1. Review

1.1. Exact Equations. Recall that the total differential of a function \( F(x, y) \), written \( dF \), is the differential form

\[
dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy.
\]

An arbitrary differential form

\[
M(x, y) \, dx + N(x, y) \, dy
\]
is called exact if there exists \( F(x, y) \) such that

\[
M(x, y) = \frac{\partial F}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial F}{\partial y}.
\]

In other words, an exact differential form is the total differential of some function \( F \). In this case, a differential equation of the form

\[
M(x, y) \, dx + N(x, y) \, dy = 0
\]
is called an exact differential equation and has solution \( F(x, y) = c \) for some constant \( c \).

1.2. Testing for exactness. The differential equation

\[
M(x, y) \, dx + N(x, y) \, dy = 0
\]
is exact if and only if

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.
\]

\( \vbigtriangleup \) Caution 1. Make sure you differentiate with the correct variable while testing for exactness. To avoid making this mistake, remember that the coefficient of \( dx \) gets differentiated with respect to \( y \) and the coefficient of \( dy \) gets differentiated with respect to \( x \).

1.3. Solving Exact Equations. Suppose \( M \, dx + N \, dy = 0 \) is exact.
1.3.1. Method I.

(1) Integrate with respect to $x$ to get
$$F(x, y) = \int M(x, y)dx + g(y).$$

(2) Differentiate both sides of the equation in step (1) with respect to $y$ and solve for $g'(y)$ to obtain
$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \left[ \int M(x, y)dx \right].$$

(3) Obtain $g(y)$ up to a constant by integrating $g'(y)$.

(4) Obtain the implicit solution $F(x, y) = C$ by plugging in $g(y)$ into the equation in step (1).

1.3.2. Method II.

(1) Compute integrals with respect to $x$ and $y$ to obtain
$$\int M(x, y)dx + g(y) \quad \text{and} \quad \int N(x, y)dy + h(x).$$

Above, $g(y)$ and $h(x)$ are some functions of $y$ and $x$ respectively.

(2) Obtain $F(x, y)$ by merging the results of the two integrals by writing
$$c = F(x, y) = (\text{common terms that appear on both integrals})$$
$$+ \text{(terms that only depend on } x)$$
$$+ \text{(terms that only depend on } y)$$

This step is clarified in the example below.

Example 1. Suppose we need to solve the exact equation
$$(1 - 2xy)dx + (4y^3 - x^2)dy = 0.$$  

Then
$$\int M(x, y)dx = x - x^2y + g(y),$$
$$\int N(x, y)dy = y^4 - x^2y + h(x).$$

We note that $-x^2y$ is the term common to both integrals, $x$ is the only term that depends on $x$ and $y^4$ is the only term that depends on $y$. Hence we must have $h(x) = x$ and $g(y) = y^4$. So the implicit solution is
$$F(x, y) = -x^2y + x + y^4 = C.$$  

1.4. Integrating Factors. Suppose the equation
$$M(x, y)dx + N(x, y)dy = 0$$
is not exact. If there exists a function $\mu(x, y)$ such that the equation
$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$
is exact, then $\mu(x, y)$ is called an integrating factor.

Remark 1. Recall that we encountered integrating factors while solving linear first order equations. In the case of linear first order ODEs, the integrating factors helped us convert the given equation to a separable equation.
1.5. Finding (Special) Integrating Factors. Let
\[ M(x, y)dx + N(x, y)dy = 0 \]
be a non-exact equation. To simplify some of the terms, we use the notation
\[ M_y = \frac{\partial M}{\partial y} \quad \text{and} \quad N_x = \frac{\partial N}{\partial x}. \]
Note that non-exactness just means \( M_y \neq N_x \) so \( M_y - N_x \neq 0 \). We check two cases and compute the integrating factor accordingly:

1. If the term \( M_y - N_x \) only depends on \( x \), then let
\[ \mu(x, y) = \mu(x) = \exp \left[ \int \frac{M_y - N_x}{N} \, dx \right] \]

2. If the term \( M_y - N_x = \frac{N_x - M_y}{M} \) only depends on \( y \), then let
\[ \mu(x, y) = \mu(y) = \exp \left[ \int \frac{M_y - N_x}{-M} \, dy \right] \]

Caution 2. If a non-exact equation had a solution, then an integrating factor must exist. This does not mean, however, that they are easy to compute! The above method does not help us find integrating factors in all cases.

1.6. Homogeneous Linear Equations. Recall that a linear differential equation is a differential equation of the form
\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = F(x). \]
The equation above is called homogeneous if \( F(x) = 0 \). The equation above has constant coefficients if each \( a_k(x) \) is a constant function. Hence a homogeneous differential equation with constant coefficients has the form

\[ \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0 \]

where \( a_0, \ldots, a_n \) are real numbers.

Fact 1. If \( y_1(x), \ldots, y_n(x) \) are solutions to a homogeneous equation and \( c_1, \ldots, c_n \) are arbitrary constants then \( y(x) = C_1 y_1(x) + \cdots C_n y_n(x) \) is also a solution.

1.7. Solutions to Homogeneous Linear Equations with Constant Coefficients. Start with an guess\(^1\) \( y = e^{rx} \). Plugging this into (1), we get
\[ 0 = a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} \]
\[ = (a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) e^{rx}. \]

Since \( e^{rx} \neq 0 \) for all \( x \), the above equation is only zero at the roots \( r_1, \ldots, r_k \) of the polynomial above. A general solution to (1) is found by adding solutions corresponding to various \( r_i \).

\(^1\)Some books often call this initial guess an Ansatz.
2. Problems

**Problem 1.** Check whether the following equations are exact. Find the general solution for those which are exact.

1. \(3x^2ydx + (x^3 + y^3)dy = 0\)
2. \((x^2 - y^2)dx + (y^2 - x^2)dy = 0\)
3. \(ve^{uv}du + ve^{uv}dv = 0\)
4. \(2xydx - x^2dy = 0\)

**Solution.**

1. We have \(M(x, y) = 3x^2y, N(x, y) = (x^3 + y^3)\). Computing the partials we get
   \[
   \frac{\partial M}{\partial y} = 3x^2, \quad \frac{\partial N}{\partial x} = 3x^2.
   \]
   Hence the equation is exact. We can now integrate to get
   \[
   \int M(x, y)dx = yx^3 + g(y)
   \]
   \[
   \int N(x, y)dy = yx^3 + \frac{y^4}{4} + h(x).
   \]
   Combining, we get the solution
   \[
   F(x, y) = yx^3 + \frac{y^4}{4} = C.
   \]

2. We have \(M(x, y) = x^2 - y^2, N(x, y) = y^2 - x^2\). Computing the partials we get
   \[
   \frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = -2x.
   \]
   Hence the equation is not exact.

3. We have \(M(u, v) = ve^{uv}, N(u, v) = ue^{uv}\). Computing the partials, we get
   \[
   \frac{\partial M}{\partial v} = ue^{uv} + e^{uv}, \quad \frac{\partial N}{\partial u} = ue^{uv} + e^{uv}.
   \]
   Hence the equation is exact. We can now integrate (using substitution)
   \[
   \int M(u, v)du = \int ve^{uv}du = e^{uv} + g(v)
   \]
   \[
   \int N(u, v)dv = \int ue^{uv}dv = e^{uv} + h(u).
   \]
   Combining, we get the solution
   \[
   F(u, v) = e^{uv} = C.
   \]

4. We have \(M(x, y) = 2xy, N(x, y) = -x^2dy\). Computing the partials we get
   \[
   \frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = -2x^2.
   \]
   Hence the equation is not exact.
Problem 2. Find a general solution to the differential equation
\[ \frac{dy}{dx} + \frac{x^2y}{x^3 + y} = 0. \]

Solution. Rearranging the equation, we obtain
\[ (x^2y)dx + (x^3 + y)dy = 0. \]
So \( M(x, y) = x^2y \) and \( N(x, y) = x^3 + y \). Computing the partials
\[ \frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = 3x^2 \]
we conclude that the given equation is not exact. We compute
\[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} - \frac{M}{-M(x, y)} = x^2 - 3x^2 - \frac{x^2}{-x^2y} = \frac{2y}{y} \]
to get a function that depends just on \( y \). So we can compute the integrating factor
\[ \mu(x, y) = \mu(y) = \exp \left[ \int \frac{2}{y} dy \right] = \exp[2 \ln(y)] = y^2. \]
Multiplying both sides of the original equation by \( \mu(x, y) \), we get an exact equation
\[ (x^2y^3)dx + (x^3y^2 + y^3)dy = 0. \]
Now,
\[ \int x^2y^3dx = \frac{x^3y^3}{3} + g(y), \quad \int x^3y^2 + y^3 dy = \frac{x^3y^3}{3} + \frac{y^4}{4}. \]
Combining these terms, we see that
\[ F(x, y) = \frac{1}{3}x^3y^3 + \frac{1}{4}y^4 = C \]
is a general solution.

Problem 3. Find a general solution to the differential equation
\[ y'' - 2y' - 3y = 0 \]

Solution. We note that the given equation is a second order homogeneous equation. Hence start with the ansatz \( y = e^{rx} \), then
\[ (r^2 - 2r - 3)e^{rx} = 0 \implies (r + 1)(r - 3) = 0 \implies r = -1, r = 3. \]
Hence a general solution is given by
\[ y(x) = C_1e^{-x} + C_2e^{3x} \]
where \( C_1 \) and \( C_2 \) are constants.

Problem 4. Find a second-order linear homogeneous ODE whose general solution
is \( y = c_1e^x + c_2e^{2x} \)

Solution. An arbitrary second-order homogeneous ODE looks like
\[ a_2y'' + a_1y' + a_0y = 0. \]
If we start with an ansatz \( y = e^{rx} \), we get
\[ (a_2r^2 + a_1r + a_0)e^{rx} = 0. \]
From the general solution we conclude that $r = 1$ and $r = 2$ must be roots of the quadratic. Hence

$$a_2 r^2 + a_1 r + a_0 = (r - 1)(r - 2) = r^2 - 3r + 2.$$ 

Hence $a_2 = 1, a_1 = -3, a_0 = 2$ and the correct differential equation is

$$y'' - 3y' + 2y = 0.$$ 

**Problem 5.** Find the solution to the differential equation

$$y'' + 4y' + 3y = 0$$

such that $y(0) = 1, y'(0) = -\frac{5}{3}$.

**Solution.** The given equation is second-order linear homogeneous so we start with the ansatz $y = e^{rx}$. Then

$$r^2 + 4r + 3 = 0 \implies (r + 3)(r + 1) = 0 \implies r = -3, r = -1.$$ 

Hence the general solution is given by

$$y = c_1 e^{-x} + c_2 e^{-3x}.$$ 

Using the initial conditions, we get

$$c_1 + c_2 = 1$$

$$-c_1 - 3c_2 = -\frac{5}{3}.$$ 

Solving for $c_1, c_2$ we get $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Hence the solution is given by

$$y(x) = \frac{2}{3} e^{-x} + \frac{1}{3} e^{-3x}.$$