MAR. 09 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

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1. Review

1.1. Power series. A **power series** about a point x_0 is an infinite sum

$$f(x) = \sum_{n=0}^{\infty} a_k (x - x_0)^k. \tag{1}$$

Above, x is a variable and a_k are constants. The power series converges at $x = c$ if the corresponding infinite sum converges. Otherwise the series is said to diverge. A series converges **absolutely** if the sum

$$\sum_{n=0}^{\infty} |a_k (x - x_0)^k| \tag{2}$$

converges. We use the following terminology for convergence:

- absolute convergence: (1) and (2) converge.
- conditional convergence: (1) converges but (2) does not.
- divergence: both (1) and (2) do not converge.

A real number ρ is the **radius of convergence** of the series (1) if for all $|x - x_0| < \rho$, the (1) converges absolutely. We distinguish two special cases:

- $\rho = 0$ if the series converges only for $x = x_0$ (i.e., the series diverges for general x).
- $\rho = \infty$ when the series converges for all real numbers x.

Caution 1. *Note that even when $\rho = 0$, the series still converges for x_0. Hence, in this case, it is incorrect to state the series diverges for all x.*

If a_n are eventually non-zero and

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = L$$

where $0 \leq L \leq \infty$, then $\rho = L$.

We can add and multiply power series using the following formulas

(1) (Addition)

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n + \sum_{n=0}^{\infty} b_n (x - x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n.$$

(2) (Multiplication)

$$\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n (x - x_0)^n \right) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where

$$c_n = \sum_{i+j=n} a_i b_j.$$

This just means that c_n is just the sum of all indices that add up to to n.

If the infinite sum in (1) converges to a values of a function $f(x)$ for all x in some positive interval around x_0, then f is said to be **analytic** at x_0. In this case, within the radius of convergence, we can compute

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x - x_0)^n = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$\int f(x) dx = \sum_{n=0}^{\infty} \int a_n (x - x_0)^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C.$$

Some common power series expansions of analytic functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

Remark 1. *We can derive the expressions for* $\sin$ *and* $\cos$ *using* $e^{it} = \cos(t) + i\sin(t)$ *and the series formula for* e^x*. This is a helpful exercise.*

1.2. **Solving differential equations using power series.** Suppose we are given a second order equation in standard form

$$y'' + p(x)y' + q(x)y = 0.$$

A point x_0 is called **ordinary** if both p and q are analytic at x_0. A point that is not ordinary is called **singular**.

If the above differential equation is given and x_0 is an ordinary point, then we can obtain a power series solution to the differential equation using the following steps

(1) Assume that y is also analytic at x_0 and is given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

(2) Expand $p(x)$ and $q(x)$ as power series around x_0.
(3) Take derivatives of y term-by-term and obtain an infinite sum corresponding to the differential equation (re-indexing whenever necessary).
(4) Starting with $n = 0$, recursively solve for a_n for all n.

1.3. **Matrices and vectors.** We use the notation $A = [a_{ij}]$ to indicate that A is a matrix with entries a_{ij}. Instead of constants, we can let a_{ij} be functions of some variable t and let

$$A(t) = [a_{ij}(t)].$$

All the usual rules for matrix operations apply for these matrices. A matrix will be called **constant** if all its entries are constant functions of t.

Such a matrix is differentiable at t_0 is all entries $a_{ij}(t)$ are differentiable at t_0. In this case the derivative of A is again a matrix given by

$$\frac{dA}{dt}(t_0) = A'(t_0) = [a'_{ij}(t_0)].$$

Similarly,

$$\int_a^b A(t)dt = \left[\int_a^b a_{ij}(t)dt \right].$$

Differentiation of matrices satisfies the following properties

(1) (Linearity) Suppose α and β are constant matrices. Then
$$\frac{d}{dt}\left[\alpha A + \beta B\right] = \alpha \frac{dA}{dt} + \beta \frac{dB}{dt}.$$

(2) (Product rule)
$$\frac{d}{dt}\left[AB\right] = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

Caution 2. *Recall that matrix multiplication is not necessarily commutative. Hence we need to be careful about the order of matrix multiplication in the product rule!*

2. Problems

Problem 1. Find the radius of convergence of

$$\sum_0^\infty \frac{(2n)!}{(n!)^2} x^n.$$

Solution. Letting $a_n = \frac{(2n)!}{(n!)^2}$ and using the ratio test,

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(2n)!}{(n!)^2} \cdot \frac{[(n+1)!]^2}{(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

So the radius of convergence is $\rho = \frac{1}{4}$.

Problem 2. Starting from the geometric series and the power series of e^x, use operations on series (substitution, addition, multiplication, differentiation, integration), to find a series for $\tan^{-1}(x)$.

Solution. We recall that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} = \sum_0^\infty (-1)^n x^{2n}.$$

Integrating both sides,

$$\tan^{-1}(x) = \sum_0^\infty \frac{(-1)^n}{2n+1} x^{2n+1} + C.$$

Using the substitution $x = 0$, we get $C = 0$.

Problem 3. Show that

$$y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$$

solves the differential equation

$$y'' - y = 0.$$

Solution. Differentiating term-by-term, we get

$$\begin{aligned} y' &= \sum_0^\infty \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_0^\infty \frac{x^{2n}}{(2n)!}, \\ y'' &= \sum_1^\infty \frac{2nx^{2n-1}}{(2n)!} = \sum_1^\infty \frac{x^{2n-1}}{(2n-1)!} = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

This shows that $y = y''$ or $y'' - y = 0$.

Problem 4. For the nonlinear initial value problem

$$y' = x + y^2, \quad y(0) = 1,$$

find the first four nonzero terms of a series solution $y = \sum_0^\infty a_n x^n$.
Solution. Since we only care about the first four terms, we may write

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3^3 + \cdots \\ y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots . \end{aligned}$$

From the initial conditions, $y(0) = a_0 = 1$. Now,

$$\begin{aligned} y^2 &= (1 + a_1 x + a_2 x^2 + a_3^3 + \cdots) \cdot (1 + a_1 x + a_2 x^2 + a_3^3 + \cdots) \\ &= 1 + 2a_1 x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_2 a_1)x^3 + \cdots \end{aligned}$$

The given differential equation implies that

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 = 1 + (2a_1 + 1)x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_2 a_1)x^3 + \cdots$$

Equating, the relevant coefficients

$$\begin{aligned} &a_1 = 1, \\ &2a_2 = 2a_1 + 1 = 3 \implies a_2 = \frac{3}{2}, \\ &3a_3 = 2a_2 + a_1^2 = 4 \implies a_3 = \frac{4}{3}. \end{aligned}$$

So,

$$y = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \cdots$$

Problem 5. Find a series solution to the initial value problem

$$(1 - x)y' - y = 0, \quad y(0) = 1.$$

Solution. We first compute the relevant power series

$$y = \sum_0^\infty a_n x^n$$
$$y' = \sum_0^\infty n a_n x^{n-1} = \sum_0^\infty (n+1) a_{n+1} x^n$$
$$xy' = \sum_0^\infty n a_n x^n$$

So, using the given differential equation, and equating the coefficient of x^n to 0, we get

$$(n+1)a_{n+1} - na_n - a_n = 0 \implies a_{n+1} = \frac{(n+1)a_n}{(n+1)} = a_n.$$

Since $y(0) = 1$, $a_0 = 1$ and $a_n = 1$ for all n. We conclude

$$y = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Problem 6. Find two independent power series $\sum a_n x^n$ solutions to $y'' - 4y = 0$ by obtaining a recursively formula for the a_n.

Solution. If $y = \sum_0^\infty a_n x^n$, then

$$y'' = \sum_2^\infty a_n n(n-1)x^{n-2} = \sum_0^\infty a_{n+2}(n+2)(n+1)x^n.$$

So, using the differential equation and equating the coefficients of x^n to zero, we obtain

$$a_{n+2}(n+2)(n+1) - 4a_n = 0 \implies a_{n+1} = \frac{4a_n}{(n+1)(n+1)}.$$

So, for even indices, we have

$$a_2 = \frac{4}{2 \cdot 1} \cdot a_0,$$
$$a_4 = \frac{4}{4 \cdot 3} \cdot \frac{4}{2 \cdot 1} \cdot a_0 = \frac{4^2}{4!} a_0, \cdots$$

Similarly, for the odd indices,

$$a_3 = \frac{4}{3 \cdot 2} a_1,$$
$$a_5 = \frac{4}{5 \cdot 4} \cdot \frac{4}{3 \cdot 2} a_1 = \frac{4^2}{5!} a_1, \cdots$$

Hence, the two independent solutions are given by series comprising of the even and odd indices respectively. That is

$$y_e = 1 + \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 + \frac{4^3}{6!}x^6 + \cdots = \sum_0^\infty \frac{4^n}{n!} x^{2n}$$
$$y_o = x + \frac{4}{3!}x^3 + \frac{4^2}{5!}x^5 + \cdots = \sum_0^\infty \frac{4^n}{(2n+1)!} x^{2n+1}$$

are the two independent solutions.