1. Review

1.1. Complex and repeated roots. Recall that to find a solution to a homogeneous linear equation

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0 \]

we started with the ansatz \( y = e^{rx} \) to get the equation

\[ p(r) e^{rx} = (a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) e^{rx} = 0 \implies p(r) = 0. \]

Above, the equation \( p(r) = 0 \) is called the auxiliary equation associated to the differential equation.

The case when \( p(r) \) is quadratic and has distinct real roots was discussed last week. We treat the remaining two cases below. If \( r_1, r_2 \) are roots of the auxiliary equation, then the different cases to determine the general solution are:

1. If both \( r_1 \) and \( r_2 \) are real and distinct then the general solution is given by

\[ y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \]

2. If \( r_1 = r_2 = r \) then the general solution is given by

\[ y(x) = c_1 e^{rx} + c_2 xe^{rx}. \]

3. If \( r_1, r_2 = \alpha + i\beta \) then the general (real) solution is given by

\[ y(x) = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x)). \]

Remark 1. If \( r_0 \in \mathbb{C} \) is a solution to \( p(r) = 0 \), then \( \bar{r}_0 \) is also a solution. Hence when \( p(r) \) is quadratic and \( r_0 \not\in \mathbb{R} \), \( r_0 \) and \( \bar{r}_0 \) are the only solutions. This explains why we only consider the case \( r_1, r_2 = \alpha \pm i\beta \) when \( p(r) = 0 \) has non-real solutions.

1.2. Method of Undetermined Coefficients. A non-homogeneous equation with constant coefficients is a differential equation of the form

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t). \]

Any solution to (1) is of the form \( y(t) = y_P(t) + y_H(t) \) where \( y_P(t) \) is any function that satisfies

\[ a_n \frac{d^n y_P}{dt^n} + a_{n-1} \frac{d^{n-1} y_P}{dt^{n-1}} + \cdots + a_1 \frac{dy_P}{dt} + a_0 y_P = f(t) \]

and \( y_H \) is the general solution to the homogeneous equation

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0. \]

We call \( y_P \) a particular solution to the in non-homogeneous equation (1).
The method of undetermined coefficients helps us make educated guesses about the particular solutions $y_P(t)$. The general strategy is

1. Make a guess for $y_P(t)$ with possibly unknown coefficients.
2. Substitute guess into (1) to determine values of the coefficients.

The main difficulty lies in finding $y_P(t)$. This is usually done by making educated guesses that (obviously) depends on $f(t)$. The sections below summarise the general strategy for determining $y_P(t)$ based on various forms of $f(t)$.

1.2.1. “Nice” $f(t)$. If $f(t)$ is “nice enough” we can make the following guesses:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>Guess for $y_P(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ce^rt$</td>
<td>$Ae^rt$</td>
</tr>
<tr>
<td>$C\sin(\lambda t)$</td>
<td>$A\sin(\lambda t) + B\cos(\lambda t)$</td>
</tr>
<tr>
<td>$C\cos(\lambda t)$</td>
<td>$A\sin(\lambda t) + B\cos(\lambda t)$</td>
</tr>
<tr>
<td>$a_m t^m + \cdots + a_1 t + a_0$</td>
<td>$A_m t^m + \cdots + A_1 t + A_0$</td>
</tr>
</tbody>
</table>

Above $A_m, \cdots, A_0, A, B$ are all constants that need to determined afterwards.

If we further assume that the given differential equation has order two then we can treat two more forms of $f(t)$.

1.2.2. $f(t) = Ct^m e^{rt}$. Guess

$$y_P(t) = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{rt},$$

where the value of $s$ depends on relationship between the associated auxiliary equation and $r$:

1. $s = 0$ if $r$ is not a root,
2. $s = 1$ if $r$ is a simple root, and
3. $s = 2$ if $r$ is a double root.

1.2.3. $f(t) = Ct^m e^{\alpha t} \cos(\beta t)$ or $f(t) = Ct^m e^{\alpha t} \sin(\beta t)$. Guess

$$y_P(t) = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{\alpha t} \cos(\beta t) + t^s (B_m t^m + \cdots + B_1 t + B_0) e^{\alpha t} \sin(\beta t),$$

where the value of $s$ depends on the relationship between the associated auxiliary equation and $\alpha + i\beta$:

1. $s = 0$ if $\alpha + i\beta$ is not a root, and
2. $s = 1$ if $\alpha + i\beta$ is a root.

**Remark 2.** You can remember the various conditions on $s$ by noting that it is just the multiplicities of $r$ and $\alpha + i\beta$ as roots of the associated auxiliary equation.

2. Problems

**Problem 1.** Find the general solution to the equation $y'' + 4y' + 13y = 0$

**Solution.** Starting with the ansatz $y = e^{rt}$, we see that the auxiliary equation is

$$r^2 + 4r + 13 = 0.$$ 

Completing the square, we see that $r_{1,2} = -2 \pm 3i$. So the general solution is given by

$$y(t) = e^{-2t} (c_1 \cos(3t) + c_2 \sin(3t)).$$
**Problem 2.** Find the solution to the initial value problem
\[ y'' - 2y' + 5y = 0; \quad y(0) = 1, y'(0) = -1. \]

*Solution.* Starting with the ansatz \( y = e^{rt} \), we see that the auxiliary equation is
\[ r^2 - 2r + 5 = 0. \]
Using the quadratic formula, \( r_{1,2} = 1 \pm 2i \).
So the general solution is
\[ y = e^t (c_1 \cos(2t) + c_2 \sin(2t)). \]
Plugging in the initial conditions, we get
\[ y(0) = 1 \implies c_1 = 1 \]
\[ y'(0) = 1 \implies c_1 + 2c_2 = -1 \implies c_2 = -1. \]
Hence the solution is given by
\[ y = e^t (\cos(2t) - \sin(2t)). \]

**Problem 3.** Find the solution to the initial value problem
\[ y'' - 2y' + 4y = 0; \quad y(0) = 1, y'(0) = 1. \]

*Solution.* Using the ansatz \( y = e^{rt} \), we see that the auxiliary equation is
\[ r^2 - 4r + 4 = (r - 2)^2 = 0. \]
Since \( r = 2 \) is a double root, the general solution is given by
\[ y = e^{2t} (c_1 x + c_2). \]
Plugging in the initial conditions
\[ y(0) = 1 \implies c_2 = 1 \]
\[ y'(0) = 1 \implies 2c_2 + c_1 = 1 \implies c_1 = -1. \]
So the solution is \( y = (1 - t)e^{2t} \).

**Problem 4.** Find the general solution to the equation \( y'' + 4y = t^2 \).

*Solution.* The auxiliary equation is
\[ r^2 + 4 = 0 \implies r_{1,2} = \pm 2i. \]
So \( y_H(t) = c_1 \cos(2t) + c_2 \sin(2t) \). Since \( f(t) = t^2 \), we must look for a particular solution of the form \( y_P(t) = At^2 + Bt + C \). Substituting into the differential equation, we get
\[ 2A + 4 \cdot (4At^2 + Bt + C) = 4At^2 + 4Bt + (2A + C) = t^2. \]
Matching the coefficients, we have \( A = \frac{1}{4}, B = 0, C = -\frac{1}{2} \). Hence the general solution is given by
\[ y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{2}. \]

**Problem 5.** Find the general solution to the differential equation
\[ y'' + y' + y = 2te^t. \]
Solution. Starting with the ansatz $y = e^{rt}$ for the homogeneous system, we get

$$r^2 + r + 1 = 0 \implies r_{1,2} = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}.$$  

Hence

$$y_{H}(t) = e^{-\frac{t}{2}} \left( c_1 \cos \left( \frac{\sqrt{3}}{2} t \right) + c_2 \sin \left( \frac{\sqrt{3}}{2} t \right) \right).$$

Since $2te^t = 2 \cdot t e^{1-t}$, we start with the guess

$$y_{P}(t) = (At + B)e^t = Ate^t + Be^t.$$  

Computing derivatives, we get

$$y'_{P}(t) = Ae^t + (At + B)e^t = Ae^t(t + 1) + Be^t$$

$$y''_{P}(t) = Ae^t(t + 2) + Be^t.$$  

Substituting into the original equation, we get

$$2te^t = 3Ate^t + (3A + 3B)e^t \implies A = \frac{2}{3}, B = -\frac{2}{3}.$$  

Hence the general solution is given by

$$y(t) = e^{-\frac{t}{2}} \left( c_1 \cos \left( \frac{\sqrt{3}}{2} t \right) + c_2 \sin \left( \frac{\sqrt{3}}{2} t \right) \right) + \frac{2}{3}e^t(t - 1).$$

**Problem 6.** Find the solution to the initial value problem

$$y'' + 4y = 2 \cos(t); \quad y(0) = 0, y'(0) = 1.$$  

Solution. Starting with the ansatz $y = e^{rt}$ for the homogeneous system, we get

$$r^2 + 4 = 0 \implies r = \pm 2i \implies y_{H}(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

Since $f(t) = 2 \cos(t)$, we can guess

$$y_{P}(t) = A \cos(t) + B \sin(t).$$  

Substituting into the differential equation

$$[-A \cos(t) - B \sin(t)] + 4[A \cos(t) + B \sin(t)] = 3A \cos(t) + 3B \sin(t) = 2 \cos(t).$$

Hence we must have $A = \frac{2}{3}$ and $B = 0$.  

The general solution is given by

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{2}{3} \cos(t).$$  

Plugging in the initial conditions

$$y(0) = 0 \implies c_1 + \frac{2}{3} = 0 \implies c_1 = -\frac{2}{3}$$

$$y'(0) = 1 \implies 2c_2 = 1 \implies c_2 = \frac{1}{2}.$$