Last time

\[ G \text{ acts on } S \]

\[ S \mapsto O_s \text{ the orbit} \]

\[ H \subseteq G, \text{ the stab. of } s \]

Then \[ G/H \cong O_s \text{ bijectively} \]

\[ G \text{ a group acts } \}

\text{compatibly with } G/H \text{ action} \]

Proof: \[ S \text{ a finite set, } G \text{ acts } \]

\[ S, \text{ the stab. of } s \subseteq S, \text{ } O_s \]

\[ \text{the orbit. Then} \]

\[ |G| = |G_s| \cdot |O_s| \]

Pf: \[ |O_s| = |G : G_s| = |G : C_s| \]

\[ = \frac{|G|}{|C_s|} \] 

In other words \[ |O_s| = [G : C_s] \]

Also: \[ |S| = |O_1| + |O_2| + \cdots + |O_k| \]

in the sum of

the sizes of the various orbits.

Leader: Will use this to prove: every group whose order is a prime has a nontrivial center.
Example $(\mathbb{Z}/10\mathbb{Z})^x \subset \mathbb{Z}/10\mathbb{Z}$

Non representatives
\[ 1, 3, 7, 9 \]

\[ \mathcal{O}_1 \rightarrow 9 \xrightarrow{y} 3 \]

Stabilizer in $(\mathbb{Z}/10\mathbb{Z})^x$

\[ \mathcal{O}_2 \rightarrow 2 \xrightarrow{y} 6 \]

Stabilizer is $\mathbb{Z}/3\mathbb{Z}$

\[ |\mathcal{O}| = 10 = 1 + 4 + 4 + 1 \]

Example

$C = D_6$

$G \cong \text{regular hexagon}$

$Q = \{ x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^3 \}$

\[ y \text{ is reflection across this axis} \]

\[ x \text{ is rotation by $\frac{2\pi}{6}$} \]
$S$ = set of line segments connecting two vertices in the hexagon

$|S| = 15$
- 6 edges
- 3 diagonals
- 6 semi-diagonals

Understand the orbits and stabilizers for the $G$ action on $S$

Other orbits of $G$

$G$ contains 3 reflections thru diagonals

3 reflections thru those lines

$|S| = 15 = (3 \text{ diags}) + (16 \text{ semi diags}) + (6 \text{ edges})$

Stab. of a diagonal must have size 4

Stab. must have size 2

Stab. 1, $x,y$

Stab. must have size 2

$G$ contains 3 reflections thru diagonals

3 reflections thru those lines

$|S| = 15 = (3 \text{ diags}) + (16 \text{ semi diags}) + (6 \text{ edges})$
Semi-direct products

- Have groups $H, K \rightarrow H \times K$
- Generalization of this constructs involves group actions
- Setup: If $K \leq H$ acts by automorphisms, meaning
  
  $k \cdot (h_1, h_2) = (k \cdot h_1, k \cdot h_2)$

- We can construct a new group within $H \times K$

Recall: $\text{Mn} \subseteq \mathbb{R}^n$, group of isometries

$T_n \leq \text{Mn}$, the subgroup of translations

$T_n = \{ t_v : v \in \mathbb{R}^n \} \ltimes (\mathbb{R}^n, +)$
$T_n = \{ t_v : v \in \mathbb{R}^n \} \times (\mathbb{R}^n, +)$.

**Facts**

- $M_n = T_n \ltimes O_n$, $O_n$ the orthogonal group
- $\varphi : M_n \rightarrow O_n$ is a gp homomorphism, with kernel $T_n$
- $t_v \varphi \rightarrow \varphi$

**Multiplication** $(t_v \varphi)(t_{v'} \varphi') = t_v \varphi(v') \varphi' \varphi$

one is using an action of $O_n$ on $T_n \times \mathbb{R}^n$

- This is an example of a semi-direct product
- This is a special case of the general setup

**The construction**

- Suppose I have gps $H$, $K$
- $\phi : K \rightarrow \text{Aut}(H) \times$ a set of isomorphisms $H \rightarrow H$

In other words, $K \ltimes H$ by gp isomorphisms.

We will write $\phi_k$ to be the value of $\phi$ applied to $k \in K$

$\Rightarrow \phi_k \in \text{Aut}(H)$

$\Rightarrow \phi_k(h_1 h_2) = \phi_k(h_1) \phi_k(h_2)$.

Define $H \rtimes_{\phi} K$ to be the set of pairs $(h, k)$ with $h \in H$, $k \in K$
with multiplication

\[(h_1, k_1) \cdot (h_2, k_2) = (h_1 \phi_{k_1}(h_2), k_1 k_2)\]

Thm 2) \(H \times \phi K \) is a sp

b) The map \( H \times \phi K \rightarrow K \) given by \((h, k) \mapsto k\) is a surjective sp homomorphism with kernel \( H \)

c) If \( \phi : K \rightarrow \text{Act}(H) \) is the identity map, then \( H \times \phi K \cong H \times K \), the direct product of \( H \times K \)