A group action $G \times S \rightarrow S$

$(g, s) \rightarrow gs$

satisfying:

1. $1 \cdot s = s \quad \forall s \in S$
2. $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s \quad \forall s \in S, \forall g_1, g_2 \in G.$

$\Rightarrow \quad m: S \rightarrow S$ is a bijection.

**Def.** If $s \in S$, then the orbit $O_s$ of $s$ is

$$O_s = \{ s' \in S : \exists g \in G \text{ so that } s' = gs \}.$$ 

The orbits are equivalence classes for the equivalence relation $s \sim s' : \text{if } \exists g \in G \text{ so that } s' = gs.$

$\Rightarrow S$ is partitioned into the orbits for the group action.

*E.g.* $S = O_{s_1} \cup O_{s_2} \cup O_{s_3}$ (disjoint union)

**Def.** A group action $G \times S \rightarrow S$ is said to be transitive

if there is one orbit. Equivalently, given $s_1, s_2 \in S \exists g \in G$ so that $s_2 = gs_1.$
Define $G S S$, $s \in S$. Then the stabilizer of $s$ is

$$G_s = \{g \in G : g s = s \}.$$ 

**Claim:** $G_s$ is a subgp of $G$.

**Pf:**
- $1 \in G_s$ b/c $1 \cdot s = s$
- If $g s = s$ then applying $g^{-1}$ gives $g^{-1}(g s) = g^{-1}(s)$
  $$\Rightarrow g^{-1} s = g^{-1}(g s) = (g^{-1} g) s = 1 \cdot s = s$$
- If $g_1 s = s$ and $g_2 s = s$ then
  $$(g_1 g_2) s = g_1 (g_2 s) = g_1 (s) = s \quad \Rightarrow g_1, g_2 \in G_s.$$ 

**Example**

$G = GL_n(\mathbb{R})$

$S = M_n(\mathbb{R})$

$G \subseteq S$ as $g \circ m = gm + g$

**Claim:** This is an action

**Pf:**
- $1 \cdot s = 1 \cdot m \cdot t_1 = 1 \cdot m \cdot 1 = m$
- $g_1 \circ (g_2 \circ m) = g_1 \circ (g_2 m + g_2) = g_1 (g_2 m + g_2) = g_1 (g_2 m + g_2) t_1 = (g_1 g_2) m \circ (g_1 g_2) t_1 = (g_1 g_2) \circ m.$

$\Rightarrow$ If $A$ is fixed, then

$$-$$
If \( l \) is fixed, then

\[
G_A = \{ g \in GL_n(\mathbb{R}) : gA^t g = A \} \subseteq GL_n(\mathbb{R})
\]

a subgroup.

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**Example:** \( S_n \cong \mathbb{Z}/2 \times \ldots \times \mathbb{Z}/2 \). The stabilizer of \( n \) is isomorphic to \( S_{n-1} \) (permutations \( A \subset \{1, 2, \ldots, n-1\} \)).

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**Prop:** \( G \leq S_n \), so \( S \) is the stabilizer of the elt \( s \).

a) If \( a, b \in G \) then \( as = bs \iff a^{-1}b \in H \iff b \in H \).

b) Suppose \( s' = as \), and let \( H' \) denote the stabilizer of \( s' \).

Then \( H' = \{ aHa^{-1} : g \in G \} \).

**Pf:**

a) \( as = bs \iff s = a^{-1}bs \iff a^{-1}b \in H \).

b) \( \forall a \in H, aHa^{-1} \leq H' \), so \( s' = ah(s) = as = s' \).

\( H' \leq aHa^{-1} \); Suppose \( g' \in H' \), then \( g' = aga^{-1} \) for some \( g \in G \).

(Pf: \( g = a^{-1}g'a \).)
\[ g \in C \quad (\text{Pf: } g = a'g'a) \]

\[ as = s' = g's' = (aga')s' = (aga')(as) = a(gs) \]

\[ \Rightarrow \quad s = gs \quad \Rightarrow g \in H \Rightarrow g' \in \text{at} \backslash a^{-1} \backslash \]

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**Recall** \( G, \; H \subseteq G \) is a subgroup. Then

\[ G \subseteq C \subseteq \frac{G}{H} \]

the set \( H \backslash \text{left } G \backslash H \) exists.

**Notation**: if \( C = aH \) is a left \( H \) coset, write \([C]\) for the associated left \( G \) \( H \).

**Action**: \( C \triangleleft \frac{G}{H} \) is \( g \cdot [C] = [gC] = \{gc : c \in C\} \)

**Proof**: \( H \subseteq C \) is a subgroup.

1. The action \( G \triangleleft \frac{G}{H} \) is transitive.
2. The stabilizer \( \text{Stab}_G(C) \) is \( H \).

**Pf**: 1. Recall that transitive means there is one \( G \) orbit:
   And indeed, if \( [C] \in \frac{G}{H} \), \( C = [aH] = a \cdot [H] \).
2. \( g \cdot [H] = [H] \quad g \cdot [H] = [gH] = [H] \)
   \( \Rightarrow g \in H. \)
Prop: $S$ a set, $G < S$, so $S$, $G_s$ is the orbit of $s$, and $H = G_s$ is the stabilizer of $s$. Then there exists a bijection $\varphi : G/H \to G_s$ defined as $\varphi([aH]) = a \cdot s$.

Moreover, $\varphi$ is compatible with the group actions on both sides $\varphi(g \cdot [C]) = g \cdot \varphi([C])$.

Proof: Our definition of $\varphi$ depends on representative $a$, a priori.

If $aH = bH$ then $\varphi([C]) = as$ as $S$ are then equal.

$aH = bH \iff b = ah$ for some $h \in H$.

$\implies bs = ah \cdot s = a \cdot s$ because $H$ stabilizes $s$.

$\varphi(g \cdot [C]) = \varphi(g \cdot [aH]) = \varphi(g \cdot ah) = (ga) \cdot s = g(a \cdot s) = g \cdot \varphi([C])$.