

PART 3: MODULAR FORMS ON G_2

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We will now move on to discussing modular forms on the exceptional group G_2 . Recall from the beginning of the semester modular forms on G_2 of weight n are very special automorphic functions $F : G_2(\mathbf{Q}) \backslash G_2(\mathbf{A}) \rightarrow \text{Sym}^{2n}(\mathbf{C})$ that satisfy

- (1) a $K_\infty = (\text{SU}(2) \times \text{SU}(2)) / \mu_2$ -equivariance condition $F(gk) = k^{-1} \cdot F(g)$ for all $g \in G_2(\mathbf{A})$ and $k \in K_\infty$ and
- (2) are annihilated by a special first order linear differential operator $\mathcal{D}_n$.

For the expert, for $n \gg 0$ the right regular action of $G_2(\mathbf{R})$ on such an F generates a quaternionic discrete series representation π_n , and F corresponds to the minimal K_∞ -type in this representation. These particular discrete series representations were singled out and studied by Benedict Gross and Nolan Wallach. However, it is not simply that π_n is a discrete series representation (and thus cohomological) that makes these F very special. Rather, it turns out that the π_n behave very similarly to the *holomorphic* discrete series representations on Hermitian symmetric spaces. And consequently, the associated automorphic functions F seem to behave like holomorphic modular forms on Hermitian symmetric spaces. In particular, these F have a robust Fourier expansion (which we will prove) and appear to be closely connected to arithmetic (we will give evidence for this). Just how *arithmetic* these modular forms are remains to be seen, although I am optimistic.

1. THE ALGEBRAIC GROUP G_2

Let us now begin describing the algebraic group G_2 . To do this, we will first construct the group PGL_2 in a slightly unusual fashion, and then mimic this construction to obtain G_2 . Throughout, k denotes an arbitrary field of characteristic 0.

We begin with some notation. For $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k)$ a two-by-two matrix, set $m' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ the cofactor matrix of m . Thus $mm' = \det(m)1_2$.

Lemma 1. *One has*

- $(m')' = m$
- $(m_1 m_2)' = m_2' m_1'$
- $m + m' = \text{tr}(m)1_2$
- $mm' = \det(m)1_2$
- If $\det(m) \neq 0$, then $m' = \det(m)m^{-1}$
- If $g \in \text{GL}_2$, then $(gm g^{-1})' = g m' g^{-1}$.

Proof. These are all immediate. □

Now, consider the algebraic group H_1 of all linear automorphisms of $M_2(k)$ that preserve the identity 1_2 , multiplication, and involution $'$:

$$H_1 := \{h \in \text{GL}(M_2) : h(1_2) = 1_2, h(xy) = h(x)h(y), h(x)' = h(x') \text{ for all } x, y \in M_2\}.$$

Denote by H the connected component of the identity of H_1 .

Lemma 2. *The group H defined above is PGL_2 .*

Proof. If $g \in \mathrm{GL}_2$, then the linear automorphism $m \mapsto gmg^{-1}$ on $M_2(k)$ is clearly in H . As the center of GL_2 acts trivially via this action, one obtains a map $\mathrm{PGL}_2 \rightarrow H$.

For the converse, denote by $M^{\mathrm{tr}=0}$ the trace 0 elements of M_2 . Note that elements of H preserve the quadratic form $\det(\cdot)$ on M_2 , and because they preserve 1_2 , elements of H preserve the orthogonal complement $(k1_2)^\perp$ of 1_2 for this quadratic form. This orthogonal complement is exactly the three-dimensional space $M_2^{\mathrm{tr}=0}$. Thus we obtain an inclusion $H \rightarrow O(M_2^{\mathrm{tr}=0}, \det)$ of H into this three-dimensional orthogonal space. As is well-known, $\mathrm{PGL}_2 = \mathrm{SO}(M_2^{\mathrm{tr}=0}, \det)$. This gives the lemma. $\square$

To construct the group G_2 , we proceed as in our above construction of PGL_2 , except with M_2 replaced by what are called the octonions. The octonions are an 8-dimensional k vector space, that comes equipped with certain extra algebraic structures, that we now describe.

We denote by Θ the octonions. As mentioned, this is an 8-dimensional k vector space that has the following extra structures:

- (1) An element $1 \in \Theta$
- (2) a multiplication $\Theta \otimes \Theta \rightarrow \Theta$ that is neither commutative nor even associative. One has $1 \cdot x = x \cdot 1 = x$ for all $x \in \Theta$
- (3) a non-degenerate quadratic form $n_\Theta : \Theta \rightarrow k$ that satisfies the important property $n_\Theta(xy) = n_\Theta(x)n_\Theta(y)$ for all $x, y \in \Theta$.
- (4) an involution $*$ that satisfies $(xy)^* = y^*x^*$, $xx^* = n_\Theta(x)1 = x^*x$, and $x + x^* \in k1$ for $x, y \in \Theta$.

We will shortly construct the octonions Θ . However, with the above definition, one defines

$$G_2 := \{g \in \mathrm{GL}(\Theta) : g1 = g, (gx)^* = g(x^*), g(xy) = g(x)g(y) \text{ for all } x, y \in \Theta\}$$

the automorphisms of Θ that preserve the various structures. Note that because $xx^* = n(x)1$, if follows from the definition of G_2 that $n_\Theta(gx) = n_\Theta(x)$ for all $x \in \Theta$.

1.1. The Cayley-Dickson construction. We will give two constructions of Θ . The first one is called the Cayley-Dickson construction. Fix $\gamma \in k^\times$. The construction proceeds as follows. One sets $\Theta = M_2 \oplus M_2$; we write elements of Θ as pairs (x, y) with $x, y \in M_2$. One sets $1 = (1_2, 0)$, $n_\Theta((x, y)) = \det(x) - \gamma \det(y)$, and $(x, y)^* = (x', -y)$. Note that $(x, y) + (x, y)^* = \mathrm{tr}(x)1$. Finally, the multiplication is given by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 + \gamma y_2' y_1, y_2x_1 + y_1x_2').$$

Observe that

$$(x, y)(x, y)^* = (x, y)(x', -y) = (xx' - \gamma y'y, -yx + yx) = (\det(x) - \gamma \det(y))(1, 0) = n_\Theta((x, y))1$$

as desired. Moreover, it is straightforward to check that $(z_1z_2)^* = z_2^*z_1^*$ for all $z_1, z_2 \in \Theta$. We must check the important property of the multiplicativity of the norm, $n_\Theta(z_1z_2) = n_\Theta(z_1)n_\Theta(z_2)$ for all $z_1, z_2 \in \Theta$.

Lemma 3. *One has $n_\Theta((x_1, y_1)(x_2, y_2)) = n_\Theta((x_1, y_1))n_\Theta((x_2, y_2))$.*

Proof. We checked this in class, but it would be a good exercise to do it yourself. To do it, use that fact that for all $x, y \in M_2$, $\det(x + y) = \det(x) + \mathrm{tr}(xy') + \det(y)$. $\square$

We have thus constructed the octonions. What makes them interesting, however, is that these algebraic structures possess automorphisms, i.e., that G_2 is not the trivial group. In fact, we can use the Cayley-Dickson construction to see that easily that $\mathrm{SL}_2 \times \mathrm{SL}_2$ maps to G_2 , as stated precisely in the following proposition.

For $g, h \in \mathrm{SL}_2$, let (g, h) act on Θ as

$$(g, h) \cdot (x, y) = (gxg^{-1}, hyg^{-1}).$$

Proposition 4. *The above action defines an map $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow G_2$ whose kernel is a diagonally embedded μ_2 .*

Proof. It is easy to see that the map $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{GL}(\Theta)$ has kernel exactly this diagonally embedded μ_2 . To check that the image is in G_2 , one must verify that this action preserves the identity 1 (immediate), the conjugation $*$ (easy), and the multiplication. We wrote this out in class. Again, you should try to write it out yourself that this action preserves the multiplication on Θ . $\square$

1.2. The Zorn model. The above construction of Θ made it clear that $\mathrm{SL}_2 \times \mathrm{SL}_2$ maps to G_2 . We will now construct Θ in a way that makes it clear that $\mathrm{SL}_2 \rightarrow G_2$. This construction is called the Zorn model of the octonions.

To define Θ this way, denote by V the defining representation of SL_3 . We assume then that V comes equipped with an identification $\wedge^3 V \rightarrow k$. This gives rise to identifications $\wedge^2 V \simeq V^\vee$ (dual representation) and $\wedge^2 V^\vee \simeq V$. The map $\wedge^2 V \rightarrow V^\vee$ is $x \wedge y \mapsto \{z \rightarrow x \wedge y \wedge z\}$. Now, define

$$\Theta = \left\{ \begin{pmatrix} a & v \\ \phi & d \end{pmatrix} : a, d \in k, v \in V, \phi \in V^\vee \right\}.$$

If $z = \begin{pmatrix} a & v \\ \phi & d \end{pmatrix}$, $z' = \begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix}$, then $n_\Theta(z) = ad - \phi(v)$, $z^* = \begin{pmatrix} d & -v \\ -\phi & a \end{pmatrix}$ and

$$zz' = \begin{pmatrix} aa' + \phi'(v) & av' + d'v - \phi \wedge \phi' \\ a'\phi + d\phi' + v \wedge v' & \phi(v') + dd' \end{pmatrix}.$$

Here $v \wedge v'$ is considered as an element of $V^\vee$, and similarly, $\phi \wedge \phi'$ is considered as an element of V . It is not so hard to check that the norm is multiplicative, and the various other properties of Θ .

Suppose $g \in \mathrm{SL}_3$ and $z = \begin{pmatrix} a & v \\ \phi & d \end{pmatrix} \in \Theta$. Define $g \cdot z = \begin{pmatrix} a & gv \\ g\phi & d \end{pmatrix}$. Here the action of g on $V^\vee$ satisfies-or is defined by-the equality $(gv, g\phi) = (v, \phi)$ for all $v \in V$ and $\phi \in V^\vee$.

Lemma 5. *The above action of SL_3 on Θ is equivariant for the multiplication on Θ , and thus defines a map $\mathrm{SL}_3 \rightarrow G_2$.*

Proof. Again, we wrote this out in class, but you should feel free to check it yourself, it is easy. $\square$

Denote by $V_7 = \Theta^{\mathrm{tr}=0}$ the trace 0 elements of the octonions. This is the orthogonal complement to $k1$ under the norm form. Just as in the case of PGL_2 above, because G_2 fixes the norm $n_\Theta(\cdot)$ and fixes 1, we obtain a map $G_2 \rightarrow \mathrm{SO}(V_7, n_\Theta)$, the special orthogonal group of this seven dimensional quadratic space.

So, we know $\mathrm{SL}_3 \leq G_2 \leq \mathrm{SO}(V_7)$ and $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \leq G_2 \leq \mathrm{SO}(V_7)$. To obtain more concrete of an understanding of G_2 , and because we will need this to do explicit computations later, we now discuss the Lie algebra of G_2 .

1.3. The Lie algebra of G_2 . We first briefly explain how the Lie algebra of G_2 sits inside that of $\mathrm{SO}(V_7)$. Recall that $\mathfrak{so}(V_7) \simeq \wedge^2 V_7$. There is a map

$$\kappa : \wedge^2 V_7 \rightarrow V_7$$

via $\kappa(x \wedge y) = \mathrm{Im}(xy)$. Here if $z \in \Theta$ then $\mathrm{Im}(z) = z - \mathrm{tr}(z)/2$ sits inside of V_7 .

Lemma 6. *The map κ is well-defined, i.e., $\mathrm{Im}(xy) = -\mathrm{Im}(yx)$ for all $x, y \in V_7$.*

Proof. Because $x, y \in V_7$, $x^* = -x$, $y^* = -y$, and thus $yx = y^*x^* = (xy)^*$. Therefore

$$\mathrm{Im}(xy) + \mathrm{Im}(yx) = \mathrm{Im}(xy + yx) = \mathrm{Im}(xy + (xy)^*) = \mathrm{Im}(\mathrm{tr}(xy)) = 0.$$

$\square$

It is easy to see that κ is G_2 -equivariant:

$$\kappa(gx \wedge y) = \kappa((gx) \wedge (gy)) = \text{Im}((gx)(gy)) = \text{Im}(g(xy)) = g\text{Im}(xy) = g\kappa(x \wedge y).$$

One also checks easily that κ is surjective. Thus $\ker \kappa$ is a $14 = 21 - 7$ -dimensional k -vector space with a natural action of G_2 .

Proposition 7. *The kernel of κ is the Lie algebra of G_2 . The Lie bracket is that induced from the Lie bracket on $\mathfrak{so}(V_7) = \wedge^2 V_7$.*

We also discussed in class the Lie algebra of $\mathfrak{so}(V)$ in more detail, and a different construction of $\mathfrak{g}_2$ via $\mathbf{Z}/3$ -grading: $\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee$, for V_3 the three-dimensional representation of SL_3 . See, for example, section 2 of my (mostly expository) paper “Modular forms on G_2 and their standard L -function” for this. One can also see Fulton and Harris, “Representation theory: A first course”, section 22.3, 22.4 for similar takes on $\mathfrak{g}_2$.

2. ARITHMETIC INVARIANT THEORY

Before moving on to modular forms on G_2 , Chen explained to us the arithmetic invariant theory of binary quadratic forms and binary cubic forms.

Recall that if we have a level one Siegel modular form F on Sp_4 , then F has a Fourier expansion $F(Z) = \sum_{T \geq 0} a_F(T) q^T$, where the sum is over half-integral symmetric 2×2 matrices $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ and $a_F(T) \in \mathbf{C}$ are the Fourier coefficients. Recall moreover that we (easily) deduced the equality $a_F(m^t T m) = a_F(T)$ for $m \in \text{GL}_2(\mathbf{Z})$.

Now half-integral matrices $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ can be considered as integral binary quadratic forms $ax^2 + bxy + cy^2$. Thus, the Fourier coefficients of a level one Siegel modular on Sp_4 gives rise to functions on the orbits $\text{GL}_2(\mathbf{Z}) \backslash \{ \text{BQFs} \}$, and it can be useful to have some way of understanding these orbits. For example, when we discussed the Rankin integral of Andrianov for the Spin L -function on GSp_4 , we used an auxiliary quadratic extension $L/\mathbf{Q}$; we will see that the orbits of $\text{GL}_2(\mathbf{Z})$ on binary quadratics are related to quadratic rings (and ideal classes) and this relationship between quadratic rings and the Fourier coefficients is lurking in the background of that Rankin integral.

It will turn out that the Fourier coefficients of modular forms on G_2 are parametrized by integral binary *cubic* forms $ax^3 + bx^2y + cxy^2 + dy^3$, and having a sense of their arithmetic invariant theory will be useful to us.

For the AIT of binary quadratics, one can see (for example) section 5.2 of Cohen’s book “A course in computational algebraic number theory”. For the AIT of binary cubics, one can see section 4 of the paper of Gan-Gross-Savin “Fourier coefficients of modular forms on G_2 ”.

3. DIFFERENTIAL OPERATORS FOR GL_2

Recall that modular forms on G_2 will be functions $f : \Gamma \backslash G_2(\mathbf{R}) \rightarrow \text{Sym}^{2n}(V_2)$ that satisfy $f(gk) = k^{-1}f(g)$ and are annihilated by a certain linear differential operator D_n . Our goal is to define and motivate this operator D_n . (It is a special case of a well-known procedure in representation theory, and frequently called the Schmid operator.) Once we have D_n , we will use it to understand the Fourier expansion of modular forms on G_2 .

So let’s first see how all this works out on SL_2 . Let f be a modular form on SL_2 of weight w , of level one say. Then since $f(z) = f(z+1)$, f has a Fourier expansion $f(z) = \sum_n a_n(y) e^{2\pi i n x}$. Now, of course we know from the holomorphy of f that $f(z) = \sum_n c_n(f) q^n$, but let’s reprove this quickly. Namely, because f is holomorphic, $\bar{\partial} f = 0$, where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Writing this out, one obtains $0 = \sum_n (2\pi i n a_n(y) + i a'_n(y)) e^{2\pi i n x}$. Thus $a'_n(y) = -2\pi n a_n(y)$ and we obtain $a_n(y) = c_n e^{-2\pi n y}$ for some constant c_n . Hence, plugging back in, $f(z) = \sum_n c_n e^{2\pi i n(x+iy)}$ as desired. We will do this

sort of argument to get the Fourier expansion of modular forms on G_2 once we have the operator D_n .

Let's now see how the operator $\bar{\partial}$ arises from group theory on SL_2 . We recall some basic facts about the Cartan decomposition. Let $\theta : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$ be the map $X \mapsto -X^t$. This has the following properties:

- (1) θ is an involution, i.e. $\theta \circ \theta$ is the identity
- (2) θ is a Lie algebra automorphism, i.e., $\theta[x, y] = [\theta(x), \theta(y)]$
- (3) The pairing $(X, Y) := -\mathrm{tr}(X\theta(Y))$ on $\mathfrak{sl}_2$ is symmetric and positive definite.

If we set $\mathfrak{k}_0 = \mathfrak{sl}_2^{\theta=1}$ the subalgebra of $\mathfrak{sl}_2$ where $\theta = 1$ and $\mathfrak{p}_0 = \mathfrak{sl}_2^{\theta=-1}$ the subalgebra of $\mathfrak{sl}_2$ where $\theta = -1$, then $\mathfrak{sl}_2 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a $\mathbf{Z}/2$ -grading. Moreover, $\mathfrak{k}_0$ is one dimensional and spanned by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathfrak{p}_0 = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a, b \in \mathbf{R} \right\}$. Finally, $\mathfrak{k}_0$ is the Lie algebra of $\mathrm{SO}(2)$, a maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$.

All these facts generalize to arbitrary $\mathfrak{g}$, so let us make a quick aside about this. Suppose $\mathfrak{g}_0$ is a semisimple real Lie algebra. An $\mathbf{R}$ -linear map $\theta : \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ is said to be a *Cartan involution* if

- (1) $\theta^2 = 1$
- (2) θ is a Lie algebra automorphism
- (3) $(X, Y) := -B(X, \theta(Y))$ is symmetric positive definite on $\mathfrak{g}_0$, where B is the Killing form

The basic facts are

- (1) Cartan involutions exist and are unique up to automorphisms of $\mathfrak{g}_0$
- (2) Denote by $\mathfrak{k}_0 = \mathfrak{g}_0^{\theta=1}$ and $\mathfrak{p}_0 = \mathfrak{g}_0^{\theta=-1}$, so that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Then this is a $\mathbf{Z}/2$ -grading on $\mathfrak{g}_0$, and $\mathfrak{k}_0$ is the Lie algebra of a maximal compact subgroup of G (Real Lie group with Lie algebra $\mathfrak{g}_0$)
- (3) If $\mathfrak{g}_0 = \mathfrak{sl}_n$, then $\theta(X) = -X^t$ is a Cartan involution

Notationally, we will write $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}$, $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C}$ and $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbf{C}$.

Going back to $\mathfrak{sl}_2$, we will see that the $\bar{\partial}$ operator comes from breaking up $\mathfrak{p}$ as a $\mathfrak{k}$ -representation. This decomposition is handled in the following lemma. Set $X^+ = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $X^- = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$.

Lemma 8. *One has $\mathfrak{p} = \mathbf{C}X^+ \oplus \mathbf{C}X^- = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Moreover, $[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X^\pm] = \pm 2iX^\pm$.*

The proof is an immediate check.

The fact that $\bar{\partial}$ is related to group theory is the content of the following proposition. Recall that if $X \in \mathfrak{g}_0$, and $\phi : G(\mathbf{R}) \rightarrow \mathbf{C}$ is smooth, the right regular action of $\mathfrak{g}_0$ on ϕ is:

$$X\phi(g) = \frac{d}{dt}\phi(ge^{tX})|_{t=0}.$$

Proposition 9. *Suppose w is an integer and $\phi : \mathrm{SL}_2(\mathbf{R}) \rightarrow \mathbf{C}$ satisfies $\phi(gk_\theta) = j(k_\theta, i)^{-w}\phi(g) = e^{iw\theta}\phi(g)$ for all $g \in \mathrm{SL}_2(\mathbf{R})$ and $k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ in $\mathrm{SO}(2)$. Assume moreover that $X^-\phi = 0$.*

Define $h_\phi : \mathfrak{h} \rightarrow \mathbf{C}$ as $h_\phi(x+iy) = y^{-w/2}\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right)$. Then h_ϕ is holomorphic.

Proof. We worked this out in class, but it would be good for you to do it yourself. Here are the key steps. First, we decompose X^- into different pieces:

$$X^- = (-i)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first piece is in $\mathfrak{n}$, the Lie algebra of the unipotent radical of the uppertriangular Borel, the second piece is in $\mathfrak{a}$, the Lie algebra of the split diagonal torus, and the third piece is in $\mathfrak{k}$.

Now, set $f_\phi(x, y) = \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right)$. Using the K -equivariance of ϕ and the definition of the right regular action, one computes

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right) &= iw f_\phi(x, y) \\ \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right) &= y \partial_y f_\phi(x, y) \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right) &= y \partial_x f_\phi(x, y). \end{aligned}$$

Thus $X^- \phi = 0$ implies that

$$(-i)y \partial_x f_\phi + y \partial_y f_\phi + \frac{i}{2}(iw) f_\phi = 0.$$

From this it is easy to deduce that $\bar{\partial} h_\phi = 0$, giving the proposition. $\square$

We also discussed the Cayley tranformation (see e.g. Bump's book "Automorphic forms and representations", section 2.2, page 154) and how the above proposition generalizes to Sp_{2n} .

4. MODULAR FORMS ON G_2

We are now (almost) in a position to define precisely modular forms on G_2 . That is, to define precisely the differential operator D_n . We first work in some generality, and then specialize to G_2 .

Thus, assume $\mathfrak{g}_0$ is a semisimple real Lie algebra, with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and complexified decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}_0$, and $K \subseteq G$ a maximal compact subgroup with Lie algebra $\mathfrak{k}_0$. Finally, assume that V is an irreducible K -representation, and $\phi : G \rightarrow V$ is a smooth function satisfying $\phi(gk) = k^{-1} \cdot \phi(g)$ for all $k \in K$ and $g \in G$. We will first define a differential operator $\tilde{D}$, and $\tilde{D}\phi$ will be valued in $V \otimes \mathfrak{p}^\vee$.

To define $\tilde{D}$, choose a basis $\{X_\alpha\}_\alpha$ of $\mathfrak{p}$ and let $\{X_\alpha^\vee\}_\alpha$ be the dual basis of $\mathfrak{p}^\vee$. Then one sets $\tilde{D}\phi(g) = \sum_\alpha (X_\alpha \phi)(g) \otimes X_\alpha^\vee$. We have the following lemma.

Lemma 10. *The operator $\tilde{D}$ is well-defined, i.e., independent of the choice of basis X_α of $\mathfrak{p}$. Moreover, one has $\tilde{D}\phi(gk) = k^{-1} \cdot \tilde{D}\phi(g)$ for all g, k . Here the k^{-1} action on the RHS is the tensor product action on $V \otimes \mathfrak{p}^\vee$.*

Proof. The element $\sum_\alpha X_\alpha \otimes X_\alpha^\vee$ in $\mathfrak{p} \otimes \mathfrak{p}^\vee$ is the identity map, when one considers $\mathfrak{p} \otimes \mathfrak{p}^\vee \simeq \mathrm{End}(\mathfrak{p})$; thus, it is independent of the choice of basis. Applying this, one sees that the operator $\tilde{D}$ is the composition

$$\tilde{D} : C^\infty(G; V) \rightarrow C^\infty(G) \otimes V \otimes \mathfrak{p} \otimes \mathfrak{p}^\vee = \mathfrak{p} \otimes C^\infty(G) \otimes V \otimes \mathfrak{p}^\vee \rightarrow C^\infty(G) \otimes V \otimes \mathfrak{p}^\vee.$$

Here the final arrow is the restriction to $\mathfrak{p}$ of the right-regular action of $\mathfrak{g}$ on $C^\infty(G)$.

We leave the proof of the second statement to the reader (we discussed it in class.) $\square$

Typically, $V \otimes \mathfrak{p}^\vee$ is a reducible K -representation, and there is natural decomposition $V \otimes \mathfrak{p}^\vee = V_+ \oplus V_-$. I won't define such a decomposition in general, but we will see it in a couple of examples. With such a decomposition, one defines $D = pr_- \circ \tilde{D}$, where pr_- is the K -equivariant projection $V \otimes \mathfrak{p}^\vee \rightarrow V_-$.

Example Suppose $G = \mathrm{Sp}_{2n}(\mathbf{R})$, $K = U(n)$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \mathfrak{gl}_n(\mathbf{C})$ and $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where $\mathfrak{p}^+ = \mathrm{Sym}^2(V_n)$ and $\mathfrak{p}^- = \mathrm{Sym}^2(V_n)^\vee$. Let $\{X_\alpha^+\}_\alpha$ be a basis of $\mathfrak{p}^+$ and $\{X_\beta^-\}_\beta$ be the dual

basis of $\mathfrak{p}^\vee$. Then

$$\tilde{D}\phi(g) = \sum_{\alpha} X_{\alpha}^{+}\phi \otimes X_{\alpha}^{+,\vee} + \sum_{\beta} X_{\beta}^{-}\phi \otimes X_{\beta}^{-,\vee}.$$

We have $V \otimes \mathfrak{p}^\vee = V \otimes \mathfrak{p}^{+,\vee} \oplus V \otimes \mathfrak{p}^{-,\vee} = V_+ \oplus V_-$. Thus

$$D\phi = \sum_{\beta} X_{\beta}^{-}\phi \otimes X_{\beta}^{-,\vee}.$$

Since the $X_{\beta}^{-,\vee}$ are linearly independent, we see that $D\phi = 0$ if and only if $X_{\beta}^{-}\phi = 0$ for all β , if and only if $X\phi = 0$ for all $X \in \mathfrak{p}^-$. As we saw in the last lecture, this condition is equivalent to the function h_{ϕ} on the Siegel upper half space of degree n being holomorphic.

As an important aside, the operator D can be used to construct the discrete series representations of G . I remarked a bit on this in class.

Moving on to G_2 , we have $K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$. The first $\mathrm{SU}(2)$ -factor is the long root $\mathrm{SU}(2)$, while the second is the short root $\mathrm{SU}(2)$. This means that $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, where the first $\mathfrak{sl}_2$ has a root spaces made up of long roots of $\mathfrak{g}_2$ and the second $\mathfrak{sl}_2$ has root spaces made up of short roots of $\mathfrak{g}_2$. Denote by $\mathbf{V}_n = \mathrm{Sym}^{2n} V_2 \boxtimes 1$ the representation of K where the long root $\mathrm{SU}(2)$ acts on $\mathrm{Sym}^{2n}(V_2)$ and the short root $\mathrm{SU}(2)$ acts on the trivial representation.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition of $\mathfrak{g}_2$. Then $\mathfrak{p} = V_2 \boxtimes \mathrm{Sym}^3(V_2)$ as a representation of $\mathfrak{k}$. Hence

$$\begin{aligned} \mathbf{V}_n \boxtimes \mathfrak{p}^\vee &= (S^{2n}(V_2) \otimes V_2) \boxtimes S^3(V_2) \\ &= S^{2n+1}(V_2) \boxtimes S^3(V_2) \oplus S^{2n-1}(V_2) \boxtimes S^3(V_2) \\ &= V_+ \oplus V_-. \end{aligned}$$

As above, for $n \geq 1$ we let $D_n = pr_- \circ \tilde{D}$ be the composition of pr_- with $\tilde{D}$ for pr_- the K -equivariant projection

$$\mathbf{V}_n \otimes \mathfrak{p}^\vee \rightarrow V_- = S^{2n-1} \boxtimes S^3(V_2).$$

Definition 11. Fix $n \geq 1$. A smooth function of moderate growth $\phi : \Gamma \backslash G_2(\mathbf{R}) \rightarrow \mathbf{V}_n$ is said to be a modular form of weight n if

- (1) $\phi(gk) = k^{-1} \cdot \phi(g)$ for all $k \in K$ and $g \in G_2(\mathbf{R})$
- (2) $D_n \phi = 0$.

We next discussed a concrete choice of Cartan involution on G_2 . See section 2.4 of my paper "Modular forms on G_2 and their standard L -function" for this.

5. THE FOURIER EXPANSION

We will now discuss the Fourier expansion of modular forms on G_2 . In class, we discussed various things from section 4 of *loc cit*. Namely, we

- (1) Discussed the $\mathbf{Z}/2$ -grading on $\mathfrak{g}_2$, i.e., $\mathfrak{g}_2 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus (V_2 \boxtimes \mathrm{Sym}^3(V_2))$
- (2) Discussed that the complexified Cartan decomposition on $\mathfrak{g}_2$ gives the above decomposition
- (3) Defined K -Bessel functions
- (4) Stated precisely the theorem on the Fourier expansion of modular forms on G_2 .

After stating precisely the above result, we sketched the proof. See section 4 of "Modular forms on G_2 and their standard L -function".

6. ODDS AND ENDS

In the final classes, we discussed a few things briefly that we did not have time to discuss in great detail. In particular

- (1) Klingen Eisenstein series: From a holomorphic modular form Φ on GL_2 of weight $3n \geq 18$, one can construct a modular form of weight n via an absolutely convergent Eisenstein series. The Eisenstein series sees Φ in its constant term.
- (2) The Standard L -function on G_2 and a Rankin-Selberg integral for it. This follows work of Gurevich-Segal and Segal. See sections 3 and 5 of “Modular forms on G_2 and their standard L -function” for details.
- (3) Finally, we briefly discussed the construction of cusp forms on G_2 from my paper “A quaternionic Saito-Kurokawa lift and cusp forms on G_2 ”

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