FEB. 02 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

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1. Review

1.1. Superposition Principle. Let

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y.$$

Suppose $y_1(t)$ is a solution to the differential equation $\mathcal{L}(y) = f_1$ and $y_2(t)$ is a solution to $\mathcal{L}(y) = f_2$. The superposition principle states that $c_1 y_1(t) + c_2 y_2(t)$ is a solution to $\mathcal{L}(y) = c_1 f_1 + c_2 f_2$.

We used the method of undetermined coefficients to solve equations where the inhomogeneity $f(t)$ had a particularly simple form. Using the superposition principle, we will be able to treat cases where the inhomogeneity $f(t)$ can be split into a sum of simpler terms.

1.2. Variation of Parameters. Suppose

$$a_2 y'' + a_1 y' + a_0 y = f(t)$$

is the given differential equation. The method of variation of parameters will help us find a particular solution to the above equation.

(1) Start by guessing a particular solution

$$y_P(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

where y_1 and y_2 are the two solutions to the homogeneous equation

$$a_2 y'' + a_1 y' + a_0 y = 0.$$

(2) Solve for v_1' and v_2' satisfying the system of equations

$$y_1 v_1' + y_2 v_2' = 0,$$
$$y_1' v_1' + y_2' v_2' = \frac{f(t)}{a_2}.$$

(3) Solve for $v_1(t)$ and $v_2(t)$ by taking integrals.

2. Problems

Problem 1. Find a particular solution to

$$y'' + 2y' + 2y = t + e^t.$$

Solution. Let $f_1(t) = t$ and $f_2(t) = e^t$. We may guess particular solutions

$$y_1(t) = A_1 t + A_0, \qquad y_2(t) = Be^t.$$

Since $f(t) = f_1(t) + f_2(t)$, the particular solution will be of the form $y_P(t) = y_1(t) + y_2(t) = A_1 t + A_0 + Be^t$. Substituting into the differential equation

$$5Be^t + 2A_1 t + (2A_1 + 2A_0) = t + e^t.$$

The non-trivial solutions for A_0, A_1, B are given by

$$B = \frac{1}{5}, A_0 = -\frac{1}{2}, A_1 = \frac{1}{2}.$$

Therefore the final particular solution is

$$y_P(t) = \frac{1}{2}(t-1) + \frac{1}{5}e^t.$$

Problem 2. Find a particular solution to

$$y'' - 3y' + 2y = e^{-t} + 3e^{2t}.$$

Solution. Note that the auxiliary equation to the corresponding homogeneous equation is given by

$$r^2 - 3r + 2 = 0 \implies (r-1)(r-2).$$

Let $f_1(t) = e^{-t}$ and $f_2(t) = e^{2t}$. Then we must guess the solutions

$$y_1(t) = Ae^{-t}$$

as -1 is not a root of the auxiliary equation. Substituting, we get

$$6Ae^{-t} = e^{-t} \implies A = \frac{1}{6}.$$

Similarly we have

$$y_2(t) = Bte^{2t}$$

as 2 is a simple root of the auxiliary equation. Substituting, we get

$$Be^{2t} = e^{2t} \implies B = 1.$$

Since $f(t) = f_1(t) + 3f_2(t)$, the final particular solution is given by

$$y_P(t) = y_1(t) + 3y_2(t) = \frac{1}{6}e^{-t} + 3te^{2t}.$$

Problem 3. Find a particular solution to using variation of parameters

$$y'' + y = \tan(t).$$

Solution. The auxiliary equation is

$$r^2 + 1 = 0 \implies r = \pm i.$$

Hence the homogeneous solution is given by

$$y_H(t) = c_1 \cos(t) + c_2 \sin(t).$$

We may set $y_1 = \cos(t)$ and $y_2 = \sin(t)$. We now need to solve

$$v_1' \cos(t) + v_2' \sin(t) = 0$$
$$-v_1' \sin(t) + v_2' \cos(t) = \tan(t)$$

Solving the above system, we get

$$v_1' = -\sin(t)\tan(t) = \cos(t) - \sec(t)$$
$$v_2' = \cos(t)\tan(t) = \sin(t).$$

Integrating, we obtain

$$v_1(t) = \int \cos(t) - \sec(t)dt = \sin(t) - \ln|\sec(t) + \tan(t)|$$
$$v_2(t) = \int \sin(t)dt = -\cos(t).$$

Therefore

$$y_P(t) = (\sin(t) - \ln|\sec(t) + \tan(t)|)\cos(t) - \cos(t)\sin(t) = \ln|\sec(t) + \tan(t)|\cos(t).$$

Problem 4. Find a particular solution to using variation of parameters

$$y'' + 2y' - 3y = e^{-t}.$$

Solution. The auxiliary equation is

$$r^2 + 2r - 3 = (r-1)(r+3) = 0 \implies r = 1, -3.$$

Hence the homogeneous solution is given by

$$y_H(t) = c_1 e^t + c_2 e^{a-3t}$$

We may set $y_1 = e^t$ and $y_2 = e^{-3t}$. We now need to solve

$$v_1' e^t + v_2' e^{-3t} = 0$$
$$v_1' e^t - 3v_2' e^{-3t} = e^{-t}$$

Solving the above system, we get

$$v_1' = \frac{1}{4}e^{-2t}$$
$$v_2' = -\frac{1}{4}e^{2t}.$$

Integrating, we obtain

$$v_1(t) = -\frac{1}{8}e^{-2t}$$
$$v_2(t) = -\frac{1}{8}e^{2t}.$$

Therefore

$$y_P(t) = -\frac{1}{8}e^{-2t}e^t - \frac{1}{8}e^{2t}e^{-3t} = -\frac{1}{4}e^{-t}.$$

Problem 5. Find a particular solution to using variation of parameters

$$y'' + 4y + \sec^2(2t).$$

Solution. The auxiliary equation is

$$r^2 + 4 = 0 \implies r = \pm 2i$$

Hence the homogeneous solution is given by

$$y_H(t) = c_1\cos(2t) + c_2\sin(2t).$$

We may set $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$. We now need to solve

$$v_1' \cos(2t) + v_2' \sin(2t) = 0$$
$$-2v_1' \sin(2t) + 2v_2' \cos(2t) = \sec^2(2t)$$

Solving the above system, we get

$$v_1' = -\frac{\sin(2t)}{2\cos^2(2t)}$$
$$v_2' = \frac{1}{2}\sec(2t).$$

Integrating, we obtain

$$v_1(t) = -\frac{1}{4\cos(2t)}$$
$$v_2(t) = \frac{1}{4}\ln|\sec(2t) + \tan(2t)|.$$

Therefore

$$y_P(t) = -\frac{1}{4} + \frac{1}{4}\ln|\sec(2t) + \tan(2t)|\sin(2t).$$