FEB. 23 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

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1. Review

Recall that the **Laplace transform** of $f(t)$ defined on $[0, \infty)$ is given by

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt.$$

If $F(s)$ is given then $f(t)$ satisfying $\mathcal{L}\{f\} = F$ is called a **Laplace inverse** of F and is denoted by $\mathcal{L}^{-1}\{F\} = f$. A function $f(t)$ is **exponential of order** α if there are constants T and M such that for all $t \geq T$,

$$|f(t)| \leq Me^{\alpha t}.$$

If α is chosen as above, then the Laplace transform $\mathcal{L}\{f\}(s)$ always exists for $s > \alpha$. Unless stated otherwise, we will assume $s > \alpha$.

1.1. **Some common Laplace transforms.** The following table lists some common Laplace transforms and the intervals where they are valid.

$f(t)$	$\mathcal{L}\{f\}(s)$	Conditions on s
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n, n > 0$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n, n > 0$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

1.2. Properties of the Laplace Transform. When it exists, the Laplace transform satisfies the following properties (α is assumed to be the exponential order of f).

(1) (Linearity)

$$\mathcal{L}\left\{c_1 f_1 + c_2 f_2\right\} = c_1 \mathcal{L}\left\{f_1\right\} + c_2 \mathcal{L}\left\{f_2\right\}$$

for constants $c_{1,2}$.

(2) (Translation in s)

$$\mathcal{L}\left\{e^{at} f(t)\right\}(s) = F(s-a)$$

for $s > \alpha + a$.

(3) (Laplace Transform of the Derivative)

$$\mathcal{L}\left\{f'\right\}(s) = s\mathcal{L}\left\{f\right\}(s) - f(0).$$

(4) (Laplace Transform of Higher-Order Derivatives)

$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n \mathcal{L}\left\{f\right\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

(5) (Derivatives of the Laplace Transform)

$$\mathcal{L}\left\{t^n f(t)\right\}(s) = (-1)^n \frac{d^n F}{ds^n}(s).$$

1.3. Inverse Laplace Transform. Suppose $F(s) = P(s)/Q(s)$ where $\deg(P) < \deg(Q)$. We can compute the Laplace inverse of F by first factoring $Q(s)$ and then using the method of partial fractions. The final Laplace inverse is computed by then using the linearity of the inverse transform. We consider three cases

(1) $Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)$ where all r_k are distinct. Then the partial expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \cdots + \frac{A_n}{s-r_n}.$$

(2) If $Q(s) = (s-r)^m$, then

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_m}{(s-r)^m}.$$

(3) If $Q(s) = [(s-\alpha)^2 + \beta^2]^m$ where $(s-\alpha)^2 + \beta^2$ cannot be reduced into linear factors with real coefficients, then

$$\frac{P(s)}{Q(s)} = \frac{C_1 s + D_1}{(s-\alpha)^2 + \beta^2} + \frac{C_2 s + D_2}{[(s-\alpha)^2 + \beta^2]^2} + \cdots + \frac{C_m s + D_m}{[(s-\alpha)^2 + \beta^2]^m}.$$

1.4. Solving differential equations using Laplace Transforms. Using the Laplace transform, we can now solve linear differential equations. The main strategy is summarised below

(1) Take the Laplace transform of both sides
(2) Use properties of the Laplace transform and initial conditions to obtain an equation involving the Laplace transform of the solution.
(3) Apply the inverse transform to obtain the final solution.

2. Problems

Problem 1. Compute

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+5}\right\}.$$

Solution. Completing the square in the denominator, we get

$$\frac{s+1}{s^2-4s+5}=\frac{s+1}{(s-2)^2+1}.$$

Since the denominator looks resembles the denominator for a translation in s by 2, we express the numerator in terms of $s-2$ to get

$$\frac{s+1}{s^2-4s+5}=\frac{s-2}{(s-2)^2+1}+\frac{3}{(s-2)^2+1}.$$

Now, using linearity and the table of Laplace transforms,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+5}\right\}&=\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}\\&=e^{2t}\cos(t)+3e^{2t}\sin(t).\end{aligned}$$

Problem 2. Solve the initial value problem using the Laplace transform

$$y'-y=e^{3t},\qquad y(0)=1.$$

Solution. Taking the Laplace transform on both sides,

$$\begin{aligned}s\mathcal{L}\{y\}-1-\mathcal{L}\{y\}&=\frac{1}{s-3}\\ \implies (s-1)\mathcal{L}\{y\}&=1+\frac{1}{s-3}\\ \implies \mathcal{L}\{y\}&=\frac{1}{s-1}+\frac{1}{(s-3)(s-1)}.\end{aligned}$$

Now, using partial fraction decomposition of the second summand,

$$\frac{1}{(s-3)(s-1)}=\frac{1}{2}\left[\frac{1}{s-3}-\frac{1}{s-1}\right].$$

Hence,

$$\mathcal{L}\{y\}=\frac{1}{2}\cdot\frac{1}{s-1}+\frac{1}{2}\cdot\frac{1}{s-3}.$$

Taking the inverse transform of the summands (using linearity),

$$y=\frac{1}{2}e^{t}+\frac{1}{2}e^{3t}.$$

Problem 3. Solve the initial value problem using the Laplace transform

$$y''+4y=\sin(t),\qquad y(0)=1,y'(0)=0.$$

Solution. Taking the Laplace transform on both sides

$$(s^2\mathcal{L}\{y\} - s) + 4\mathcal{L}\{y\} = \frac{1}{s^2+1} \implies \mathcal{L}\{y\} = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}.$$

It is easier to simply the first summand using partial fractions by setting $u = s^2$. In which case

$$\frac{1}{(u+1)(u+4)} = \frac{1}{3}\cdot\frac{1}{u+1} - \frac{1}{3}\cdot\frac{1}{u+4}.$$

Substituting, we obtain

$$\mathcal{L}\{y\} = \frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\cdot\frac{1}{s^2+4} + \frac{s}{s^2+4}.$$

Taking the inverse transform

$$y = \frac{1}{3}\sin(t) - \frac{1}{6}\sin 2t + \cos(2t).$$

Problem 4. Solve the initial value problem using the Laplace transform

$$y'' - 2y' + 2y = 2e^t, \qquad y(0) = 0, y'(0) = 1.$$

Solution. Taking the Laplace transform on both sides,

$$(s^2\mathcal{L}\{y\} - 1) - 2s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{2}{s-1} \implies \mathcal{L}\{y\} = \frac{s+1}{(s^2-2s+2)(s-1)}.$$

Using partial fractions and completing the square for the irreducible quadratic, we get

$$\mathcal{L}\{y\} = \frac{2}{s-1} - \frac{2(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}.$$

Now, taking the inverse transform

$$y = 2e^t - 2e^t\cos(t) + e^t\sin(t).$$