1. Review

1.1. Power series. A power series about a point $x_0$ is an infinite sum

$$ f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. $$

Above, $x$ is a variable and $a_n$ are constants. The power series converges at $x = c$ if the corresponding infinite sum converges. Otherwise the series is said to diverge.

A series converges absolutely if the sum

$$ \sum_{n=0}^{\infty} |a_n (x - x_0)^n| $$

converges. We use the following terminology for convergence:

- absolute convergence: (1) and (2) converge.
- conditional convergence: (1) converges but (2) does not.
- divergence: both (1) and (2) do not converge.

A real number $\rho$ is the radius of convergence of the series (1) if for all $|x - x_0| < \rho$, the (1) converges absolutely. We distinguish two special cases:

- $\rho = 0$ if the series converges only for $x = x_0$ (i.e., the series diverges for general $x$).
- $\rho = \infty$ when the series converges for all real numbers $x$.

\textbf{Caution 1.} Note that even when $\rho = 0$, the series still converges for $x_0$. Hence, in this case, it is incorrect to state the series diverges for all $x$.

If $a_n$ are eventually non-zero and

$$ \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = L $$

where $0 \leq L \leq \infty$, then $\rho = L$.

We can add and multiply power series using the following formulas

(1) (Addition)

$$ \sum_{n=0}^{\infty} a_n (x - x_0)^n + \sum_{n=0}^{\infty} b_n (x - x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n. $$

(2) (Multiplication)

$$ \left( \sum_{n=0}^{\infty} a_n (x - x_0)^n \right) \cdot \left( \sum_{n=0}^{\infty} b_n (x - x_0)^n \right) = \sum_{n=0}^{\infty} c_n (x - x_0)^n $$
where
\[ c_n = \sum_{i+j=n} a_i b_j. \]

This just means that \( c_n \) is just the sum of all indices that add up to \( n \).

If the infinite sum in (1) converges to a values of a function \( f(x) \) for all \( x \) in some positive interval around \( x_0 \), then \( f \) is said to be analytic at \( x_0 \). In this case, within the radius of convergence, we can compute
\[
f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}
\]
\[
\int f(x) dx = \sum_{n=0}^{\infty} \int a_n (x-x_0)^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-x_0)^{n+1} + C.
\]

Some common power series expansions of analytic functions:

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]
\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \]
\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \]
\[ \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 + \cdots \]
\[ \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \]

**Remark 1.** We can derive the expressions for \( \sin \) and \( \cos \) using \( e^{it} = \cos(t) + i \sin(t) \) and the series formula for \( e^z \). This is a helpful exercise.

1.2. **Solving differential equations using power series.** Suppose we are given a second order equation in standard form
\[ y'' + p(x) y' + q(x)y = 0. \]

A point \( x_0 \) is called ordinary if both \( p \) and \( q \) are analytic at \( x_0 \). A point that is not ordinary is called singular.

If the above differential equation is given and \( x_0 \) is an ordinary point, then we can obtain a power series solution to the differential equation using the following steps

1. Assume that \( y \) is also analytic at \( x_0 \) and is given by
\[ y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n. \]

2. Expand \( p(x) \) and \( q(x) \) as power series around \( x_0 \).

3. Take derivatives of \( y \) term-by-term and obtain an infinite sum corresponding to the differential equation (re-indexing whenever necessary).

4. Starting with \( n = 0 \), recursively solve for \( a_n \) for all \( n \).
1.3. **Matrices and vectors.** We use the notation \( A = [a_{ij}] \) to indicate that \( A \) is a matrix with entries \( a_{ij} \). Instead of constants, we can let \( a_{ij} \) be functions of some variable \( t \) and let

\[ A(t) = [a_{ij}(t)]. \]

All the usual rules for matrix operations apply for these matrices. A matrix will be called **constant** if all its entries are constant functions of \( t \).

Such a matrix is differentiable at \( t_0 \) if all entries \( a_{ij}(t) \) are differentiable at \( t_0 \). In this case the derivative of \( A \) is again a matrix given by

\[ \frac{dA}{dt}(t_0) = A'(t_0) = [a'_{ij}(t_0)]. \]

Similarly,

\[ \int_a^b A(t) \, dt = \left[ \int_a^b a_{ij}(t) \, dt \right]. \]

Differentiation of matrices satisfies the following properties

1. **(Linearity)** Suppose \( \alpha \) and \( \beta \) are constant matrices. Then

\[ \frac{d}{dt} [\alpha A + \beta B] = \alpha \frac{dA}{dt} + \beta \frac{dB}{dt}. \]

2. **(Product rule)**

\[ \frac{d}{dt} [AB] = A \frac{dB}{dt} + \frac{dA}{dt} B. \]

**Caution 2.** Recall that matrix multiplication is not necessarily commutative. Hence we need to be careful about the order of matrix multiplication in the product rule!

2. **Problems**

**Problem 1.** Find the radius of convergence of

\[ \sum_{n=0}^{\infty} \frac{(2n)!}{[n!]^2} x^n. \]

**Solution.** Letting \( a_n = \frac{(2n)!}{[n!]^2} \) and using the ratio test,

\[ \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{(2n)!}{(n!)^2} \cdot \frac{(n+1)^2}{(2n+2)!} = \lim_{n \to \infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}. \]

So the radius of convergence is \( \rho = \frac{1}{4} \).

**Problem 2.** Starting from the geometric series and the power series of \( e^x \), use operations on series (substitution, addition, multiplication, differentiation, integration), to find a series for \( \tan^{-1}(x) \).

**Solution.** We recall that

\[ \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \]
Integrating both sides,

$$\tan^{-1}(x) = \sum_{0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1} + C.$$  

Using the substitution $x = 0$, we get $C = 0$.

**Problem 3.** Show that

$$y = \sum_{0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}$$  

solves the differential equation

$$y'' - y = 0.$$  

**Solution.** Differentiating term-by-term, we get

$$y' = \sum_{0}^{\infty} \frac{(2n + 1)x^{2n}}{(2n + 1)!} = \sum_{0}^{\infty} \frac{x^{2n}}{(2n)!},$$

$$y'' = \sum_{1}^{\infty} \frac{2nx^{2n-1}}{(2n)!} = \sum_{1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = \sum_{0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}.$$  

This shows that $y = y''$ or $y'' - y = 0$.

**Problem 4.** For the nonlinear initial value problem

$$y' = x + y^2, \quad y(0) = 1,$$  

find the first four nonzero terms of a series solution $y = \sum_{0}^{\infty} a_n x^n$.

**Solution.** Since we only care about the first four terms, we may write

$$y = a_0 + a_1 x + a_2 x^2 + a_3^2 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots.$$  

From the initial conditions, $y(0) = a_0 = 1$. Now,

$$y' = (1 + a_1 x + a_2 x^2 + a_3^2 + \cdots) \cdot (1 + a_1 x + a_2 x^2 + a_3^2 + \cdots)$$

$$= 1 + 2a_1 x + (2a_2 + a_1^2) x^2 + (2a_3 + 2a_2 a_1) x^3 + \cdots.$$  

The given differential equation implies that

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 = 1 + (2a_1 + 1)x + (2a_2 + a_1^2) x^2 + (2a_3 + 2a_2 a_1) x^3 + \cdots.$$  

Equating, the relevant coefficients

$$a_1 = 1,$$  

$$2a_2 = 2a_1 + 1 = 3 \implies a_2 = \frac{3}{2},$$

$$3a_3 = 2a_2 + a_1^2 = 4 \implies a_3 = \frac{4}{3}.$$  

So,

$$y = 1 + x + \frac{3}{2} x^2 + \frac{4}{3} x^3 + \cdots.$$  

**Problem 5.** Find a series solution to the initial value problem

$$(1 - x)y' - y = 0, \quad y(0) = 1.$$
Solution. We first compute the relevant power series
\[ y = \sum_{n=0}^{\infty} a_n x^n \]
\[ y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \]
\[ xy' = \sum_{n=0}^{\infty} n a_n x^n \]

So, using the given differential equation, and equating the coefficient of \( x^n \) to 0, we get
\[(n+1)a_{n+1} - na_n - a_n = 0 \implies a_{n+1} = \frac{(n+1)a_n}{(n+1)} = a_n.\]

Since \( y(0) = 1 \), \( a_0 = 1 \) and \( a_n = 1 \) for all \( n \). We conclude
\[ y = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}. \]

Problem 6. Find two independent power series \( \sum a_n x^n \) solutions to \( y'' - 4y = 0 \) by obtaining a recursively formula for the \( a_n \).
Solution. If \( y = \sum_{n=0}^{\infty} a_n x^n \), then
\[ y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n. \]
So, using the differential equation and equating the coefficients of \( x^n \) to zero, we obtain
\[ a_{n+2}(n+2)(n+1) - 4a_n = 0 \implies a_{n+1} = \frac{4a_n}{(n+1)(n+1)}. \]

So, for even indices, we have
\[ a_2 = \frac{4}{2 \cdot 1} \cdot a_0, \]
\[ a_4 = \frac{4}{4 \cdot 3} \cdot \frac{4}{2 \cdot 1} \cdot a_0 = \frac{4^2}{4!} a_0, \cdots \]

Similarly, for the odd indices,
\[ a_3 = \frac{4}{3 \cdot 2} \cdot a_1, \]
\[ a_5 = \frac{4}{5 \cdot 4} \cdot \frac{4}{3 \cdot 2} a_1 = \frac{4^2}{5!} a_1, \cdots \]

Hence, the two independent solutions are given by series comprising of the even and odd indices respectively. That is
\[ y_e = 1 + \frac{4}{2!} x^2 + \frac{4^2}{4!} x^4 + \frac{4^3}{6!} x^6 + \cdots = \sum_{n=0}^{\infty} \frac{4^n}{n!} x^{2n} \]
\[ y_o = x + \frac{4}{3!} x^3 + \frac{4^2}{5!} x^5 + \cdots = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n+1} \]
are the two independent solutions.