Mathematics 103B Final Exam

Instructions: This is a closed book, closed note exam. You have 3 hours to read the questions, solve the questions, and write your solutions. Please time yourself, and mark your starting and ending time on your solutions. Asking others for help, using the internet to help you, consulting your notes or lectures, or spending longer than 3 hours is academic misconduct.

Once your 3 hours is up, please scan and upload your solutions to Gradescope. The time it takes to upload your solutions to Gradescope is not part of the 3 hours you have for the exam. You must upload your solutions to Gradescope by **2:30pm on Friday June 11th, US Pacific time**. Do not wait until the last minute, in case you have technical difficulties!

For the questions below, please write your solutions clearly and be sure to prove your answers.

1. (20 points) Does there exist a commutative ring $S$ and a ring homomorphism $\varphi : \mathbb{Q}(\sqrt{2}) \to S$ with $\ker(\varphi) = \mathbb{Z}[\sqrt{2}]$?

2. (20 points) Let $R = \mathbb{Z}[x]/\langle x^2 \rangle$. Prove that the group of units of $R$ is infinite. (Recall that a unit of $R$ is an element $u \in R$ that has a multiplicative inverse.)

3. (20 points) For a commutative ring $R$ with identity, let $\varphi : \mathbb{Z} \to R$ be the ring homomorphism determined by $\varphi(1) = 1$. Give an explicit example of a commutative ring $R$ with identity that has the following properties:
   - The principal ideals $\langle \varphi(5) \rangle$ and $\langle \varphi(7) \rangle$ of $R$ are maximal;
   - $R$ is not an integral domain.

4. (20 points) Determine whether $\mathbb{Z}[\sqrt{7}]/\langle 5 \rangle$ is a field.

5. (20 points) Let $R = \mathbb{Z}[i]$ and $I = (2 + i)$. What is the size of $R/I$?

6. (20 points) Let $F = \mathbb{Z}_{19}[t]$ be the fraction field of the polynomial ring $\mathbb{Z}_{19}[t]$. Give an explicit example of a field $E$ that is an infinite extension of $F$, i.e., a extension field $E$ of $F$ that is infinite-dimensional as an $F$ vector space.

7. (20 points) Suppose $R$ is a commutative ring with identity of characteristic 5, and $|R| = 5^3 = 125$, i.e., $R$ has 125 elements. Suppose moreover that $S$ is a commutative ring with identity, and $\varphi : R \to S$ is a surjective ring homomorphism with $\varphi(1) = 1$. What are the possible sizes of the ring $S$, i.e., what are the possibilities for the number of elements of $S$?

8. (20 points) What is the minimal polynomial of $2^{1/5}$ over $\mathbb{Q}$?

9. (20 points) This question has two parts.
   (a) (10 points) Suppose $p$ is a prime number. Prove that $\mathbb{Q}(p^{2/5}) = \mathbb{Q}(p^{1/5})$.
   (b) (10 points) Using part (a) or otherwise, prove that $x^5 - p^2$ is irreducible over $\mathbb{Q}$.

10. (20 points) Let $E$ denote the splitting field of $x^3 - 5$ over $\mathbb{Q}$. What is the degree $[E : \mathbb{Q}]$?