Generators: \( \langle s, t \rangle \) \( 7,7,9 \), \( 7,7 \), \( 7,0 \) \n
Recall: \( S_3 \) sym sp \n\n\[
\begin{align*}
x & = (1,2,3) \\
y & = (1,2)
\end{align*}
\]
\[
\begin{align*}
x^3 & = 1 \\
yx & = x^2 y \\
y^2 & = 1
\end{align*}
\]

Using these can “simplify” an expression in \( x \)'s and \( y \)'s

eg. \( y^3 x^5 y x \rightarrow \) into explicitly an

an elt: \( \{ 1, x, x^2, y, xy, x^2 y^3 \} \)

\[\cong S_3\]

Free Group
Set \( S = \{ a, b, c, d \} \)

\( F_S = \text{free gp on } S \)

Elts of \( F_S \) are words in \( a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1} \)

eg. \( a b c a d c^{-1} a \)

- \( a b c a d c^{-1} a \)
- \( a b c a d c^{-1} a \)
- \( b c b c b c b c b c b c \)
- \( a b c a d c^{-1} a \)

Operation: concatenation of words

eg. \( (a^{-1} b c) \cdot (b c b c) = a^{-1} b c b c b c b c b c \)

Rel's between symbols cancel when \( x \) and \( x^{-1} \) are some \( x \in S \)

eg. \( (a^{-1} b c) \cdot (c^{-1} b^{-1} a^{-5} d a) \)
\[\begin{align*}
\text{e.g. } (a^{-10}bc) & \cdot (c^{-1}b^{-1}a^5 da) \\
& = a^{-10} b^{-1} c^{-1} a^5 da \\
& = a^{-10} b^{-1} a^5 da \\
& = a^{-5} da
\end{align*}\]

\underline{Identity}: I \quad \text{"empty word"}

**Then**: \( F_S \) is a gp. \quad \text{Called the "Free group on } S\text{".}

For example, clear how to make inverses:

- Invers of \( abcb^{-1}c^{-5}a^{-1} \) in \( d^{-1}c^{-1}b^{-1}c^{-1}b^{-1}a^{-1} \)

If \(|S| = 1\), then \( F_S \) is infinite cyclic (in particular, an abelian gp)

If \(|S| \geq 2\), then \( F_S \) is huge and in nonabelian.

**Lemma**: Suppose \( G \) a gp. \( \bigcap_{g \in I} N_g \) is a set of normal subgps

**If** \( G \). Then \( \bigcap_{g \in I} N_g = N \) is again a normal subgp of \( G \).

**Pf:** Suppose \( n \in N \), \( g \in G \). \( gng^{-1} \in N \).

Well \( n \in N \Rightarrow n \in N_g \forall g \).

\[gng^{-1} \in N_g \forall g \Rightarrow \text{ } N_g \text{ is normal.}\]

\[\Rightarrow \quad gng^{-1} \in \bigcap_{g \in I} N_g = N. \quad \Box\]
Suppose $S$ is a set, $S = \{a, b, c, d\}$.

$\mathbb{F}_S$: the free gp on $S$.

**Question:** Can we construct a gp $G$ that has elts $a, b, c, d$ that satisfy the rels: $a^5 = 1$, $bc = cb$, $abcd = 1$, $c^2d^2 = 1$?

Consider every normal subgp of $\mathbb{F}_S$ that contains $a^5$, $bc\ 5^{-1}c^{-1}$, $abcd$, $c^2d^2$.

Call these subgps $N_\alpha$, $\alpha \in I$.

$N = \bigcap_{\alpha \in I} N_\alpha = \text{intersection of every normal subgp that contains}$

$N$ is normal.

Define $G := \mathbb{F}_S/N$ the quotient gp, which we can do b/c $N$ is normal.

Let $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ be the images of the elts $a, b, c, d$ from $\mathbb{F}_5$ in $G$.

Then $\overline{a^5} \in G$, $\overline{a^5} = 1$ in $G$ b/c $a^5$ is in $N$.

Similarly, $\overline{bc\ 5^{-1}c^{-1}}$ in $1$ in $G$ b/c $bc\ 5^{-1}c^{-1}$ is in $N$.

$G = \langle \overline{a}, \overline{b}, \overline{c}, \overline{d} \mid \overline{a^5} = 1, \overline{bc} = \overline{cb}, \overline{abcd} = 1, \overline{c^2d^2} = 1 \rangle$

The group $G$ satisfies the rels $a^5 = 1$, $bc = cb$, $abcd = 1$, $c^2d^2 = 1$. 
This is a relation for the $G$ we just defined.

This process is called constructing a $G$ by generalizes relations.

**Example**

$G = \langle x, y \mid x^3 = 1, y^2 = 1, yx = x^2y \rangle$

$G = \mathbb{F}_3\langle x, y \rangle / \langle x^3, y^2, yxy^{-1}x^{-2} \rangle$

"smallest" normal subgroup of $\mathbb{F}_3\langle x, y \rangle$ that contains

$x^3, y^2, yxy^{-1}x^{-2}$

$G$ is isomorphic to $S_3$.

**Example**

$\langle x, y \mid x^5 = 1, y^2 = 1 \rangle$ : huge group (infinite)

$\langle x, y \mid x^5 = 1, y^2 = 1, yx = x^3y \rangle$ : finite

$x^3yxyx^{-1} = x^3(x^3y)yx^{-1}$

$= x^6y^2x^{-1}$

$= x \cdot 1 \cdot x^{-1}$

$= 1$
Dihedral group $n \geq 1$

$$D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

- Turns at: $D_n$ has order $2n$
  - $\exists \varphi : D_n \to \text{GL}_2(\mathbb{R})$ an injective homomorphism

Corresponding Theorem

- $\varphi : G \to G'$, surjective group homomorphism.
- Relates subgroups $A \leq G$ with subgroups $A' \leq G'$.

Proof: $\varphi : G \to G'$ be a group homomorphism, w/ kernel $K$.

- $H' \leq G'$ is a subgroup. Let $H = \varphi^{-1}(H')$ be the inverse image.
- $H \leq G$ if $\{ h \in G : \varphi(h) \in H' \}$.

Thus, $H$ is a subgroup of $G$ that contains $K$.

- If $H'$ is normal, then $H$ is normal.
- If $\varphi$ is surjective and $H$ is normal, then $H'$ is normal.

Proof: $H \leq K = \varphi^{-1}(1) \leq \varphi^{-1}(H')$

- $1 \in H$.
  - $x, y \in H$. $\Rightarrow \varphi(x), \varphi(y) \in H'$.
    - $\varphi(xy) = \varphi(x)\varphi(y) \in H' \Rightarrow \varphi(1) \in H'$.
\[ x'y' \in H' \]

- \( x \in H \Rightarrow \varphi(x) \in H' \)
  \[ \varphi(x') = \varphi(x)^{-1} \in H' \text{ b/c } H' \text{ is closed under inverse} \]
  \[ x' \in H \]

Suppose \( H' \) normal: \( h \in H, g \in G \):

\[ g(h^{-1}) = \frac{\varphi(g) \varphi(h) \varphi(g)^{-1}}{\in H'} \text{ b/c } H' \text{ is normal} \]

\[ g h g^{-1} \in H. \Rightarrow H' \text{ is normal. } \]

Suppose \( \varphi \) is surjective \& \( H \) is normal \( \Rightarrow \) \( H' \) is normal. \( \blacksquare \)

Then \( \varphi': G \rightarrow G' \) is surjective \& homomorphism w/ kernel \( K \).

Then

```
{ } \xrightarrow{\{ \}} \{ \}
```

These maps are inverse to one another and induce a bijection between the two sets above.

If \( H \) is normal, so is \( H' \). Moreover, \( |H'| = |K||H'| \).

PF: We've essentially checked this theorem already.
PF: We've essentially checked this theorem already.

Still need to check: \( \varphi'( \varphi(H) ) = H \)

\[ \varphi( \varphi'(H') ) = H' \]

\[ \varphi( \varphi^{-1}(H') ) = H' \quad \text{True for any surjective map of sets } \varphi \]

\[ \varphi'( \varphi(H) ) = H' \quad \text{Claim: } H \subseteq \varphi^{-1}(\varphi(H)) \]

Conversely, must check: \( \varphi^{-1}(\varphi(H)) \leq H \).

Suppose \( x \in \varphi^{-1}(\varphi(H)) \), \( \Rightarrow \varphi(x) \in \varphi(H) \)

\[ \Rightarrow \exists h \in H \text{ s.t. } \varphi(x) = \varphi(h). \]

\[ \Rightarrow \varphi(h^-1 x) = 1 \Rightarrow h^{-1} x \in K. \]

But \( K \subseteq H \Rightarrow h^{-1} x \in H \Rightarrow x \in H = H \quad \blacksquare \)