Last time: \[ n \geq 1 \]

\[ D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle \]

Classify subgps of \( O_2 \) that are finite:

either \( \cong C_n \)

\( = D_n \)

Thm: The gp \( D_n \) has order \( 2n \), and \( \exists \) injective homomorphism \( D_n \rightarrow C_2 \).

Lemm: Suppose \( S = \{ s_1, s_2, \ldots \} \) a set, and \( \mathcal{F} \) denote the free gp on \( S \). Suppose moreover \( G' \) is a gp, and have a collection of elts \( \{ g'_s \}_{s \in S} \). Then \( \exists \) a unique gp homomorphism \( \varphi: \mathcal{F} \rightarrow G' \) with \( \varphi(s) = g'_s \) \( \forall s \in S \).

Ex: \( S = \{ x, y \} \)

\( G' = S_3 = \langle x, y \mid x^3 = 1, y^2 = 1 \rangle \)

Then \( \mathcal{F} \)

\( \cong S_3 \)

\[ x \rightarrow \alpha \]

\[ y \rightarrow \beta \]

\[ x^{-3} \rightarrow x^{-3} y x^3 \]

\[ X^{-3} YX^3 Y^{-1} \rightarrow x^{-3} y x^3 y^3 \]

Pf of Thm

First step \( |D_n| \leq 2n \)
First step \( |D_n| \leq 2^n \)

Second step: \( \varphi : D_n \to O_2 \) and prove \( \varphi(D_n) \geq 2^n \)

\[ \Rightarrow 2^n = |D_n| \geq |\varphi(D_n)| \geq 2^n \]

\[ \Rightarrow |D_n| = 2^n \text{ and } \varphi \text{ is injective} \]

**Step 1** Can use the relators \( \langle x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle \)

to put any word in \( x \) and \( y \) into the form

\[ x^j y^k \quad 0 \leq j \leq n-1, \quad 0 \leq k < n-1 \]

\[ \{x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y \} \subseteq 2n \text{ elt's} \]

**Pf.** Use \( yg = x^{-1}y \) to move all the \( y \)'s in the word to the right.

Thus \( x^n = 1, y^2 = 1 \) to see that the power of \( x \) is between 0 and \( n-1 \) and the power of \( y \) is 0 or 1.

\[ \Rightarrow |D_n| \leq 2^n . \]

Define \( X = P_{2\pi} = \begin{pmatrix} \cos(2\pi n) & -\sin(2\pi n) \\ \sin(2\pi n) & \cos(2\pi n) \end{pmatrix} \in O_2 \)

\[ Y = I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O_2 \]

(lemma a free group \( \exists \varphi : \mathbb{F}_{2n} \to O_2 \), up to homomorphism.)
Lemma a free group $F = \{x, y\} \rightarrow \mathbb{C}^2$ is homom.

$x \mapsto X$

$y \mapsto Y$

Note: $X^n = 1$, $Y^2 = 1$, $YXY^{-1} = X^{-1}$

$\Rightarrow X^n \in \ker \phi$, $y^2 \in \ker \phi$, $yx^{-1} \in \ker \phi$

$\Rightarrow \phi$ induces $\overline{\phi} : \overline{F} = \{x, y\} \rightarrow \mathbb{C}^2$

$D_n = \langle x^n, y^2, yxy^{-1}x \rangle$

$\overline{F}[x, y]/(\ker \overline{\phi})$

$\ker \overline{\phi} = \langle x^n, y^2, yxy^{-1}x \rangle$

$\overline{F}[x, y]/(\ker \overline{\phi})$ is the smallest normal subgroup containing these elts.

$2n \geq |D_n| > |\overline{\phi}(D_n)| = \langle x, y \rangle \leq \mathbb{C}^2$

Subgrp of $\mathbb{C}^2$ gend by $x, y$.

Claim: $\langle x, y \rangle$ has size at least $2n$

$\overline{\phi}(D_n) = \{1, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y \}$

$\Rightarrow 2n$

$\Rightarrow$ These $2n$ elts are distinct.

DFi
The 2n elts are distinct.

Indeed, the \( x_j \) are distinct from the \( x_k \) as seen by taking determinants.

The \( x^j, x^k \) for \( j \neq k \) in \( \{0, 1, \ldots, n-1\} \) are distinct because \( X \) has order \( n \).

Case of Part: \( \varphi: D_n \to \mathbb{Z}_2 \)

\[ x \mapsto 2 \pi i \]

\[ y \mapsto \varphi = (c_{0,1}) \]

is an injective gp homom.

**Proof:** \( D_3 \cong S_3 \).

**Pf:** \( D_3 = \langle x, y | x^3 = 1, y^2 = 1 \rangle \cong S_3 = \langle x, y | x^3 = 1, y^2 = 1 \rangle \)

For \( n > 3 \), \( D_n \not\cong S_n \)

\[ \frac{2n}{n!} \]

Thus, \( S_3 \) is a finite subgp.

a) If \( G \leq S_3 \), then \( G \cong C_n \) for some \( n \).

b) If \( G \not\cong S_3 \), then \( G \cong D_n \) for some \( n \).

\( \Gamma' \leq (\mathbb{R}, +) \) is a subgp. \( \Gamma' \) is said to be discrete
Define $\Gamma \leq (\mathbb{R}, +)$ is a subgroup $\Gamma$ is said to be discrete if $\exists \varepsilon > 0$ so that $\forall \gamma \in \Gamma$, $\gamma \neq 0$ then $|\gamma| > \varepsilon$.

Lemma: $\mathbb{R}_+^\times$ is discrete subgroup. Then $\exists a > 0$ so that $\Gamma = \mathbb{Z}a$ or $\Gamma < \mathbb{Z}a$.

Proof: If $\Gamma < \mathbb{Z}a$ then done.

Otherwise, $\exists a' \in \Gamma$, $a' \neq 0$. $-a' \in \Gamma$.

WLOG can assume $a' > 0$.

Claim: $\exists$ finitely many $x$ in $\Gamma$ with $0 \leq x \leq a'$.

Proof: There exists $\varepsilon > 0$ so that if $b, c \in \Gamma$, $b \neq c$ then $|b - c| > \varepsilon$. $\Rightarrow$ In any bounded interval of $\mathbb{R}$ there's at most finitely many elts of $\Gamma$. In particular, finitely many in $[0, a']$.

Claim: $\exists$ a smallest positive elt $a \in \Gamma$.

$\Rightarrow \mathbb{Z}a \leq \Gamma$ (clear).

Claim: $\Gamma \leq \mathbb{Z}a$.

Proof: Suppose $b \in \Gamma$ $\Rightarrow \exists c \in \mathbb{R}$ with $b = ca$.

$r = \frac{c}{\gamma}$ for $\gamma \in \mathbb{Z}$ and $0 < r < 1$. 


\[ r = m + r_0 \quad \text{for } m \in \mathbb{Z} \quad \text{and } r_0 < 1 \]
\[ \text{(e.g., } -3.7 = -4 + 0.3) \]

\[ b = r_0 a = (m + r_0) a = ma + r_0 a \]
\[ \Rightarrow b - ma = r_0 a \quad \quad 0 \leq r_0 a < a \]
\[ \therefore r_0 a = 0. \]
\[ \Rightarrow b = ma \checkmark. \]

**Pf of Thm**

a) \( G \subseteq SO_2 \). Define

\[ \Gamma = \{ \alpha \in \mathbb{R}^2 : \rho_\alpha \in G \} \]

\( \text{rotation by } \alpha \)

- \( \Gamma \subseteq (\mathbb{R}^+)^2 \) is a subgroup

- \( \Gamma \) is discrete \( \left( \ldots, 0, 1, 1, 1, \ldots \right) \) because \( G \) is finite

- \( 2\pi \in \Gamma \) because \( \rho_{2\pi} = 1 \).

**Lemma:** \( \Gamma = Za \) for some \( a > 0 \).

\[ 2\pi = n \cdot a \quad \text{for some int. } n \]

\[ \Rightarrow a = 2\pi / n \quad \text{for some } n. \]

\[ \Rightarrow G = C_n = \langle \rho_{2\pi/n} \rangle \subseteq SO_2 \checkmark. \]
\[ G = C_n = \langle P_{2\pi/n} \rangle \leq SO_2 \]