JAN. 26 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

ABHIK PAL

1. Review

1.1. Complex and repeated roots. Recall that to find a solution to a homogeneous linear equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

we started with the ansatz $y = e^{rx}$ to get the equation

$$p(r) e^{rx} = (a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) e^{rx} = 0 \implies p(r) = 0.$$

Above, the equation $p(r) = 0$ is called the **auxiliary equation** associated to the differential equation.

The case when $p(r)$ is quadratic and has distinct real roots was discussed last week. We treat the remaining two cases below. If r_1, r_2 are roots of the auxiliary equation, then the different cases to determine the general solution are:

(1) If both r_1 and r_2 are real and distinct then the general solution is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(2) If $r = r_1 = r_2$ then the general solution is given by

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}.$$

(3) If $r_{1,2} = \alpha + i\beta$ then the general (real) solution is given by

$$y(x) = e^{\alpha x} \left(c_1 \cos(\beta x) + c_2 \sin(\beta x) \right).$$

Remark 1. *If $r_0 \in \mathbf{C}$ is a solution to $p(r) = 0$, then $\bar{r_0}$ is also a solution. Hence when $p(r)$ is quadratic and $r_0 \notin \mathbf{R}$, r_0 and $\bar{r_0}$ are the only solutions. This explains why we only consider the case $r_{1,2} = \alpha \pm i\beta$ when $p(r) = 0$ has non-real solutions.*

1.2. Method of Undetermined Coefficients. A non-homogeneous equation with constant coefficients is a differential equation of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t). \tag{1}$$

Any solution to (1) is of the form $y(t) = y_P(t) + y_H(t)$ where $y_P(t)$ is any function that satisfies

$$a_n \frac{d^n y_P}{dt^n} + a_{n-1} \frac{d^{n-1} y_P}{dt^{n-1}} + \cdots + a_1 \frac{dy_P}{dt} + a_0 y_P = f(t)$$

and y_H is the general solution to the homogeneous equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0.$$

We call y_P a **particular solution** to the in non-homogeneous equation (1).

The method of undetermined coefficients helps us make educated guesses about the particular solutions $y_P(t)$. The general strategy is

(1) Make a guess for $y_P(t)$ with possibly unknown coefficients.
(2) Substitute guess into (1) to determine values of the coefficients.

The main difficulty lies in finding $y_P(t)$. This is usually done by making educated guesses that (obviously) depends on $f(t)$. The sections below summarise the general strategy for determining $y_P(t)$ based on various forms of $f(t)$.

1.2.1. *"Nice" $f(t)$.* If $f(t)$ is "nice enough" we can make the following guesses:

$f(t)$	Guess for $y_P(t)$
Ce^{rt}	Ae^{rt}
$C\sin(\lambda t)$	$A\sin(\lambda t) + B\cos(\lambda t)$
$C\cos(\lambda t)$	$A\sin(\lambda t) + B\cos(\lambda t)$
$a_m t^m + \cdots + a_1 t + a_0$	$A_m t^m + \cdots + A_1 t + A_0$

Above $A_m, \cdots, A_0, A, B$ are all constants that need to determined afterwards.

If we further assume that the given differential equation has order two then we can treat two more forms of $f(t)$.

1.2.2. *$f(t) = Ct^m e^{rt}$.* Guess

$$y_P(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}.$$

where the value of s depends on relationship between the associated auxiliary equation and r:

(1) $s = 0$ if r is not a root,
(2) $s = 1$ if r is a simple root, and
(3) $s = 2$ if r is a double root.

1.2.3. *$f(t) = Ct^m e^{\alpha t}\cos(\beta t)$ or $f(t) = Ct^m e^{\alpha t}\sin(\beta t)$.* Guess

$$y_P(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t).$$

where the value of s depends on the relationship between the associated auxiliary equation and $\alpha + i\beta$:

(1) $s = 0$ if $\alpha + i\beta$ is not a root, and
(2) $s = 1$ if $\alpha + i\beta$ is a root.

Remark 2. *You can remember the various conditions on s by noting that it is just the multiplicities of r and $\alpha + i\beta$ as roots of the associated auxiliary equation.*

2. Problems

Problem 1. Find the general solution to the equation $y'' + 4y' + 13y = 0$

Solution. Starting with the ansatz $y = e^{rt}$, we see that the auxiliary equation is

$$r^2 + 4r + 13 = 0.$$

Completing the square, we see that $r_{1,2} = -2 \pm 3i$. So the general solution is given by

$$y(t) = e^{-2t}\left(c_1\cos(3t) + c_2\sin(3t)\right).$$

Problem 2. Find the solution to the initial value problem

$$y'' - 2y' + 5y = 0; \qquad y(0) = 1, y'(0) = -1.$$

Solution. Starting with the ansatz $y = e^{rt}$, we see that te auxiliary equation is

$$r^2 - 2r + 5 = 0.$$

Using the quadratic formula

$$r_{1,2} = 1 \pm 2i.$$

So the general solution is

$$y = e^t \left(c_1 \cos(2t) + c_2 \sin(2t)\right).$$

Plugging in the initial conditions, we get

$$y(0) = 1 \implies c_1 = 1$$
$$y'(0) = 1 \implies c_1 + 2c_2 = -1 \implies c_2 = -1.$$

Hence the solution is given by

$$y = e^t(\cos(2t) - \sin(2t)).$$

Problem 3. Find the solution to the initial value problem

$$y'' - 2y' + 4y = 0; \qquad y(0) = 1, y'(0) = 1.$$

Solution. Using the ansatz $y = e^{rt}$, we see that the auxiliary equation is

$$r^2 - 4r + 4 = (r - 2)^2 = 0.$$

Since $r = 2$ is a double root, the general solution is given by

$$y = e^{2t}(c_1 x + c_2).$$

Plugging in the initial conditions

$$y(0) = 1 \implies c_2 = 1$$
$$y'(0) = 1 \implies 2c_2 + c_1 = 1 \implies c_1 = -1.$$

So the solution is $y = (1 - t)e^{2t}$.

Problem 4. Find the general solution to the equation $y'' + 4y = t^2$.
Solution. The auxiliary equation is

$$r^2 + 4 = 0 \implies r_{1,2} = \pm 2i.$$

So $y_H(t) = c_1 \cos(2t) + c_2 \sin(2t)$. Since $f(t) = t^2$, we must look for a particular solution of the form $y_P(t) = At^2 + Bt + C$. Substituting into the differential equation, we get

$$2A + 4 \cdot (At^2 + Bt + C) = 4At^2 + 4Bt + (2A + C) = t^2.$$

Matching the coefficients, we have $A = \frac{1}{4}$, $B = 0$, $C = -\frac{1}{2}$. Hence the general solution is given by

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{2}.$$

Problem 5. Find the general solution to the differential equation

$$y'' + y' + y = 2te^t.$$

Solution. Starting with the ansatz $y = e^{rt}$ for the homogeneous system, we get

$$r^2 + r + 1 = 0 \implies r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Hence

$$y_H(t) = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

Since $2te^t = 2 \cdot t^1 e^{1\cdot t}$, we start with the guess

$$y_P(t) = (At + B)e^t = Ate^t + Be^t.$$

Computing derivatives, we get

$$y_P'(t) = Ae^t + (At + B)e^t = Ae^t(t + 1) + Be^t$$
$$y_P''(t) = Ae^t(t + 2) + Be^t.$$

Substituting into the original equation, we get

$$2te^t = 3Ate^t + (3A + 3B)e^t \implies A = \frac{2}{3}, B = -\frac{2}{3}.$$

Hence the general solution is given by

$$y(t) = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) + \frac{2}{3}e^t(t - 1)$$

Problem 6. Find the solution to the initial value problem

$$y'' + 4y = 2\cos(t); \qquad y(0) = 0, y'(0) = 1.$$

Solution. Starting with the ansatz $y = e^{rt}$ for the homogeneous system, we get

$$r^2 + 4 = 0 \implies r = \pm 2i \implies y_H(t) = c_1 \cos(2t) + c_2 \cos(2t).$$

Since $f(t) = 2\cos(t)$, we can guess

$$y_P(t) = A\cos(t) + B\sin(t).$$

Substituting into the differential equation

$$[-A\cos(t) - B\sin(t)] + 4[A\cos(t) + B\sin(t)] = 3A\cos(t) + 3B\sin(t) = 2\cos(t).$$

Hence we must have $A = \frac{2}{3}$ and $B = 0$.

The general solution is given by

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{2}{3}\cos(t).$$

Plugging in the initial conditions

$$y(0) = 0 \implies c_1 + \frac{2}{3} = 0 \implies c_1 = -\frac{2}{3}$$
$$y'(0) = 1 \implies 2c_2 = 1 \implies c_2 = \frac{1}{2}.$$