MAR. 02 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

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1. Review

1.1. Laplace Transform and Discontinuous Functions. A **unit step function** $u(t)$ is a function defined by

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

We may tweak the location and size of the step using the transforms

- $u(t-a)$ has the step at a, and
- $Mu(t)$ has a step with height M.

The **rectangular window function** $\Pi_{a,b}(t)$ is a function that is 1 on the interval (a, b) and zero elsewhere and is defined by

$$\Pi_{a,b}(t) = u(t-a) - u(t-b).$$

If $f(t)$ is any piece-wise continuous function we may write it as a sum of products of the "pieces" with $\Pi_{a,b}(t)$ and $u(t)$. If $a \geq 0$, then the Laplace transforms of these products is given by

$$\mathcal{L}\left\{f(t-a)u(t-a)\right\}(s) = e^{-as}\mathcal{L}\left\{f\right\}(s)$$

and the Laplace inverse of $e^{-as}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a).$$

1.2. Convolutions. The **convolution** of two piecewise continuous functions $f(t)$ and $g(t)$ is given by the integral

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv.$$

We may check that convolution satisfies the following properties

(1) $f * g = g * f$,
(2) $f * (g + h) = f * g + f * h$,
(3) $(f * g) * h = f * (g * h)$,
(4) $f * 0 = 0$.

Convolution of two functions behaves especially well with respect to the Laplace transform. If f, g are exponential of order α, then

$$\mathcal{L}\left\{f * g\right\}(s) = \mathcal{L}\left\{f\right\}(s) \cdot \mathcal{L}\left\{g\right\}(s).$$

Similarly,

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\}(t) = (\mathcal{L}^{-1}\left\{F\right\} * \mathcal{L}^{-1}\left\{G\right\})(t).$$

1.2.1. *Solutions using Impulse Response Function.* Suppose we are given the second-order constant coefficient initial value problem

$$ay'' + by' + cy = g; \qquad y(0) = y_0, \quad y'(0) = y_1.$$

Then the unique solution to the differential equation is given by

$$y(t) = (h * g)(t) + y_k(t)$$

where $h(t)$ is the **impulse response function**

$$h(t) = \mathcal{L}^{-1}\left\{H\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}(t)$$

and y_k is the unique solution to the homogeneous initial value problem

$$ay'' + by' + cy = 0; \qquad y(0) = y_0, \quad y'(0) = y_1.$$

Above, we call $H(s)$ the **transfer function**.

1.3. **Dirac Delta.** The **Dirac delta distribution** can be (loosely speaking) thought of as a "function" that is zero everywhere except at the origin, i.e.,

$$\delta(t) = \begin{cases} +\infty, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

We also require $\delta(t)$ to satisfy

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

By translating the argument, we may "pick up" different values of f,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a).$$

The Laplace transform of the delta distribution is given by

$$\mathcal{L}\left\{\delta(t-a)\right\}(s) = e^{-as}$$

for $a \geq 0$.

2. Problems

Problem 1. Compute $\mathcal{L}\left\{f\right\}$ where

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2-t, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

Solution. We first express $f(t)$ in terms of $u(t)$.

$$\begin{aligned} f(t) &= t\Pi_{0,1}(t) + (2-t)\Pi_{1,2}(t) \\ &= t[u(t) - u(t-1)] + (2-t)[u(t-1) - u(t-2)] \\ &= tu(t) - u(t-2) - 2u(t-1) + (t-2)u(t-2). \end{aligned}$$

Using the relevant formula for $\mathcal{L}\left\{f(t-a)u(t-a)\right\}$ for various a and using linearity,

$$\mathcal{L}\left\{f\right\}(s) = \mathcal{L}\left\{tu(t)\right\} - \mathcal{L}\left\{u(t-2)\right\} - 2\mathcal{L}\left\{u(t-1)\right\} + \mathcal{L}\left\{(t-2)u(t-2)\right\} = \frac{1}{s^2}\left(1 - 2e^{-s} + e^{2s}\right)$$

Problem 2. Compute $\mathcal{L}\{f\}$ where $f(t) = |\sin(t)|$ for $t \geq 0$.

Solution. Using periodicity, we may write

$$f(t) = |\sin(t)| = (-1)^n \sin(t), \quad n\pi \leq t \leq (n+1)\pi = \sum_{n=0}^{\infty} (-1)^n \sin(t)\Pi_{n\pi,(n+1)\pi}(t).$$

For a fixed n, we have

$$\begin{aligned}\mathcal{L}\left\{\sin(t)\Pi_{n\pi,(n+1)\pi}(t)\sin(t)\right\} &= \mathcal{L}\{\sin(t)u(t-n\pi)\} - \mathcal{L}\{\sin(t)u(t-(n+1)\pi)\} \\ &= e^{-sn\pi}\mathcal{L}\{\sin(t+n\pi)\} - e^{-s(n+1)\pi}\mathcal{L}\{\sin(t+(n+1)\pi)\} \\ &= (-1)^n e^{-sn\pi}\mathcal{L}\{\sin(t)\} - (-1)^{n+1}e^{-s(n+1)\pi}\mathcal{L}\{\sin(t)\} \\ &= (-1)^n e^{-sn\pi}\left(\frac{1+e^{-s\pi}}{1+s^2}\right).\end{aligned}$$

Hence, we conclude

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \sum_{n=0}^{\infty}(-1)^n \mathcal{L}\left\{\sin(t)\Pi_{n\pi,(n+1)\pi}(t)\right\} = \sum_{n=0}^{\infty} e^{-sn\pi}\left(\frac{1+e^{-s\pi}}{1+s^2}\right) \\ &= \left(\frac{1+e^{-s\pi}}{1+s^2}\right)\sum_{n=0}^{\infty} e^{-sn\pi} = \left(\frac{1+e^{-s\pi}}{1+s^2}\right)\frac{1}{1-e^{-s\pi}}.\end{aligned}$$

In the last step we treat the sum as a convergent geometric series with $r = e^{-s\pi}$.

Problem 3. Solve the initial value problem

$$y'' + 2y + 2y = h(t); \qquad y(0) = 0, \quad y'(0) = 1$$

where

$$h(t) = \begin{cases} 1, & \pi \leq t \leq 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

Solution. Writing $h(t) = \Pi_{\pi,2\pi}(t) = u(t-\pi) - u(t-2\pi)$, we see that

$$\mathcal{L}\{h(t)\}(s) = \frac{e^{-s\pi} - e^{-2s\pi}}{s}.$$

If $Y = \mathcal{L}\{y\}$ using the initial values,

$$(s^2Y - 1) + 2(sY) + 2Y = \frac{e^{-s\pi} - e^{-2s\pi}}{s} \implies Y = \frac{1}{(s+1)^2+1}\left[1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s}\right]$$

Using partial fractions, we may write the denominator above as

$$\frac{1}{((s+1)^2+1)s} = \frac{1}{(s^2+2s+2)s} = \frac{-\frac{1}{2}(s+1) - \frac{1}{2}}{(s+1)^2+1} + \frac{\frac{1}{2}}{s}.$$

Taking the inverse Laplace transform, we get

$$\begin{aligned} y &= e^{-t}\sin(t) \\ &\quad + \frac{1}{2}\left[1 - e^{-(t-\pi)}(\sin(t-\pi) + \cos(t-\pi))\right] u(t-\pi) \\ &\quad - \frac{1}{2}\left[1 - e^{-(t-2\pi)}(\sin(t-2\pi) + \cos(t-2\pi))\right] u(t-2\pi) \\ &= e^{-t}\sin(t) \\ &\quad + \frac{1}{2}\left[1 + e^{-(t-\pi)}(\sin(t) + \cos(t))\right] u(t-\pi) \\ &\quad - \frac{1}{2}\left[1 - e^{-(t-2\pi)}(\sin(t) + \cos(t))\right] u(t-2\pi). \end{aligned}$$

Problem 4. Solve the initial value problem

$$y'' - 3y' + 2y = tu(t); \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution. Note that we have

$$\mathcal{L}\{tu(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}.$$

Taking $\mathcal{L}\{\cdot\}$ on both sides and setting $\mathcal{L}\{y\} = Y$, we get

$$(s^2Y - s) - 3(sY - 1) + 2Y = \frac{1}{s^2} \implies Y = \frac{s-3}{(s-2)(s-1)} + \frac{1}{s^2(s-2)(s-1)}.$$

Using partial fractions

$$Y = \frac{s^3 - 3s^2 + 1}{s^2(s-2)(s-1)} = \frac{1}{s-1} - \frac{\frac{3}{4}}{s-2} + \frac{\frac{3}{4}}{s} + \frac{\frac{1}{2}}{s^2}.$$

Computing $\mathcal{L}^{-1}\{Y\}$, we get

$$y = \mathcal{L}^{-1}\{Y\} = e^t - \frac{3}{4}e^{2t} + \frac{3}{4} + \frac{t}{2}.$$

Problem 5. Use convolution to compute

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\}.$$

Solution.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} \cdot \frac{s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &= e^{-t} * \cos(2t) = \int_0^t e^{-(t-v)}\cos(2v)dv \\ &= e^{-t}\int_0^t e^{-v)}\cos(2v)dv = e^{-t}\left[\frac{e^t}{5}(\cos(2t) + 2\sin(2t)) - \frac{1}{5}\right] \\ &= \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) - \frac{1}{5}e^{-t}. \end{aligned}$$

Problem 6. Use convolution to compute
$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}.$$

Solution.
$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t) * \sin(t) \\ &= \int_0^t \sin(t-v)\sin(v)dv = \frac{1}{2}\int_0^t \cos(t-2v) - \cos(t)dv \\ &= \frac{\sin(t)}{2} - \frac{t}{2}\cos(t).\end{aligned}$$

Problem 7. Solve the initial value problem
$$y'' + 2y' + y = \delta(t) + u(t-1); \qquad y(0) = 1, \quad y'(0) = 1.$$

Solution. Taking the Laplace transform of the right hand side, we get
$$\mathcal{L}\{\delta(t) + u(t-1)\} = 1 + \frac{e^{-s}}{s}.$$
With $Y = \mathcal{L}\{y\}$, we see that
$$(s^2Y - 1) + 2sY + Y = (s^2 + 2s + 1)Y - 1 = 1 + \frac{e^{-s}}{s}.$$
So,
$$Y = \frac{2}{(s+1)^2} + e^{-s}\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right].$$
Taking the inverse transform,
$$y(t) = 2te^{-t} + \left[1 - e^{-(t-1)} - (t-1)e^{-(t-1)}\right]u(t-1) = 2te^{-t} + [1 - te^{1-t}]u(t-1).$$