Last time: Finite subgps of $\mathbb{O}_2$

Recall: $M_2$ denotes the gp of isometries of $\mathbb{R}^2$

will classify the finite subgps of $M_2$

**Lemma:** Suppose $P_1, \ldots, P_n \in \mathbb{R}^2$, $P = \frac{P_1 + P_2 + \ldots + P_n}{n}$.

If $f \in M_2$, then

\[
f(P) = \frac{f(P_1) + \ldots + f(P_n)}{n}
\]

**Pf:** Any elt $g \in M_2$ is from $g = t_a P e$ for some $a \in \mathbb{R}^2$, $e \in \mathbb{R}$. To prove the lemma, suffice to prove it for $P e$ and $t_a$ individually.

- Clear for $P e$ because $P e$ is linear
- For $t_a$, have

\[
t_a \left( \frac{P_1 + \ldots + P_n}{n} \right) = \frac{(P_1 + a) + \ldots + (P_n + a)}{n} = \frac{P_1 + \ldots + P_n}{n} + a
\]

\[
= P + a = t_a(P)
\]

Thus Suppose $G \leq M_2$ is finite subgp. Then $\exists P \in \mathbb{R}^2$ s.t.

$g(P) = P$ for every $g \in G$.

**Pf:** Take arbitrary pt $Q \in \mathbb{R}^2$. Define
Take arbitrary pt \( Q \in \mathbb{R}^2 \). Define
\[
P = \frac{1}{|G|} \sum_{g \in G} g(Q).
\]

Then, if \( h \in G \), then
\[
h(p) = \frac{1}{|G|} \sum_{g \in G} hg(Q)
\]
\[
= \frac{1}{|G|} \sum_{g' \in G} g'(Q) = P.
\]

Cor: Suppose \( G \leq M_2 \) is finite. Then \( G \cong C_n \) or \( D_n \) for some \( n \).

Pf: By Thm, \( \exists P \in \mathbb{R}^2 \) so that \( g(P) = P \) for all \( g \in G \).

Consider \( G' = T_pG T_{-p} \subseteq M_2 \).

Then if \( g' \in G' \), \( g'(O) = 0 \); if \( g' \not\in G' \), then
\[
g' = T_{-p}g T_p
\]
for some \( g \in G \). Thus
\[
g'(O) = T_{-p}g(P) = T_{-p}(P) = O.
\]

\[\Rightarrow G' \leq O_2 \text{ is finite} \Rightarrow G' \cong C_n \text{ or } D_n.\]

But \( G \cong G' \) so \( G \cong C_n \) or \( D_n \) some \( n \).

Def: \( G \leq M_2 \) a subgp is said to be discrete if
\[\exists \varepsilon > 0 \text{ so that} \]
\[\text{if } t_a \in G \text{ and } t_a \neq 1 \text{ then } ||a|| > \varepsilon.\]
• if $t_a \in G$ and $t_a \neq 1$ then $||a|| > \varepsilon$

• if $t_a p \in G$ with $p \neq 1$ then $|tp| > \varepsilon$.

(Comment: $G$ does not contain points obtained by rotation by $2 \pi$.

Note: Why no condition on glide reflections?

Reason: If $t_u r_l$ is a glide reflection,

(i.e., reflection across the line $l$ and $u$ is parallel to $l$)

then $(t_u r_l)^2 = t_u r_l t_u r_l = t_u r_l r_l = t_u r_l$

So, by the condition above, if $t_u \neq 1$, $|2| > \varepsilon$

so $|u| > \frac{\varepsilon}{2}$.

Will do: Some work towards classifying the discrete subgps of $M_2$.

Recall: $\varphi : M_2 \to O_2$ gp homom.

$\varphi(t_a) = \rho$, $\varphi(p) = p$

Suppose $G \subseteq M_2$ is discrete. Set $\overline{G} = \varphi(G) \subseteq O_2$

$\overline{G}$ is called the "point gp".

Set $L = \{ v \in \mathbb{R}^2 : t_v \in G \} = \ker(\varphi)$

Have: $G$ discrete $\Rightarrow \exists \varepsilon > 0$ so that $v \in L$ and $v \neq 0$ then $|v| > \varepsilon$. 


**H ave:** \( n \) discrete \( \iff \exists \varepsilon > 0 \text{ so that } \|v\| > 0 \text{ then } \|v\| \geq \varepsilon \).

**Def:** Suppose \( M \subseteq (\mathbb{R}^n, +) \) is a subgp. \( M \) is said to be 

\[ \text{discrete} \iff \exists \varepsilon > 0 \text{ so that } \forall m, n \in M, m \neq n \text{ then } \|m - n\| \geq \varepsilon. \]

**Thus:** If \( G \subseteq M \) is discrete, then \( L \subseteq (\mathbb{R}^2, +) \) is discrete.

**Thm:** Suppose \( L \subseteq (\mathbb{R}^2, +) \) is discrete. Then either

- \( L = \{0\} \)
- \( L = \mathbb{Z}a = \{ma : m \in \mathbb{Z}\} \text{ for some } a \in \mathbb{R}^2, a \neq 0 \)
- \( L = \mathbb{Z}a + \mathbb{Z}b = \{ma + nb : m, n \in \mathbb{Z}\} \text{ for some lin. ind. elts } a, b \in \mathbb{R}^2 \)

**L emma:** Suppose \( L \subseteq (\mathbb{R}^2, +) \) is discrete.

a) A bounded region of \( \mathbb{R}^2 \) contains finitely many elts of \( L \)

b) If \( L \neq \{0\} \), then \( \exists a \in L, a \neq 0 \) so that

\[ \|a\| \leq \|v\| \text{ for all } v \in L, v \neq 0. \]

Such a vector \( a \) is said to be of mininum length.

**P f:** a) Any sufficiently small sq. contains at most one elt of \( L \) because if \( v_1, v_2 \in L, v_1 \neq v_2 \text{ then } \|v_1 - v_2\| > \varepsilon \)

a) follows because a bounded region is covered by finitely many such small squares.
a) follows because a bounded region is covered by finitely many such small squares.

b) This follows from part a).

**Lemma:** Suppose $u, w \in \mathbb{R}^2$ are a basis. Define $L = \mathbb{Z}u + \mathbb{Z}w$

Then any vector $v \in \mathbb{R}^2$ can be written uniquely as $v = x + v_0$, where $x \in L$ and

$v_0 \in \{xu + \beta w : 0 \leq \alpha < 1, 0 \leq \beta < 1\}$.

**Proof:** Because $u, w$ is a basis,

$v = ru + sw$ for $r, s \in \mathbb{R}$

$=(m + \alpha)u + (n + \beta)w$ with $m, n \in \mathbb{Z}$

$0 \leq \alpha, \beta < 1.$

$(m + \alpha)u + (n + \beta)w = m_u + n_w + (\alpha u + \beta w)$.

$L = e_L + v_0$

**Proof** $L \subseteq (\mathbb{R}^2, +)$ is discrete.

a) If $L = \{0\}$ we're done.

b) Suppose $\exists a' \in L$, $a' \neq 0$. Either
b) Suppose \( \exists a' \in L, a' \neq 0 \). Either
   
   1) \( L \subseteq Ra' \) or
   2) \( L \nsubseteq Ra' \).

   In case 1, \( L \) is isomorphic to a discrete subgroup of \( R \)
   \( \Rightarrow L = \mathbb{Z}_a \) for some \( a \in Ra' \).

   In case 2, \( L \) contains a basis of \( R^2 \).

   By applying a linear operator \( S \) to \( L \), can assume
   \[
   L \cap Ra' = \mathbb{Z} \cdot (\begin{pmatrix} 1 \\ 0 \end{pmatrix}), \text{we choose so that } S(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
   \]

   Consider \( T = \{ b' = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \in L : b'_2 > 0 \} \).

   **Claim:** \( T \) is nonempty. Moreover, \( \exists b \in T, b = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \) so that
   \[
   b'_2 \leq b'_2 \text{ for all } b' = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \in T.
   \]

   **Proof:** \( T \) is nonempty since \( L \) contains a basis of \( R^2 \)
   and \( b' \in L \Rightarrow -b' \in L \).

   Note that if \( b' \in T \) then \( b' + m(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} b'_1 + m \\ b'_2 \end{pmatrix} \in T \).

   By choosing \( m \) appropriately, \( 0 \leq b'_1 + m < 1 \).

   Thus, it's clear that \( T \) is a
by choosing \( m \) appropriately, we get a index of finite 
sets of \( L \) in this region.

\[ \exists \ b = \left( b_1 \right) \ \text{with} \ 0 < b_1 < 1 \quad \text{and} \quad b_2 > 0 \quad \text{and minimal.} \]

Claim: \( L = \mathbb{Z} a + \mathbb{Z} b \), \( a = (1) \)

Pf: Clear that \( \mathbb{Z} c + \mathbb{Z} b \subseteq L \). Suppose \( c \in L \).

Then by Lemma, \( c = (ma + nb) + \alpha a + \beta b \) with \( m, n \in \mathbb{Z} \), \( 0 \leq \alpha, \beta < 1 \).

Then \( \frac{c - (ma + nb)}{c} = \alpha a + \beta b = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \beta b_2 \end{pmatrix} = \begin{pmatrix} 1 + \beta b_1 \\ \beta b_2 \end{pmatrix} \in \mathbb{T} \).

But: \( \beta b_2 < b_2 \Rightarrow \beta = 0. \)

\[ \Rightarrow c - (ma + nb) = \alpha a \in \mathbb{R} a \cap L = \mathbb{Z} a \]

\[ \Rightarrow \alpha = 0. \]

\[ \Rightarrow c = ma + nb \in \mathbb{Z} a + \mathbb{Z} b. \]