on $\mathrm{SO}(V)$, in the sense of [Pol20a]. We also prove a certain archimedean result that is crucial to deducing the algebraicity of the Fourier coefficients of the lift $\theta(f;\phi)$ on $\mathrm{SO}(V)$.

3.1. **Test data.** In this subsection we define a certain $\mathbb{V}_w$-valued Schwartz function ϕ_∞ on $\mathbf{X}(\mathbf{R})$, and prove various properties of it that are crucial to what follows. Specifically, for $y_1, y_2 \in V$ we set

$$\phi_\infty(y_1, y_2) := \phi_\infty(y_1e_1 + y_2e_2) = p_K(y_1 \wedge y_2)^w e^{-2\pi(||y_1||^2+||y_2||^2)}.$$

Note that $p_K(y_1\wedge y_2) \in \mathbb{V}_1$, and we consider $p_K(y_1\wedge y_2)^w$ as an element of $\mathbb{V}_w$ via the K_V^0-equivariant map $Sym^w\mathbb{V}_1 \to \mathbb{V}_w$. Note also that there is a 2π in the exponential factor–as opposed to just a π–because we have a factor of $\frac{1}{2}$ in our definition of $||y||^2$. Immediately from the definition, one has

$$\phi_\infty((y_1e_1 + y_2e_2)gk) = k^{-1}\phi_\infty((y_1e_1 + y_2e_2)g)$$

for $g \in \mathrm{SO}(V)(\mathbf{R})$ and $k \in K_V^0$.

Denote by V_+ the four-dimensional subspace of V where ι acts by 1 and V_- is the n-dimensional subspace of V where ι acts by -1. For $y \in V = V_+ \oplus V_-$, write $y = y_+ + y_-$, where $y_+ \in V_+$ and $y_- \in V_-$. Then we have

$$\phi_\infty(y_1, y_2) = p_K(y_{1,+} \wedge y_{2,+})^w e^{-2\pi(||y_{1,+}||^2+||y_{2,+}||^2)}e^{-2\pi(||y_{1,-}||^2+||y_{2,-}||^2)}. \tag{7}$$

Recall the notion of a *pluriharmonic* function from [KV78, page 4].

Lemma 3.1.1. *The $\mathbb{V}_w$-valued polynomial $x_1e_1+x_2e_2 \mapsto p_K(x_1\wedge x_2)^w$ on $V_+\otimes X$ is pluriharmonic.*

We remark that if $m \in \mathrm{GL}_2(\mathbf{C})$, one has

$$x_1(e_1m) + x_2(e_2m) \mapsto \det(m)^w p_K(x_1 \wedge x_2)^w. \tag{8}$$

Proof. Note that the polynomial $x_1e_1 + x_2e_2 \mapsto p_K(x_1\wedge x_2)^w$ is of degree $2w$. As $2w$ is the smallest integer k for which $\mathbb{V}_w$ can occur in $Sym^k(X \otimes V_+)$, the pluriharmonicity follows from [KV78, Corollary 5.4]. ☐

Let $K_W \subseteq \mathrm{Sp}(W)(\mathbf{R})$ be the maximal compact subgroup that fixes the inner product on W for which e_1, e_2, f_1, f_2 is an orthonormal basis. Then $K_W \simeq U(2)$ via the map $\left(\begin{smallmatrix} A & B \\ -B & A\end{smallmatrix}\right) \mapsto A + iB$. Denote by $\mathcal{H}_2$ the Siegel upper half-space of degree two, and write $j : \mathrm{Sp}(W)(\mathbf{R}) \times \mathcal{H}_2 \to \mathbf{C}$ for the usual factor of automorphy $j(g,Z) = \det(cZ + d)$ for $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Then if $k \in K_W \simeq U(2)$, the representation $\det(\cdot)^N$ of $U(2)$ is realized as $k \mapsto j(k,i)^{-N}$.

Corollary 3.1.2. *For $k \in K_W$, one has $\omega_{\psi,\mathbf{Y}}(1,k)\phi_\infty(y_1,y_2) = j(k,i)^{-(w+2-n/2)}\phi_\infty(y_1,y_2)$.*

Proof. This follows from (7), (8) and the pluriharmonicity of Lemma 3.1.1. In more detail, in order to prove the corollary, it suffices to compute the differential of the action of K_W on $\phi_\infty(y_1,y_2)$. This action can be computed in a manner completely analogous to the proof of [LV80, Lemma 2.5.14 and Proposition 2.5.15], where the case of SL_2 and harmonic functions is treated. ☐

3.2. **Poincare series.** We now define the Poincare series. Classically, the level one holomorphic Poincare series of weight N (of exponential type) are defined as

$$P_N(Z;T) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,Z)^{-N}e^{2\pi i(T,\gamma Z)}.$$

Here T is a fixed positive-definite half-integral symmetric matrix, $\Gamma = \mathrm{Sp}_4(\mathbf{Z})$, $\Gamma_\infty = \{\gamma = \left(\begin{smallmatrix} 1 & V \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma\}$ and $N > 4$. The sum converges absolutely to a *cuspidal* holomorphic Siegel modular form of weight N. See [Kli90, Chapter III, page 90].

Adellically, and for general level, one proceeds as follows. Fix T a positive definite two-by-two rational symmetric matrix and an integer $N > 4$. Define $\mu^+_{T,\infty}$ on $\mathrm{Sp}(W)(\mathbf{R})$ as $\mu^+_{T,\infty}(h) = j(h,i)^{-N} e^{2\pi i (T, h\cdot i)}$. For p a finite prime, suppose that $\mu^+_{T,p}$ on $\mathrm{Sp}(W)(\mathbf{Q}_p)$ is a function supported on $N_Y(\mathbf{Q}_p)\,\mathrm{Sp}(W)(\mathbf{Z}_p)$ that satisfies $\mu^+_{T,p}(n_Y(s)h) = \psi_p((T,s))\mu^+_{T,p}(h)$ for all $h \in \mathrm{Sp}(W)(\mathbf{Q}_p)$ and all $n_Y(s) \in N_Y(\mathbf{Q}_p)$. Assume moreover that at all but finitely many primes, $\mu^+_{T,p}$ is the function $\mu^+_{T,p}(n_Y(s)k) = \psi_p((T,s))$ for $s \in Sym^2(\mathbf{Q}_p)$ and $k \in \mathrm{Sp}_4(\mathbf{Z}_p)$. With these assumptions, we define μ^+_T on $\mathrm{Sp}(W)(\mathbf{A})$ as $\mu^+_T(h) = \prod_v \mu^+_{T,v}(h_v)$. Then $\mu^+_T(n_Y(s)h) = \psi((T,s))\mu^+_T(h)$ for all $h \in \mathrm{Sp}(W)(\mathbf{A})$ and $s \in N_Y(\mathbf{A})$, and

$$P(h; \mu^+_T) = \sum_{\gamma \in N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{Q})} \mu^+_T(\gamma h)$$

is a Poincare series that corresponds to a cuspidal holomorpic Siegel modular form on $\mathrm{Sp}(W)$ of weight N. The cuspidality can again be deduced from the work and results of [Kli90, Chapter III], especially the boundedness result [Kli90, Chapter III, page 90, (ii)]. We now set $\mu_{-T}(h) = \overline{\mu^+_T(h)}$ and $P(h;\mu_{-T}) = \sum_{\gamma \in N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{Q})} \mu_{-T}(\gamma h)$, as in subsection 2.4. Then $\mu_{-T}(n_Y(s)h) = \psi(-(T,s))\mu_{-T}(h)$ and $P(h;\mu_{-T}) = \overline{P(h;\mu^+_T)}$ is the complex conjugate of a holomorphic Siegel modular form on weight N. We will compute the θ-lift of $P(h;\mu_{-T})$.

To do this, we compute the following integral. For $X \in \wedge^2 V \simeq \mathfrak{so}(V)$ such that $p_K(X) \neq 0$, define

$$A_w(X) = \frac{p_K(X)^w}{||p_K(X)||^{2w+1}}.$$

Here $||\cdot||$ is the K_V-invariant norm on $\mathfrak{so}(V)$ induced from the Cartan involution θ_ι, where $\theta_\iota(y_1 \wedge y_2) = \iota(y_1) \wedge \iota(y_2)$. The function A_w will play an important role in what follows. See also [Pol19, Section 6], [Pol20b, section 2.1] where the function A_w appears in the construction of degenerate Heisenberg Eisenstein series.

Proposition 3.2.1. *Suppose $N = w + 2 - n/2$ is the weight of the Poincare series and $y_1, y_2 \in V$ with $S(y_1,y_2) = T > 0$. Then $p_K(y_1 \wedge y_2) \neq 0$ and*

$$I_\infty(y_1,y_2;T) := \int_{N_Y(\mathbf{R})\backslash \mathrm{Sp}(W)(\mathbf{R})} \mu_{-T}(h)\omega_{\psi,\mathbf{Y}}(1,h)\phi_\infty(y_1e_1 + y_2e_2)\,dh$$

is equal to $A_w(y_1 \wedge y_2)$ up to a nonzero constant which is independent of y_1, y_2.

Proof. By construction, $\mu_{-T}(hk) = j(k,i)^N \mu_{-T}(h)$ for all $h \in Sp(W)(\mathbf{R})$ and $k \in K_W \simeq U(2)$. By Corollary 3.1.2, $\omega_{\psi,\mathbf{Y}}(1,hk)\phi_\infty(y_1,y_2) = j(k,i)^{-N}\omega_{\psi,\mathbf{Y}}(1,h)\phi_\infty(y_1,y_2)$. Thus applying the Iwasawa decomposition, we obtain

$$\begin{aligned} I_\infty(y_1,y_2;T) &= \int_{M_Y(\mathbf{R})} \delta_P^{-1}(h)\mu_{-T}(h)\omega_{\psi,\mathbf{Y}}(1,h)\phi_\infty(y_1,y_2)\,dh \\ &= p_K(y_1\wedge y_2)^w \int_{\mathrm{GL}_2(\mathbf{R})} |\det(m)|^{-3+w+N+\dim(V)/2} e^{-2\pi(T + R(y_1,y_2), m\,{}^tm)}\,dm. \end{aligned}$$

Here

$$R(y_1,y_2) = \frac{1}{2}\begin{pmatrix} (y_1,y_1)_r & (y_1,y_2)_r \\ (y_1,y_2)_r & (y_2,y_2)_r \end{pmatrix}$$

and $(y,y')_r := (y,\iota(y'))$ is the positive definite majorant on V. We have also used that $4|n$ so that $w+N$ is even and thus $\det(m)^{w+N} = |\det(m)|^{w+N}$.

Note that $w+N+\dim(V)/2-3=2w+1$. Thus

$$I_\infty(y_1,y_2;T)=\frac{p_K(y_1\wedge y_2)^w}{|\det(T+R(y_1,y_2))|^{w+1/2}}\left(\int_{\mathrm{GL}_2(\mathbf{R})}|\det(m)|^{2w+1}e^{-2\pi\,\mathrm{tr}(m\,{}^tm)}\,dm\right).$$

We remark that this latter integral is a so-called Siegel integral. The proposition thus follows from the following lemma. □

Lemma 3.2.2. *Suppose $y_1,y_2\in V$ and $S(y_1,y_2)>0$. Then $2||p_K(y_1\wedge y_2)||^2=\det(S(y_1,y_2)+R(y_1,y_2))$, and in particular, $p_K(y_1\wedge y_2)\neq 0$.*

Proof. One has $\frac{1}{2}(y,y')+\frac{1}{2}(y,\iota(y'))=(y_+,y'_+)$. Thus

$$\det(S(y_1,y_2)+R(y_1,y_2))=\det\left(\left(\begin{array}{cc}(y_{1,+},y_{1,+}) & (y_{1,+},y_{2,+})\\ (y_{1,+},y_{2,+}) & (y_{2,+},y_{2,+})\end{array}\right)\right).$$

This determinant is

$$||y_{1,+}\wedge y_{2,+}||^2=2||p_K(y_{1,+}\wedge y_{2,+})||^2=2||p_K(y_1\wedge y_2)||^2$$

giving the lemma. □

3.3. **The function A_w.** In this subsection, we prove that the function $g\mapsto A_w(y_1\wedge y_2 g)$ is quaternionic, i.e., that it is annihilated by D_w. This is the key step in showing that the theta lifts $\theta(f;\phi)$ of cuspidal Siegel modular forms f on $\mathrm{Sp}(W)$ are quaternionic modular forms on $\mathrm{SO}(V)$.

We now recall the differential operator D_w [Pol20a, Definition 1.1.1] that defines modular forms of weight w. Thus suppose G is a group of quaternionic type with Cartan involution Θ, maximal compact subgroup K surjecting onto $\mathrm{SU}(2)/\mu_2$, and Cartan decomposition $Lie(G)\otimes\mathbf{C}=\mathfrak{k}\oplus\mathfrak{p}$, in obvious notation. Then $\mathfrak{p}=V_2\boxtimes W$ as representation of K, where V_2 denotes the standard representation of $\mathrm{SU}(2)$ and W denotes a particular symplectic vector space.

Denote by $\{X_i\}_i$ a basis of $\mathfrak{p}$ and $\{X_i^\vee\}_i$ the dual basis of the linear dual $\mathfrak{p}^\vee$. Additionally, note that one has a K-equivariant surjection $pr_-:\mathbb{V}_w\otimes\mathfrak{p}^\vee\to Sym^{2w-1}(V_2)\boxtimes W$. The operator D_w is defined as

$$D_wF(g)=\sum_i pr_-\left(X_iF(g)\otimes X_i^\vee\right).$$

Here X_iF denotes the right-regular action and the pr_- is applied to the "coefficients" $\mathbb{V}_w\otimes\mathfrak{p}^\vee$.

Now, for $Z=y_1\wedge y_2$ with $S(y_1,y_2)>0$, define $B_{w,Z}(g):\mathrm{SO}(V)(\mathbf{R})\to\mathbb{V}_w$ as $B_{w,Z}(g)=A_w(Zg)$. It is clear that $B_{w,Z}(gk)=k^{-1}B_{w,Z}(g)$ for all $g\in\mathrm{SO}(V)(\mathbf{R})$ and $k\in K_V^0$. The following theorem is crucial to all that follows.

Theorem 3.3.1. *Suppose $w\geq 2$. With notation as above, the function $B_{w,Z}(g)$ is quaternionic, i.e., $D_wB_{w,Z}(g)\equiv 0$.*

Proof. We begin with a simple lemma. For $L\in\mathfrak{su}_2\otimes\mathbf{C}\simeq\mathfrak{sl}_2(\mathbf{C})\simeq Sym^2(V_2)$, denote by $(L,L)\in\mathbf{C}$ the $\mathrm{SL}_2(\mathbf{C})$-invariant quadratic form on L. Thus, if $L\in\mathfrak{su}_2$, then $(L,L)=||L||^2$.

Lemma 3.3.2. *Suppose $X\in\mathfrak{so}(V)$, and $X\cdot B_{w,Z}(g)=\frac{d}{dt}\left(B_{w,Z}(ge^{tX})\right)|_{t=0}$ denotes the right regular action. Set $Z'=Zg$. Then*

$$X\cdot B_{w,Z}(g)=$$
$$\frac{p_K(Z')^{w-1}}{||p_K(Z)'||^{2w+3}}\left(wp_K([Z',X])(p_K(Z'),p_K(Z'))-(2w+1)p_K(Z')(p_K([Z',X]),p_K(Z'))\right).$$

Proof. This follows immediately from the definitions. Note that the quantity outside the parentheses is an element of $\mathbb{V}_{w-1}$, the quantity inside the parenthesis is an element of $\mathbb{V}$, and the product is considered as an element of $\mathbb{V}_w$. □

To continue computing the derivative $D_w B_{w,Z}(g)$, we introduce more convenient coordinates on $V\otimes \mathbf{C}$, as follows. First, $V_+\otimes \mathbf{C}$ is a non-degenerate split quadratic space of dimension four, so it it can be identified with $(M_2 = V_2^{(1)}\otimes V_2^{(2)}, \det)$ the space of 2×2 matrices M_2 with determinant as quadratic form. Here $V_2^{(1)}$, $V_2^{(2)}$ are two copies of the two-dimensional representation of SL_2. We denote by $e^{(j)}, f^{(j)}$ a fixed symplectic basis of $V_2^{(j)}$ for $j=1,2$, and we identify M_2 with $V_2^{(1)}\otimes V_2^{(2)}$ as in [Pol20a, section A.3]. Specifically,

$$ae^{(1)}\otimes e^{(2)} + b(-f^{(1)}\otimes e^{(2)}) + ce^{(1)}\otimes f^{(2)} + df^{(1)}\otimes f^{(2)} \mapsto \begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

In terms of the elements u_1, u_2, v_1^+, v_2^+ defined in paragraph 2.2.2, up to scaling, the map $V^+\otimes\mathbf{C}\to V_2^{(1)}\otimes V_2^{(2)}$ is given by

- $u_1 - iu_2 \mapsto e^{(1)}\otimes e^{(2)}$
- $u_1 + iu_2 \mapsto f^{(1)}\otimes f^{(2)}$
- $v_1^+ - iv_2^+ \mapsto e^{(1)}\otimes f^{(2)}$
- $v_1^+ + iv_2^+ \mapsto -f^{(1)}\otimes e^{(2)}$.

Now we choose an isometry

$$V\otimes\mathbf{C} \simeq V_2^{(1)}\otimes V_2^{(2)}\oplus V' = \left(V_2^{(1)}\otimes e^{(2)}\right)\oplus V'\oplus\left(V_2^{(1)}\otimes f^{(2)}\right), \tag{9}$$

so that $V_+\otimes\mathbf{C}$ maps over to $V_2^{(1)}\otimes V_2^{(2)}$ and $V_-\otimes\mathbf{C}$ maps over to V'.

From now on, we write $e=e^{(2)}$ and $f=f^{(2)}$. Fix

$$w_1 = w_1^e e + w_1^v + w_1^f f \text{ and } w_2 = w_2^e e + w_2^v + w_2^f f$$

in $V\otimes\mathbf{C}$, written in terms of the decomposition (9), so that $w_j^e, w_j^f\in V_2^{(1)}$ and $w_j^v\in V'$ for $j=1,2$. Set

$$p = w_1^e w_2^f - w_2^e w_1^f = p_K(w_1\wedge w_2),$$

considered as an element of $\mathbb{V}_1 = Sym^2(V_2)$. If $X\in\wedge^2(V\otimes\mathbf{C})$, write

$$X(p) := p_K([w_1\wedge w_2, X])$$

for shorthand.

Set $W_J = V_2^{(2)}\otimes V'$. Denote by $\{X_\gamma\}_\gamma$ a basis of $\mathfrak{p}\otimes\mathbf{C} = V_2^{(1)}\otimes W_J$ and $\{X_\gamma^\vee\}_\gamma$ the dual basis of $\mathfrak{p}^\vee$. With this notation,

$$X_\gamma\cdot B_{w,Z}(g) = \frac{p^{w-1}}{||p||^{2w+3}}\left(w(p,p)X_\gamma(p) - (2w+1)(X_\gamma(p),p)p\right).$$

Contracting with $X_\gamma^\vee$, one obtains $\frac{1}{||p||^{2w+3}}$ times

$$wp^{w-2}\left((w-1)(p,p)\langle p, X_\gamma^\vee\rangle X_\gamma(p) + (p,p)\langle X_\gamma(p), X_\gamma^\vee\rangle p - (2w+1)(X_\gamma(p),p)\langle p, X_\gamma^\vee\rangle p\right).$$

Theorem 3.3.1 thus follows from the following proposition. $\square$

Proposition 3.3.3. *Let the notation be as above. Then*

$$\sum_\gamma (w-1)(p,p)\langle p, X_\gamma^\vee\rangle X_\gamma(p) + (p,p)\langle X_\gamma(p), X_\gamma^\vee\rangle p - (2w+1)(X_\gamma(p),p)\langle p, X_\gamma^\vee\rangle p \tag{10}$$

is 0 *as an element of* $Sym^3(V_2)\otimes W$.

Proof. Let $\{v_\alpha\}_\alpha$ be a basis of V_2 and $\{w_\beta\}_\beta$ a basis of W_J so that $X_\gamma = v_\alpha \otimes w_\beta$ is a basis of $\mathfrak{p}\otimes \mathbf{C} = V_2 \otimes W_J$. Of course, V_2 is two-dimensional, so the sum over α has two terms.

To check the vanishing in (10), we can fix β and sum over α. Then the required vanishing follows from the following three lemmas.

For $w_1, w_2 \in V$, set

$$\begin{aligned}(w_1 \wedge w_2)_{\mathfrak{p}} &= w_1^- \wedge w_2^+ + w_1^+ \wedge w_2^- \\ &= w_1^v \wedge (w_2^e e + w_2^f f) + (w_1^e e + w_1^f f) \wedge w_2^v \\ &= w_1^e \otimes (e w_2^v) + w_1^f \otimes (f w_2^v) - w_2^e \otimes (e w_1^v) - w_2^f \otimes (f w_1^v)\end{aligned}$$

This is an element of $\mathfrak{p} = V_2^{(1)} \otimes (V_2^{(2)} \otimes V')$.

The $\langle\, , \rangle_W$ below denote W_J-contractions. Thus $\langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W$ denotes a W_J-contraction between w_β, an element of W_J, and $(w_1 \wedge w_2)_{\mathfrak{p}}$, an element of $V_2^{(1)} \otimes W_J$, so that $\langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W \in V_2^{(1)}$.

Note that if $X_\gamma = v_\alpha w_\beta$, then

$$\begin{aligned}X_\gamma(p) &= p_K([w_1 \wedge w_2, v_\alpha \wedge w_\beta]) \\ &= p_K([(w_1^e e + w_1^f f) \wedge w_2^v + w_1^v \wedge (w_2^e e + w_2^f f), v_\alpha w_\beta]) \\ &= \langle w_\beta, e w_2^v\rangle_W w_1^e v_\alpha - \langle w_\beta, e w_1^v\rangle_W w_2^e v_\alpha + \langle w_\beta, f w_2^v\rangle_W w_1^f v_\alpha - \langle w_\beta, f w_1^v\rangle_W w_2^f v_\alpha.\end{aligned}$$

Thus $X_\gamma(p) = \langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W v_\alpha$. For ease of notation, set $v_p^\beta = \langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W$, which is an element of $V_2 = V_2^{(1)}$.

Lemma 3.3.4. *Let the notation be as above. Then*

$$\sum_\alpha \langle X_\gamma(p), X_\gamma^\vee\rangle = 3\langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W w_\beta^\vee.$$

Proof. Let α_1, α_2 be our basis of $V_2 = V_2^{(1)}$. Then

$$\begin{aligned}\langle X_\gamma(p), X_\gamma^\vee\rangle &= \langle v_{p,\beta} v_\alpha, v_\alpha^\vee w_\beta^\vee\rangle \\ &= (v_{p,\beta} + \langle v_{p,\beta}, v_\alpha^\vee\rangle) w_\beta.\end{aligned}$$

Taking $\alpha = \alpha_1, \alpha_2$ and summing up gives $3 v_p^\beta w_\beta^\vee$, which is the statement of the lemma. $\square$

Lemma 3.3.5. *Let the notation be as above. Then*

$$\sum_\alpha \langle p, X_\gamma^\vee\rangle X_\gamma(p) = 2\langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W p w_\beta^\vee.$$

Proof. This follows immediately from the fact that $\sum_j \langle p, v_{\alpha_j}^\vee\rangle v_{\alpha_j} = 2p$. $\square$

Lemma 3.3.6. *Let the notation be as above. Then*

$$\sum_\alpha (p, X_\gamma(p))\langle p, X_\gamma^\vee\rangle = (p,p)\langle w_\beta, (w_1 \wedge w_2)_{\mathfrak{p}}\rangle_W w_\beta^\vee.$$

Proof. We have

$$(p, X_\gamma(p))\langle p, X_\gamma^\vee\rangle = (p, v_p^\beta v_\alpha)\langle p, v_\alpha^\vee\rangle w_\beta^\vee.$$

We thus must check the identity $\sum_\alpha (p, v_p^\beta v_\alpha)\langle p, v_\alpha^\vee\rangle = (p,p) v_p^\beta$. To do this, first note that for any $u, v, X \in V_2$, one has

$$\langle u, X\rangle v - \langle v, X\rangle u = \langle u, v\rangle X. \tag{11}$$

Now, we claim

$$\sum_\alpha (p, v_p^\beta v_\alpha) v_\alpha^\vee = -\langle p, v_p^\beta \rangle.$$

To check this, we may assume $p = \ell_1 \ell_2$ is a product of two linear factors, and then the left-hand side is

$$\begin{aligned}\sum_\alpha (p, v_p^\beta v_\alpha) v_\alpha^\vee &= (\ell_1\ell_2, v_p^\beta e) f - (\ell_1 \ell_2, v_p^\beta f) e \\ &= (\langle \ell_1, v_p^\beta\rangle \langle \ell_2, e\rangle + \langle \ell_1, e\rangle\langle \ell_2, v_p^\beta\rangle) f - (\langle \ell_1, v_p^\beta\rangle\langle \ell_2, f\rangle + \langle \ell_1, f\rangle \langle \ell_2, v_p^\beta\rangle) e \\ &= \langle \ell_1, v_p^\beta\rangle(-\ell_2) + \langle \ell_2, v_p^\beta\rangle(-\ell_1) \\ &= -\langle p, v_p^\beta\rangle.\end{aligned}$$

Thus, to conclude, we must check that $\langle p, \langle v_p^\beta, p\rangle\rangle = (p,p)v_p^\beta$. For this, linearizing, it suffices to check that

$$\langle \ell_1\ell_2, \langle v, \ell_1'\ell_2'\rangle\rangle + \langle \ell_1'\ell_2', \langle v, \ell_1\ell_2\rangle\rangle = 2(\ell_1\ell_2, \ell_1'\ell_2')v$$

for all $\ell_1, \ell_2, \ell_1', \ell_2', v \in V$. Expanding the left-hand side, one checks this using the identity (11) four times. This proves the proposition, and with it, Theorem 3.3.1. □

□

Corollary 3.3.7. *Suppose f is weight $N = w + 2 - n/2$ holomorphic Siegel modular cusp form, with $w \geq 2\dim(V)$ even. Denote by $\theta(f;\phi)$ the theta lift of $\overline{f}$ to $\mathrm{SO}(V)$, where $\phi = \phi_f \otimes \phi_\infty$. Then $\theta(f;\phi)$ is a quaternionic modular form on $\mathrm{SO}(V)$ of weight w.*

Proof. As explained in subsection 2.4, it suffices to check that the theta lift of Poincare series are quaternionic, because the Poincare series span the space of cusp forms. Now, the finiteness lemma below, Lemma 3.4.1, implies that the sum (6) is finite. Thus, by Lemma 2.4.1 and Proposition 3.2.1, $\theta(\cdot, \mu_{-T})(g)$ is a nonzero constant times a sum

$$\sum_{y_1, y_2 \in V, S(y_1,y_2) = -T} \widetilde{\phi}_f((y_1 e_1 + y_2 e_2)g) A_w((y_1 \wedge y_2)g),$$

where

$$\widetilde{\phi}_f(y_1' e_1 + y_2' e_2) = \int_{N_Y(\mathbf{A}_f)\backslash \mathrm{Sp}(W)(\mathbf{A}_f)} \mu_T(h) \omega_{\psi, \mathbf{Y}}(1,h)\phi(y_1'e_1 + y_2' e_2)\, dh.$$

By Theorem 3.3.1, each term is annihilated by D_w. Since the sum converges absolutely and uniformly on compact subsets, the corollary follows. □

3.4. **Finiteness lemma.** We require the following lemma, which was used in the proof of Corollary 3.3.7 above.

Lemma 3.4.1. *Suppose $\phi \in S(\mathbf{X}(\mathbf{A}))$, and $\mu_T(h)$ is as defined in subsection 3.2. Then for $w \geq 2\dim V$ the sum*

$$\sum_{y_1, y_2 \in V(\mathbf{Q})} \int_{N_Y(\mathbf{Q})\backslash \mathrm{Sp}(W)(\mathbf{A})} |\mu_T(h)||\omega_\psi(g,h)\phi(y_1 e_1 + y_2 e_2)|\, dh$$

is finite.

Proof. The quantity of the lemma is bounded by a constant times

$$\sum_{y_1, y_2 \in \Lambda'} ||p_K(y_1 \wedge y_2)||^w \int_{\mathrm{GL}_2(\mathbf{R})} |\det(m)|^{2w+1} e^{-2\pi(T + R(y_1,y_2), m\, {}^t m)}\, dm$$

for some lattice Λ' in $V(\mathbf{R})$. This follows from the same manipulations as in the proof of Proposition 3.2.1. Thus we must check the convergence of

$$\sum_{y_1,y_2\in\Lambda'} \frac{||p_K(y_1\wedge y_2)||^w}{|\det(T+R(y_1,y_2))|^{w+1/2}}.$$

Note that T is fixed, and both T and $R(y_1,y_2)$ are positive definite. Moreover, $||p_K(y_1\wedge y_2)|| \leq C_0 \det(R(y_1,y_2))^{1/2}$ for some constant C_0. Now,

$$\frac{|\det(R(y_1,y_2))|^{w/2}}{|\det(T+R(y_1,y_2))|^{w+1/2}} \leq \frac{|\det(T+R(y_1,y_2))|^{w/2}}{|\det(T+R(y_1,y_2))|^{w+1/2}} = \frac{1}{|\det(T+R(y_1,y_2))|^{(w+1)/2}}.$$

Moreover, by making a linear transformation of $V\oplus V$, we can assume without loss of generality that $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\det(T+R(y_1,y_2)) = (1+||y_1||^2)(1+||y_2||^2) - \frac{1}{4}(y_1,y_2)_r^2 \geq 1+||y_1||^2+||y_2||^2$$

by Cauchy-Schwartz. Thus

$$\sum_{y_1,y_2\in\Lambda'} \frac{||p_K(y_1\wedge y_2)||^w}{|\det(T+R(y_1,y_2))|^{w+1/2}} \leq C' \sum_{y\in\Lambda''} \frac{1}{(1+||y||^2)^{(w+1)/2}}$$

for some constant $C'>0$ and lattice $\Lambda''\subseteq V\oplus V$. This latter sum converges if $(w+1) > 2\dim V$, giving the lemma. □

3.5. **The Fourier transform of A_w.** In this subsection, we analyze a certain Fourier transform integral of the function $B_{w,Z}(g)$. The result is essential to proving that the θ-lifts have algebraic Fourier coefficients.

For $y_1, y_2 \in V_0$, what we need to compute is

$$\int_{N_{y_1,y_2}(\mathbf{R})} A_w(y_1\wedge y_2 ng)\chi_{y_1,y_2}^{-1}(n)\,dn.$$

Here $N_{y_1,y_2} = (\mathrm{Stab}(y_1,y_2)\cap N_U)\backslash N_U$ is 4-dimensional and abelian. As a function of g, the above integral is quaternionic, i.e it is annihilated by D_w. Thus by the multiplicity one result of [Pol20a], we know that, on $\mathrm{SO}(V)(\mathbf{R})^0$, the integral is $C_{y_1,y_2,w}W_{\chi_{y_1,y_2}}(g)$ for some constant $C_{y_1,y_2,w}$ depending on y_1,y_2,w. Here $W_{\chi_{y_1,y_2}}$ denotes the function written as $\mathcal{W}^{\chi_{y_1,y_2}}$ in [Pol20a, Theorem 1.2.1]. We extend $W_{\chi_{y_1,y_2}}$ to $\mathrm{SO}(V)(\mathbf{R})$ by the formula $W_{\chi_{y_1,y_2}}(\iota g) = (-1)^w W_{\chi_{y_2,y_1}}(g)$ if $\iota = \mathrm{diag}(\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right), 1, \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right))$. It is in fact true that, with this definition, $W_{\chi_{y_1,y_2}}$ is nonzero on precisely one connected component of $\mathrm{SO}(V)(\mathbf{R})$. Thus, assuming y_1,y_2 are ordered so that $W_{\chi_{y_1,y_2}}$ is nonzero on the connected component of the identity, we must compute the above integral for $g=1$ so that we can obtain $C_{y_1,y_2,w}$.

To pin down the normalizations, we proceed to compute the following:

$$J(y_1,y_2;w) := \int_{M_2(\mathbf{R})} A_w((y_1+r_{11}b_{-1}+r_{12}b_{-2})\wedge(y_2+r_{21}b_{-1}+r_{22}b_{-2}))\psi^{-1}(\mathrm{tr}(r))\,dr.$$

Here again $y_1,y_2\in V_0$ and $r = \left(\begin{smallmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{smallmatrix}\right) \in M_2(\mathbf{R})$. Then it is clear that $J(y_1,y_2;w) = J(y_1^+,y_2^+;w)$. As V_0^+ is two-dimensional, denote by v_1^+, v_2^+ an orthonormal basis. Thus $(y_1^+,y_2^+) = (v_1^+,v_2^+)m$ for a unique $m\in \mathrm{GL}_2(\mathbf{R})$, and we'd like to compute $J'(m;w) := J((v_1^+,v_2^+)m;w)$ as a function of m.

We have $J'(m;w)$

$$= \int_{M_2(\mathbf{R})} A_w((m_{11}v_1^+ + m_{21}v_2^+ + r_{11}b_{-1} + r_{21}b_{-2}) \wedge (m_{12}v_1^+ + m_{22}v_2^+ + r_{12}b_{-1} + r_{22}b_{-2})) \times \psi^{-1}(\mathrm{tr}(r))\, dr$$

$$= \frac{\det(m)^w}{|\det(m)|^{2w+1}} |\det(m)|^2 \int_{M_2(\mathbf{R})} A_w((v_1^+ + r_{11}b_{-1} + r_{21}b_{-2}) \wedge (v_2^+ + r_{12}b_{-1} + r_{22}b_{-2})) \times \psi^{-1}(\mathrm{tr}(rm))\, dr. \tag{12}$$

Here we have made the variable change $r \mapsto rm$ and the $|\det(m)|^2$ comes from the Jacobian for this change of variables. Note that from the final expression one obtains that $J'(m;w)$ is not identically 0 as a function of m for $w >> 0$, because the last integral is a Fourier transform integral. Specifically, this integral converges absolutely for $w \geq 8$:

Lemma 3.5.1. *Suppose $w \geq 8$. Then the integral*

$$\int_{M_2(\mathbf{R})} A_w((v_1^+ + r_{11}b_{-1} + r_{21}b_{-2}) \wedge (v_2^+ + r_{12}b_{-1} + r_{22}b_{-2})g)\psi^{-1}(\mathrm{tr}(rm))\, dr \tag{13}$$

converges absolutely and defines a quaternionic function of g.

Proof. As $A_w(Z) = \frac{p_K(Z)^w}{||p_K(Z)||^{2w+1}}$, we have

$$||A_w(Z)|| = \frac{1}{||p_K(Z)||^{w+1}}.$$

Because $||p_K(Zg)|| \geq C_g ||p_K(Z)||$ for some constant $C_g > 0$ depending on g (but not on X), the absolute convergence of the integral for general g follows from the case of $g = 1$.

Now, considering the case $g = 1$, we have

$$||A_w((v_1^+ + r_{11}b_{-1} + r_{21}b_{-2}) \wedge (v_2^+ + r_{12}b_{-1} + r_{22}b_{-2}))|| = \frac{1}{\det((\begin{smallmatrix}1&0\\0&1\end{smallmatrix}) + m\,{}^tm)^{(w+1)/2}}.$$

But

$$\det(1_2 + m\,{}^tm) = 1 + \mathrm{tr}(m\,{}^tm) + \det(m)^2 \geq 1 + m_{11}^2 + m_{12}^2 + m_{21}^2 + m_{22}^2$$

and thus

$$||A_w((v_1^+ + r_{11}b_{-1} + r_{21}b_{-2}) \wedge (v_2^+ + r_{12}b_{-1} + r_{22}b_{-2}))|| \leq \frac{1}{(1 + m_{11}^2 + m_{12}^2 + m_{21}^2 + m_{22}^2)^{(w+1)/2}}.$$

The absolute converge of the integral for $w \geq 8$ follows.

To see that the function of g defined by (13) is quaternionic, note that by Lemma 3.3.2, if $X \in \mathfrak{so}(V)$, then $||X \cdot B_Z(g)|| \leq C_{X,g}||p_K(Z)||^{-(w+1)}$, for some constant $C_{X,g} > 0$ depending on X and g. Thus, as above,

$$\int_{M_2(\mathbf{R})} ||X A_w((v_1^+ + r_{11}b_{-1} + r_{21}b_{-2}) \wedge (v_2^+ + r_{12}b_{-1} + r_{22}b_{-2})g)||\, dr$$

converges absolutely if $w \geq 8$. The quaternionicity of (13) follows. $\square$

Set $M = \mathrm{diag}(m, 1, {}^tm^{-1}) \in \mathrm{SO}(V)$. For ease of notation, denote

$$n(r) = n\left(v_1^+(r_{11}b_{-1} + r_{21}b_{-2}) + v_2^+(r_{12}b_{-1} + r_{22}b_{-2})\right)$$

so that $v_1^+ n(r) = v_1^+ + r_{11}b_{-1} + r_{21}b_{-2}$ and $v_2^+ n(r) = v_2^+ + r_{12}b_{-1} + r_{22}b_{-2}$. Then

$$\int_{M_2(\mathbf{R})} A_w(v_1^+ \wedge v_2^+ n(r)M)\psi^{-1}(\mathrm{tr}(r))\,dr = \int_{M_2(\mathbf{R})} A_w(v_1^+ \wedge v_2^+ n(m^{-1}r))\psi^{-1}(\mathrm{tr}(r))\,dr$$
$$= |\det(m)|^2 \int_{M_2(\mathbf{R})} A_w(v_1^+ \wedge v_2^+ n(r))\psi^{-1}(\mathrm{tr}(mr))\,dr. \tag{14}$$

Here the first equality is because

$$(b_{-1}, b_{-2})m^{-1}r = \left({}^t m^{-1}\begin{pmatrix} b_{-1} \\ b_{-2}\end{pmatrix}\right)^t r.$$

Combining (12) and (14), we obtain

$$J'(m;w) = \frac{\det(m)^w}{|\det(m)|^{2w+1}}\int_{M_2(\mathbf{R})} A_w(v_1^+ \wedge v_2^+ n(r)M)\psi^{-1}(\mathrm{tr}(r))\,dr.$$

We claim that $J'(m;w) = C_{v_1^+,v_2^+} W_{\chi_{y_1,y_2}}(1)$. Indeed, we have

$$\begin{aligned} J'(m;w) &= C_{v_1^+,v_2^+,w}\frac{\det(m)^w}{|\det(m)|^{2w+1}} W_{\chi_{v_1^+,v_2^+}}(M) \\ &= \left(C_{v_1^+,v_2^+,w}\frac{\det(m)^w}{|\det(m)|^{2w+1}}\right)\nu(M)^m|\nu(M)|W_{\chi_{(v_1^+,v_2^+)m}}(1) \\ &= C_{v_1^+,v_2^+,w}W_{\chi_{y_1^+,y_2^+}}(1) \\ &= C_{v_1^+,v_2^+,w}W_{\chi_{y_1,y_2}}(1) \end{aligned}$$

where the second equality follows from the explicit formula Theorem 1.2.1 of [Pol20a], the third equality from the fact that $\nu(M) = \det(m)$, and the fourth equality again follows from Theorem 1.2.1 of *loc cit*, because we are evaluating $W_{\chi_{y_1^+,y_2^+}}$ at the element $g = 1$.

Putting it all together, we have proved the following. Set

$$J(y_1,y_2,g;w) = \int_{M_2(\mathbf{R})} A_w\left(((y_1 + r_{11}b_{-1} + r_{12}b_{-2}) \wedge (y_2 + r_{21}b_{-1} + r_{22}b_{-2}))\,g\right)\psi^{-1}(\mathrm{tr}(r))\,dr.$$

Proposition 3.5.2. *For $w \geq 8$, there is a nonzero constant $C_{v_1^+,v_2^+,w}$ (independent of y_1, y_2, g) so that $J(y_1,y_2,g;w) = C_{v_1^+,v_2^+,w}W_{\chi_{y_1,y_2}}(g)$.*

4. Modular forms

In this section we put together the work of the previous sections to obtain global results. In particular, we prove Corollary 4.2.4, which implies Theorem 1.0.2 of the introduction.

Suppose f is a cuspidal automorphic form on $\mathrm{Sp}(W)$, corresponding to a holomorphic Siegel modular form of weight N. For symmetric positive definite $T \in M_2(\mathbf{Q})$, define the function $a_f(T)(h)$, for $h_f \in \mathrm{Sp}(W)(\mathbf{A}_f)$, by the equality

$$f(h_f h_\infty) = \sum_{T>0} a_f(T)(h_f)j(h_\infty, i)^{-N}e^{2\pi i(T, h_\infty\cdot i)}.$$

Proposition 4.0.1. *Suppose as above that f is a cuspidal automorphic form on* $\mathrm{Sp}(W)$ *corresponding to a holomorphic Siegel modular form of weight $N = w + 2 - n/2$, with $w \geq 2\dim(V)$. Suppose $\phi' \in S(\mathbf{X}(\mathbf{A}_f))$, set $\phi = F(\phi') \in S(\mathbf{X}_U(\mathbf{A}_f))$ the partial Fourier transform, and denote*

$\theta(f,\phi'\otimes\phi_\infty)$ *the theta-lift of* $\overline{f}$ *using* ϕ' *and the special archimedean test function* ϕ_∞*. Then for* $g\in \mathrm{SO}(V)(\mathbf{R})$ *we have the following Fourier expansion:*

$$\theta(f,\phi'\otimes\phi_\infty)_Z(g)=\phi(f,\phi'\otimes\phi_\infty)_{deg}(g)+C_{v_1^+,v_2^+,w}\sum_{y_1,y_2\in V_0(\mathbf{Q}):S(y_1,y_2)>0}a(y_1,y_2;\phi;f)W_{\chi_{-y_1,-y_2}}(g)$$

with notation as follows: $\theta(f,\phi'\otimes\phi_\infty)_{deg}$ *denotes the sum of all the degenerate terms in the Fourier expansion of* $\theta(f,\phi'\otimes\phi_\infty)_Z$*, i.e., the sum of the Fourier terms* $\theta(f,\phi'\otimes\phi_\infty)_{\chi_{y_1,y_2}}$ *with* $y_1,y_2\in V_0$ *with* $\dim\mathrm{Span}\{y_1,y_2\}\leq 1$*; and*

(15)

$$a(y_1,y_2;\phi;f)=\int_{N_Y(\mathbf{A}_f)\backslash \mathrm{Sp}(W)(\mathbf{A}_f)}\omega_{\psi,\mathbf{Y}_U}(1,h)\phi(b_1f_1+b_2f_2+y_1e_1+y_2e_2)\overline{a_f(S(y_1,y_2))(h)}\,dh.$$

Proof. First, by Proposition 2.1.2, we can compute the theta-lift using the partial Fourier transform $F(\phi'\otimes\phi_\infty)$. Then from Proposition 2.3.1 we have

$$\theta(f,\phi\otimes F(\phi_\infty))_{\chi_{-y_1,-y_2}}(1)=\int_{N_Y(\mathbf{A})\backslash \mathrm{Sp}(W)(\mathbf{A})}\omega_{\psi,\mathbf{Y}_U}(1,h)\phi(z_{y_1,y_2})\overline{f}_{-S(y_1,y_2)}(h)\,dh.$$

This integral factors into an archimedean and finite adelic part. The finite adelic part gives the right-hand side of (15). By Corollary 3.1.2 and the definition of the action $\omega_{\psi,\mathbf{Y}_U}$ on $S(\mathbf{X}_U)$, one has $\omega_{\psi,Y_U}(1,k)F(\phi_\infty)=j(k,i)^{-N}F(\phi_\infty)$ for $k\in K_{\mathrm{Sp}(W),\infty}\simeq U(2)$. Applying Iwasawa again, the archimedean part gives

$$D(y_1,y_2):=\int_{M_Y(\mathbf{R})}\delta_P(h)^{-1}\mu_{-S(y_1,y_2)}(h)\omega_{\psi,\mathbf{Y}_U}(1,h)F(\phi_\infty)(z_{y_1,y_2})\,dh.$$

By Lemma 2.3.3, this is

$$D(y_1,y_2)=\int_{M_Y(\mathbf{R})}\delta_P(h)^{-1}\mu_{-S(y_1,y_2)}(h)|\det(h)|_X^{n/2+2}\times\left(\int_{X^2(\mathbf{R})}\psi(\langle z_1,f_1\rangle+\langle z_2,f_2\rangle)\phi_\infty((y_1e_1+y_2e_2+b_{-1}z_1+b_{-2}z_2)h)\,dz_1\,dz_2\right)dh.$$

Now, because $||y+b||^2=||y||^2+||b||^2$ for $y\in V_0$ and $b\in U^\vee$, it is easy to see that the double integral $D(y_1,y_2)$ converges absolutely. Thus changing the order of integration, one obtains

$$D(y_1,y_2)=\int_{M_2(\mathbf{R})}A_w((y_1+r_{11}b_{-1}+r_{12}b_{-2})\wedge(y_2+r_{21}b_{-1}+r_{22}b_{-2}))\psi(\mathrm{tr}(r))\,dr$$

by Proposition 3.2.1. But now, by Proposition 3.5.2, this is $C_{v_1^+,v_2^+,w}W_{\chi_{-y_1,-y_2}}(1)$. The proposition follows because by Corollary 3.3.7, the theta-lift is a quaternionic modular form. □

4.1. **Level one.** Putting everything together, we have the following theorem which describes the Fourier coefficients more precisely in the level one setting.

Theorem 4.1.1. *Suppose that* $F(Z)=\sum_{T>0}a_F(T)e^{2\pi i(T,Z)}$ *is a classical level one cuspidal Siegel modular form on* Sp_4 *of weight* $N=w+2-n/2$*, and assume that* $w\geq 2\dim(V)$ *is even. Denote by* f *the automorphic function on* $\mathrm{Sp}_4(\mathbf{A})$ *associated to* F *so that* $f(g)=j(g,i)^{-N}F(g\cdot i)$ *if* $g\in\mathrm{Sp}_4(\mathbf{R})$ *and let* $\theta(f)$ *be the theta-lift of the complex conjugate* $\overline{f}$ *with all unramified data. That is* $\theta(f)=\theta(f;\phi)$ *where* $\phi=\phi_f\otimes\phi_\infty$ *and* ϕ_f *is the characteristic function of* $\widehat{\mathbf{Z}}\otimes\Lambda^2$ *for the even unimodular lattice* $\Lambda=U(\mathbf{Z})\oplus V_0(\mathbf{Z})\oplus U^\vee(\mathbf{Z})$*. Then for* $y_1,y_2\in V_0(\mathbf{Z})$ *with* $\dim\mathrm{Span}\{y_1,y_2\}=2$*, one has that the Fourier coefficient* $a_{\theta(f)}(y_1,y_2)$ *is given by*

$$a_{\theta(f)}(y_1,y_2)=\sum_{r\in\mathrm{GL}_2(\mathbf{Z})\backslash(\mathrm{GL}_2(\mathbf{Q})\cap M_2(\mathbf{Z}))\ s.t.\ (y_1,y_2)r^{-1}\in V_0(\mathbf{Z})^2}|\det(r)|^{w-1}\overline{a_F({}^tr^{-1}S(y_1,y_2)r^{-1})}.$$

Proof. Suppose F is as in the statement of the theorem, and $h \in M_Y(\mathbf{A}_f)$. We can write $h = h_{\mathbf{Q}} h_\infty^{-1} k$ for $h \in M_Y(\mathbf{A}_f)$, $h_{\mathbf{Q}} \in M_Y(\mathbf{Q})$, $h_\infty \in M_Y(\mathbf{R})$ the archimedean part of $h_{\mathbf{Q}}$ and $k \in M_Y(\widehat{\mathbf{Z}})$. Then

$$a_F(T)(h) = a_F(T)(h_{\mathbf{Q}} h_\infty^{-1} k) = a_F(T \cdot h_{\mathbf{Q}}) \det(h_\infty)_X^{-N}.$$

Plugging this in to (15), one obtains

$$a(y_1,y_2) = \int_{M_Y(\mathbf{A}_f)} |\det(h)_X|^{n/2-3+N} \phi((b_1 f_1 + b_2 f_2 + y_1 e_1 + y_2 e_2)h)\overline{a_F(S(y_1,y_2)\cdot h_{\mathbf{Q}})}\, dh. \tag{16}$$

Note that $n/2 - 3 + N = w - 1$. Set $h = \mathrm{diag}(r^{-1}, r^t)$. Because ϕ_f is the characteristic function of the lattice $(U(\mathbf{Z}) \otimes W(\mathbf{Z}) \oplus V_0(\mathbf{Z}) \otimes X(\mathbf{Z})) \otimes \widehat{\mathbf{Z}}$, $\phi((b_1 f_1 + b_2 f_2)h) \neq 0$ if and only if $r \in M_2(\widehat{\mathbf{Z}})$, and then $\phi((y_1 e_1 + y_2 e_2)h) \neq 0$ if and only if $(y_1,y_2)r^{-1} \in V_0(\widehat{\mathbf{Z}})$. The theorem follows. ☐

4.2. **Modular forms on G_2.** Assume now that V is 8-dimensional. By Rallis [Ral84, Chapter I, section 3], the theta-lift $\theta(f)$ is a cusp form on $\mathrm{SO}(V)$. Consequently, the degenerate terms $\theta(f;\phi' \otimes \phi_\infty)_{deg}$ in Proposition 4.0.1 are 0 in this case. Thus, we may normalize the θ-lift by dividing by $C_{v_1^+,v_2^+,w}$, and see that the Fourier coefficients $a(y_1,y_2;\phi)$ are algebraic numbers if the Fourier coefficients of f are.

In this case that V is 8-dimensional, it turns out that restricting cuspidal modular forms from $\mathrm{SO}(V)$ to G_2 again produces a *cuspidal* modular form. This is proven in Corollary 4.2.3 below, of which the main step is the following lemma.

To setup the lemma, we define a few notations. Set $E = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, $W_E = \mathbf{R} \oplus E \oplus E \oplus \mathbf{R}$, and $W_{\mathbf{R}} = \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$. Denote by (a,b,c,d) an element of W_E, so that $a,d \in \mathbf{R}$ and $b,c \in E$. Define a projection $W_E \to W_{\mathbf{R}}$ by $(a,b,c,d) \mapsto (a, \mathrm{tr}(b)/3, \mathrm{tr}(c)/3, d)$, where $\mathrm{tr}(b) = b_1 + b_2 + b_3$ if $b = (b_1,b_2,b_3)$ and similarly for $\mathrm{tr}(c)$. The spaces W_E, $W_{\mathbf{R}}$ carry actions of

$$\mathrm{GL}_2(E)^0 = \{(g_1,g_2,g_3) \in \mathrm{GL}_2(E) : \det(g_1) = \det(g_2) = \det(g_3)\},$$

resp. $\mathrm{GL}_2(\mathbf{R})$, see [Pol18, section 4]. Embedding $\mathrm{GL}_2(\mathbf{R})$ into $\mathrm{GL}_2(E)^0$ diagonally, the projection $W_E \to W_{\mathbf{R}}$ is $\mathrm{GL}_2(\mathbf{R})$-equivariant.

If $v = (a,b,c,d) \in W_E$, we recall from [Pol18, Example 4.4.2] the element $S_{W_E}(v) \in M_2(E) = M_2(\mathbf{R}) \times M_2(\mathbf{R}) \times M_2(\mathbf{R})$. Specifically, if $b = (b_1,b_2,b_3)$, set $b^\# = (b_2b_3, b_3b_1, b_1b_2) \in E$, and similarly for $c^\#$. Then

$$S_{W_E}(v) = \begin{pmatrix} b^\# - ac & ad - bc - \mathrm{tr}(ad-bc)/2 \\ ad - bc - \mathrm{tr}(ad-bc)/2 & c^\# - db \end{pmatrix}.$$

In terms of $M_2(\mathbf{R}) \times M_2(\mathbf{R}) \times M_2(\mathbf{R})$, we write $S_{W_E}(v) = (q_1,q_2,q_3)$, with

$$q_1 = \begin{pmatrix} b_2b_3 - ac_1 & \frac{-b_1c_1 + b_2c_2 + b_3c_3 - ad}{2} \\ \frac{-b_1c_1 + b_2c_2 + b_3c_3 - ad}{2} & c_2c_3 - db_1 \end{pmatrix}. \tag{17}$$

The elements q_2, q_3 are given by the same expression as (17) except with the indices rotated modulo 3.

As the q_i are symmetric 2×2 matrices, we may also consider them as binary quadratic forms. If $v \in W_E$ we write $v > 0$ to mean that each of the q_j above is positive definite. Similarly, if $v' \in W_{\mathbf{R}}$, we write $v' = (a',b',c',d') > 0$ if

$$S_{W_{\mathbf{R}}}(v') = \begin{pmatrix} (b')^2 - a'c' & \frac{b'c' - a'd'}{2} \\ \frac{b'c' - a'd'}{2} & (c')^2 - d'b' \end{pmatrix} \in M_2(\mathbf{R})$$

is positive definite.

Lemma 4.2.1. *As above, denote by $E = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, and suppose that $v = (a, b, c, d) \in W_E = \mathbf{R} \oplus E \oplus E \oplus \mathbf{R}$ is positive definite, i.e., $v > 0$. Let $v' = (a', b', c', d') := (a, \mathrm{tr}(b)/3, \mathrm{tr}(c)/3, d) \in W_{\mathbf{R}}$ be the image of the GL_2-equivariant projection $W_E \to W_{\mathbf{R}}$. Then $v' > 0$.*

Remark 4.2.2. The binary cubic form associated to (a', b', c', d')–in the sense of [GGS02] and [Pol20a, Section 1.2.2]–is $a'z^3 + 3b'z^2 + 3c'z + d'$. Thus, if a, b, c, d are all integral, then so is the binary cubic form associated to (a', b', c', d').

Proof. As just mentioned, $v > 0$ means that each of the three binary quadratic forms associated to v is positive-definite.

Now, note that if $p(x, y)$ is a real binary cubic form, then a GL_2-translate of it is divisible by x. Indeed, this follows from the fact that every real cubic polynomial has a real root. Thus, by GL_2 equivariance, we may assume $d = 0$. Now, consider $S_{W_E}(v) = \begin{pmatrix} b^\# - ac & * \\ * & c^\# \end{pmatrix}$. Because we assume $v > 0$, we obtain that $c^\#$ is positive definite, i.e., each of its entries are positive. It follows that c is either positive-definite or negative-definite. By multiplying by -1 if necessary, we can assume $c > 0$.

Now, because $c > 0$, $\mathrm{tr}(c) > 0$ and thus by acting by unipotent elements in $\mathrm{GL}_2(\mathbf{R})$, we can assume $\mathrm{tr}(b) = 0$. But if $\mathrm{tr}(b) = 0$, then either $b = 0$ and $b^\# = 0$, or one component of $b^\#$ is negative. Because $v > 0$, $b^\# - ac > 0$ and thus since $c > 0$ we must have $a < 0$. Hence $v' = (a, 0, c', 0)$ with $a < 0$ and $c' > 0$. Therefore $v' > 0$, as desired. □

Corollary 4.2.3. *Suppose that f is a cuspidal modular form on $\mathrm{SO}(4,4)$ of weight w, and denote by f' the restriction of f to G_2. Then f' is a cuspidal modular form on G_2 of weight w.*

Proof. Because the inclusion $G_2 \to \mathrm{SO}(V)$ is compatible with the Cartan involutions on these groups, it follows easily from the definitions that f' is a modular form of weight w. Indeed, the K-equivariance and moderate growth of f' are immediate.

To see that $D_w f' = 0$, one argues as follows. Denote by $W_{\mathbf{R}}^\perp$ the subspace of W_E that is the orthogonal complement to $W_{\mathbf{R}}$ under the symplectic form on W_E. Denote by $\mathfrak{p}$ and $\mathfrak{p}_{G_2}$ the subspaces of the complexified Lie algebras of $\mathrm{SO}(4,4)$ and G_2, respectively, where the Cartan involution acts as multiplication by -1. Then $\mathfrak{p} \simeq V_2 \otimes W_E$ and $\mathfrak{p}_{G_2} \simeq V_2 \otimes W_{\mathbf{R}}$. Select a basis for $\mathfrak{p}$ that extends a basis for $\mathfrak{p}_{G_2}$, and write down the equation $D_w f = 0$ using this basis. The equation $D_w f = 0$ splits into two pieces corresponding to the coefficients being in $Sym^{2w-1}(V_2) \otimes W_{\mathbf{R}}$ and $Sym^{2w-1} \otimes W_{\mathbf{R}}^\perp$. Because of the compatibility of the Cartan involutions, the piece with coefficients in $Sym^{2w-1}(V_2) \otimes W_{\mathbf{R}}$ exactly says that $D_w f' = 0$.

For the cuspidality of f', first note that the map $G_2 \to \mathrm{SO}(V) = \mathrm{SO}(4,4)$ factors through $\mathrm{Spin}(8)$. Or relatedly, the image of the real points $G_2(\mathbf{R})$ sits in the connected component of the identity of $\mathrm{SO}(4,4)(\mathbf{R})$, as $G_2(\mathbf{R})$ is connected. Because of either of these facts, it follows from [Pol20a, Theorem 1.2.1] that the only Fourier coefficients of f that contribute to f' are those that correspond to $v \in W_E$ with $S_{W_E}(v) > 0$. That is, all three binary quadratic forms associated to v must be positive definite.[2] From Lemma 4.2.1, one sees that f' only has nonzero Fourier coefficients associated to $v' \in W_{\mathbf{R}}$ with $v' > 0$. Consequently, f' is cuspidal, as desired. □

In case $F(Z)$ is a level one Siegel modular form of weight $w \geq 16$ as in Theorem 4.1, one obtains the following corollary.

To set up the corollary, suppose $v \in W_E$ and $S_{W_E}(v)$ is the triple of binary quadratics associated to v. There are three nontrivial maps $\mathrm{Spin}(8) \to \mathrm{SO}(V)$. Choosing one such map defines an identification $W_E \approx V_0^2$. Under this identification, if $v \in W_E$ corresponds to $(y_1, y_2) \in V_0^2$, then

[2]Note that on $\mathrm{SO}(V)$, because $\mathrm{SO}(V)(\mathbf{R})$ is not connected, this is no longer true: on a non-neutral component of $\mathrm{SO}(V)(\mathbf{R})$, one of the binary quadratic forms will be negative definite, not positive definite.

$S(y_1, y_2)$ is one of the three binary quadratic forms from $S_{W_E}(v)$. See [Pol20a, Appendix A]. Without loss of generality, assume that $S(y_1, y_2) = S_{W_E}(v)_1$, the first quadratic form.

Corollary 4.2.4. *Suppose $F(Z) = \sum_{T>0} a_F(T)q^T$ is a level one Siegel modular form on $\mathrm{Sp}(4)$ of even weight $w \geq 16$. For $p(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ an integral binary cubic form that factors into three distinct linear factors over $\mathbf{R}$, define*

$$a_{\theta(f)}(p) := \sum_{\widetilde{p}} \sum_{r \in \mathrm{GL}_2(\mathbf{Z}) \backslash (\mathrm{GL}_2(\mathbf{Q}) \cap M_2(\mathbf{Z})) : \widetilde{p}r^{-1} \text{ integral}} |\det(r)|^{w-1} \overline{a_F({}^t r^{-1} S_{W_E}(\widetilde{p})_1 r^{-1})} \tag{18}$$

where the sum is over positive definite integral boxes $\widetilde{p} = (a, (b_1, b_2, b_3), (c_1, c_2, c_3), d)$ with $b_1 + b_2 + b_3 = b$ and $c_1 + c_2 + c_3 = c$. Then there is a cuspidal modular form $\theta(f)|_{G_2}$ on G_2 with p^{th} Fourier coefficient $a_{\theta(f)}(p)$. Moreover, the Fourier coefficient of $\theta(f)|_{G_2}$ corresponding to $p(x,y) = x^3 - 3xy^2$ is $a_F((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}))$.

It is not immediately obvious that the sum (18) is finite. However, this is true, and follows from Proposition 5.1.2 below. Granting this finiteness for now, Corollary 4.2.4 implies Theorem 1.0.2 of the introduction.

Remark 4.2.5. While there are three maps $\mathrm{Spin}(8) \to \mathrm{SO}(V)$, these three maps are equal on G_2. Thus, even though the formula (18) appears to depend on the fact that the first quadratic form $S_{W_E}(\widetilde{p})_1$ has been singled out, in fact it does not.

To see this explicitly, denote by C_3 the cyclic group of 3 elements. This group acts by permuting the indices $(1, 2, 3)$ of elements $\widetilde{p}$. If σ is in C_3, p a binary cubic form and $\widetilde{p}$ sits above p, then one sees immediately that $\sigma(\widetilde{p})$ also sits above p. In this way, one sees explicitly that (18) is equal to the same expression if $S_{W_E}(\widetilde{p})_1$ is replaced by $S_{W_E}(\widetilde{p})_2$ or $S_{W_E}(\widetilde{p})_3$.

Proof of Corollary 4.2.4. The only thing that remains to be proved is the final statement. For this, suppose that $v = (a, (b_1, b_2, b_3), (c_1, c_2, c_3), d)$ is a $2 \times 2 \times 2$ integer box with $v > 0$, and $v' = (a, \mathrm{tr}(b)/3, \mathrm{tr}(c)/3), d) = (-1, 0, 1, 0)$. Then $a = -1$ and $d = 0$. Because c is positive-definite and integral with $\mathrm{tr}(c) = 3$, we must have $c = (1, 1, 1)$. Moreover, because $\mathrm{tr}(b) = 0$, b is integral, and $b^{\#} + c > 0$, one must have $b = 0$. Thus the only box v lying above $(-1, 0, 1, 0)$ is $\widetilde{p} = (-1, (0, 0, 0), (1, 1, 1), 0)$. Because $S(\widetilde{p}) = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, the corollary follows. $\square$

5. Algebraicity

In this section we prove results on the algebraicity of the theta lift from $\mathrm{Sp}(4)$ to $\mathrm{SO}(V)$ and G_2. In particular, we prove that the lift from Sp_4 to $\mathrm{SO}(V)$ preserves algebraicity of the Fourier coefficients if the Schwartz-Bruhat data is algebraic. This is clear in the level one case from Theorem 4.1.1. We also prove that the restriction from $\mathrm{SO}(4,4)$ to G_2 preserves algebraicity of the Fourier coefficients.

5.1. Restriction to G_2.

We begin with some remarks on the "positive definiteness" condition on the Fourier coefficients of modular forms on $\mathrm{Spin}(8)$ and G_2. Let $E = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ and for v a $2 \times 2 \times 2$ cube, i.e., an element of W_E, recall that we denote by $S_{W_E}(v) \in M_2(E)$ the triple of binary quadratic forms associated to v. Similarly recall that if $v' \in W_{\mathbf{R}}$ then $S_{W_{\mathbf{R}}}(v)$ denotes the binary quadratic associated to v defined in section 4. The positive semi-definiteness condition for the nonvanishing of Fourier coefficients of modular forms on $\mathrm{Spin}(8)$ and G_2 from [Pol20a] is equivalent to positive semi-definiteness of the binary quadratics $S_{W_E}(v)$, $S_{W_{\mathbf{R}}}(v')$, respectively. We begin by proving this claim. Let us note that this claim can be deduced easily from [Wal03] in the case of non-degenerate v, i.e., v of rank 4.

Let $\mathfrak{h}$ denote the upper half complex plane in $\mathbf{C}$. For an element $v = (a, b, c, d) \in W_E$, with $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$, and $Z = (z_1, z_2, z_3) \in \mathfrak{h}^3$ denote by

$$\begin{aligned} j_v(Z) &= aN(Z) + (b, Z^{\#}) + (c, Z) + d \\ &= az_1z_2z_3 + b_1z_2z_3 + b_2z_3z_1 + b_3z_1z_2 + c_1z_1 + c_2z_2 + c_3z_3 + d \end{aligned}$$

the cubic polynomial on $\mathfrak{h}^3$ associated to v.

Proposition 5.1.1. *Suppose $v \in W_E$ is nonzero. Then the polynomial $j_v(Z)$ has no zeros on $\mathfrak{h}^3$ if and only if $S_{W_E}(v) \in M_2(E)$ is positive semi-definite.*

Proof. The conditions on $S_{W_E}(v)$ and $j_v(Z)$ are both equivariant with respect to the action of the group $\mathrm{GL}_2(E)^0$. So, it suffices to check the equivalence of the conditions on special orbit representatives.

Assume first that v is of rank 4. Then there is an $\mathrm{GL}_2(E)^0$-translate of v of the form $v_1 = (-1, 0, c, 0)$. One has that $S(v_1)$ is positive-definite if and only if c is, i.e., if and only if c_1, c_2, c_3 are all positive. On the other hand, $j_{v_1}(Z) = c_1z_1 + c_2z_2 + c_3z_3 - z_1z_2z_3$. Restricting to elements $z_j = iy_j$ on the imaginary axis, one obtains

$$j_{v_1}(iY) = i\left(c_1y_1 + c_2y_2 + c_3y_3 + y_1y_2y_3\right).$$

If any of the c_j are negative, it is easy to see that this expression can be made to be 0 for some positive y_j's. Conversely, if all $c_j > 0$, then $j_{v_1}(Z)$ is never 0 on $\mathfrak{h}^3$; this follows from, for example, [Pol20a, Proposition 10.0.1].

Now assume that v is rank $1, 2$ or 3. Then v has an $\mathrm{GL}_2(E)^0$-translate of the form $v_1 = (0, 0, c, d)$; see, for example, [Pol20a, Lemma 10.0.2]. Then $S_{W_E}(v_1)$ is positive semi-definite if and only if $c^{\#}$ is. In other words, $S_{W_E}(v_1)$ is positive semi-definite if and only if all of the c_j are either nonpositive or nonnegative. It is easy to see that $j_{v_1}(Z)$ is nonvanishing on $\mathfrak{h}^3$ under exactly the same condition. This completes the proof. $\square$

Suppose F is a modular form on Spin(8) of weight w and f its restriction to G_2. We claim that the Fourier coefficients of f are finite sums of Fourier coefficients of F. It follows that if F has algebraic Fourier coefficients then so does f. For the finiteness of the sum, that follows from the following proposition.

Proposition 5.1.2. *Suppose $v' \in W_{\mathbf{R}}$ has $S_{W_{\mathbf{R}}}(v')$ positive semi-definite, and denote also by v' its image in W_E under the inclusion $W_{\mathbf{R}} \to W_E$. Denote by $W_{\mathbf{R}}^{\perp}$ the subspace of W_E that is the orthogonal complement to $W_{\mathbf{R}}$ under the symplectic form on W_E. Then the set of $x \in W_{\mathbf{R}}^{\perp} \subseteq W_E$ with $S_{W_E}(v' + x)$ positive semi-definite is compact.*

Proof. Assume first that v' is rank four. By equivariance, we can assume without loss of generality that $v' = (-1, 0, 1, 0)$. Then $v' + x = (-1, b, c, 0)$ with $b_1 + b_2 + b_3 = 0$ and $c_1 + c_2 + c_3 = 1$.

Now, for $S_{W_E}(v'+x)$ to be positive semi-definite, we require that $b^{\#} + c \geq 0$ and $c^{\#} \geq 0$. Because $c^{\#} \geq 0$ and the sum of the c_j's are positive, we have that each $c_j \geq 0$. Thus the set of such c_j's with sum equal to 1 is compact.

For each such fixed $c = (c_1, c_2, c_3)$ non-negative with $c_1 + c_2 + c_3 = 1$, we claim that the set of $b = (b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$ and $b^{\#} + c \geq 0$ is compact. Indeed, we have $|b_1b_2| \leq c_3$, $|b_2b_3| \leq c_1$ and $|b_3b_1| \leq c_2$. Assume, without loss of generality, that b_1 and b_2 are either both nonnegative or both nonpositive. Writing $b_3 = -(b_1 + b_2)$, we have

$$c_1 + c_2 \geq |b_2||b_1 + b_2| + |b_1||b_1 + b_2| = (|b_1| + |b_1|)^2.$$

Thus the sum $|b_1| + |b_2|$ is bounded and $b_3 = -b_1 - b_2$, so the set of b's is compact. This completes the proof in the case that v' is of rank 4.

For the case v' with smaller rank, we may assume $v' = (0,0,c_0,d)$ and $v'+x = (0,b,c,d)$ with $b_1+b_2+b_3 = 0$ and $c_1+c_2+c_3 = c_0$. Now, for $S_{W_E}(v'+x)$ to be positive semi-definite, we require $b^\# \geq 0$. This implies that all of the b_j's are nonpositive or nonnegative. Combined with $b_1+b_2+b_3 = 0$ gives $b = 0$. Now, we have that $c^\# - db = c^\# \geq 0$. Again, this implies that all the c_j's are nonpositive or nonnegative. Because the sum $c_1+c_2+c_3 = c_0$ is fixed, the set of such c' is compact. This completes the proof. □

5.2. **Algebraicity of the theta lift.** In this subsection we prove that the non-degenerate Fourier coefficients of the theta lift are appropriately algebraic. More precisely, we prove the following statement.

Lemma 5.2.1. *Suppose f is a Siegel modular form on Sp_4 with Fourier coefficients in a field E containing the cyclotomic extension of $\mathbf{Q}$. Suppose moreover that $\phi = \phi_f \otimes \phi_\infty$ with $\phi_f \in S(\mathbf{X})(\mathbf{A}_f)$ is E-valued. Then the non-degenerate Fourier coefficients of $\theta(\phi; f)$ are E-valued.*

Proof. The non-degenerate Fourier coefficients of the theta-lift are given by (15). The claim is that this integral is in fact a finite sum. To see this, we have that both functions of h, $a_f(S(y_1,y_2))(h)$ and $\omega(1,h)\phi(b_1f_1+b_2f_2+y_1e_1+y_2e_2)$ are right invariant by a compact open subgroup U of the maximal compact subgroup $\mathrm{Sp}_4(\widehat{\mathbf{Z}})$ of $\mathrm{Sp}_4(\mathbf{A}_f)$. Applying the Iwasawa decomposition, it suffices to check that the integral (16) is a finite sum.

To see that (16) represents a finite sum, write $h = \mathrm{diag}(r^{-1}, r^t)$ with $r \in \mathrm{GL}_2(\mathbf{A}_f)$. Let $U_M = U \cap M_Y(\mathbf{A}_f)$, so that U_M can be identified with a compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$. We claim that the set of $r \in \mathrm{GL}_2(\mathbf{A}_f)$ with

$$\phi((b_1f_1+b_2f_2+y_1e_1+y_2e_2)h) = \phi((b_1(f_1r^t)+b_2(f_2r^t)+y_1(e_1r^{-1})+y_2e_2(r^{-1})) \neq 0$$

is a finite number of U_M cosets. Without loss of generality, assume that

$$\phi((b_1(f_1r^t)+b_2(f_2r^t)+y_1(e_1r^{-1})+y_2e_2(r^{-1})) = \phi_1((b_1(f_1r^t)+b_2(f_2r^t))\phi_2(y_1(e_1r^{-1})+y_2e_2(r^{-1}))$$

with ϕ_1 a characteristic function on $W(\mathbf{A}_f)^2$ and ϕ_2 a characteristic function on $V_0(\mathbf{A}_f)^2$.

The condition from ϕ_1, f_1, f_2 and ϕ_2, y_1, y_2 implies that there is some $t \in \mathrm{GL}_1(\mathbf{A}_f) \cap M_1(\widehat{\mathbf{Z}})$ so that $r \in t^{-1}M_2(\widehat{\mathbf{Z}})$ and that $(y_1,y_2)r^{-1} \in t^{-1}V_0(\widehat{\mathbf{Z}})$. This second condition implies that $\det(r)$ is bounded in terms of y_1, y_2 and t. Combining with $r \in t^{-1}M_2(\widehat{\mathbf{Z}})$ gives a finite number of U_M cosets, as desired. □

6. Nonvanishing

In this section, we prove that the theta lifts $\theta(\phi; f)$ from Sp_4 to $\mathrm{SO}(V)$ and G_2 can be made nonzero for appropriate choices of finite-adelic data. In particular, we deduce Theorem 1.0.1 of the introduction.

For the nonvanishing to $\mathrm{SO}(V)$, this follows from the following lemma.

Lemma 6.0.1. *Suppose f on Sp_4 is a nonzero Siegel modular form and let the notation be as in Proposition 4.0.1. Then there are y_1, y_2, an $\mathrm{Sp}_4(\mathbf{A}_f)$-translate of f and a $\mathbf{Q}$-valued Schwartz-Bruhat data ϕ on $\mathbf{X}_U(\mathbf{A}_f)$ so that the integral (16) is nonzero.*

Proof. Because f is nonzero, there are y_1, y_2 so that $a_f(S(y_1,y_2))(h)$ is nonzero as a function on $\mathrm{Sp}_4(\mathbf{A}_f)$. By applying an $\mathrm{Sp}_4(\mathbf{A}_f)$-translate of f if necessary, we may assume that this function is nonzero at $h = 1$.

We can now take appropriate Schwartz-Bruhat data to make (16) a positive number times $a_f(S(y_1,y_2))(1)$. To see this, note that there is a sufficiently divisible positive integer N so that $a_f(S(y_1,y_2))(h)$ is right invariant under a compact open subgroup

$$K_N = \{g \in \mathrm{Sp}_4(\mathbf{A}_f) : g \in 1 + NM_4(\widehat{\mathbf{Z}})\}.$$

Now, define $\phi = \phi_1 \otimes \phi_2$ so that $\phi_1 \in S((U \otimes W)(\mathbf{A}_f))$ is the characteristic function of $(f_1, f_2) + NW(\widehat{\mathbf{Z}})^2$ and $\phi_2 \in S((V_0 \otimes X)(\mathbf{A}_f))$ is the characteristic function of $V_0(\widehat{\mathbf{Z}})^2$.

Suppose $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in 2×2 block form. That $\phi_1(b_1(f_1 h) + b_2(f_2 h)) \neq 0$ implies that $c \equiv 0 \pmod N$ and $d \equiv 1 \pmod N$. It is not difficult to check that this, in turn, implies that $h \in N_Y(\mathbf{A}_f)K_N$. The lemma follows. □

We finally check the nonvanishing of the restriction from D_4 to G_2.

Lemma 6.0.2. *Suppose f is a smooth automorphic function on $G = \mathrm{SO}(4,4) \supseteq G_2$. Then there is a $G(\mathbf{A}_f)$-translate of f whose restriction to G_2 is nonzero.*

Proof. Indeed, $G(\mathbf{Q})G(\mathbf{A}_f)$ is dense in $G(\mathbf{A})$. It follows that there is a $G(\mathbf{A}_f)$-translate of f that is nonvanishing at $g = 1$. The lemma follows. □

Combining Proposition 5.1.2 and Lemmas 5.2.1, 6.0.1 and 6.0.2 we obtain Theorem 1.0.1 of the introduction.

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