JAN. 19 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

ABHIK PAL

1. Review

1.1. Exact Equations. Recall that the **total differential** of a function $F(x, y)$, written dF, is the differential form

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

An arbitrary differential form

$$M(x, y)dx + N(x, y)dy$$

is called **exact** if there exists $F(x, y)$ such that

$$M(x, y) = \frac{\partial F}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial F}{\partial y}.$$

In other words, an exact differential form is the total differential of *some* function F. In this case, a differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact differential equation** and has solution $F(x, y) = c$ for some constant c.

1.2. Testing for exactness. The differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Caution 1. *Make sure you differentiate with the correct variable while testing for exactness. To avoid making this mistake, remember that the coefficient of dx gets differentiated with respect to y and the coefficient of dy gets differentiated with respect to x.*

1.3. Solving Exact Equations. Suppose $M\ dx + N\ dy = 0$ is exact.

1.3.1. *Method I.*

(1) Integrate with respect to x to get
$$F(x, y) = \int M(x, y)dx + g(y).$$
(2) Differentiate both sides of the equation in step (1) with respect to y and solve for $g'(y)$ to obtain
$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\left[\int M(x, y)dx\right].$$
(3) Obtain $g(y)$ up to a constant by integrating $g'(y)$.
(4) Obtain the implicit solution $F(x, y) = C$ by plugging in $g(y)$ into the equation in step (1).

1.3.2. *Method II.*

(1) Compute integrals with respect to x and y to obtain
$$\int M(x, y)dx + g(y) \quad \text{and} \quad \int N(x, y)dy + h(x).$$
Above, $g(y)$ and $h(x)$ are some functions of y and x respectively.
(2) Obtain $F(x, y)$ by merging the results of the two integrals by writing
$$\begin{aligned} c = F(x, y) = &(\text{common terms that appear on both integrals}) \\ &+ (\text{terms that only depend on } x) \\ &+ (\text{terms that only depend on } y) \end{aligned}$$
This step is clarified in the example below.

Example 1. *Suppose we need to solve the exact equation*
$$(1 - 2xy)dx + (4y^3 - x^2)dy = 0.$$
Then
$$\int M(x, y)dx = x - x^2y + g(y),$$
$$\int N(x, y)dy = y^4 - x^2y + h(x).$$
We note that $-x^2y$ is the term common to both integrals, x is the only term that depends on x and y^4 is the only term that depends on y. Hence we must have $h(x) = x$ and $g(y) = y^4$. So the implicit solution is
$$F(x, y) = -x^2y + x + y^4 = C.$$

1.4. **Integrating Factors.** Suppose the equation
$$M(x, y)dx + N(x, y)dy = 0$$
is not exact. If there exists a function $\mu(x, y)$ such that the equation
$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$
is exact, then $\mu(x, y)$ is called an **integrating factor**.

Remark 1. *Recall that we encountered integrating factors while solving linear first order equations. In the case of linear first order ODEs, the integrating factors helped us convert the given equation to a separable equation.*

1.5. **Finding (Special) Integrating Factors.** Let

$$M(x,y)dx + N(x,y)dy = 0$$

be a non-exact equation. To simplify some of the terms, we use the notation

$$M_y = \frac{\partial M}{\partial y} \quad \text{and} \quad N_x = \frac{\partial N}{\partial x}.$$

Note that non-exactness just means $M_y \neq N_x$ so $M_y - N_x \neq 0$. We check two cases and compute the integrating factor accordingly:

(1) If the term

$$\frac{M_y - N_x}{N}$$

only depends on x, then let

$$\mu(x,y) = \mu(x) = \exp\left[\int \left(\frac{M_y - N_x}{N}\right) dx\right]$$

(2) If the term

$$\frac{M_y - N_x}{-M} = \frac{N_x - M_y}{M}$$

only depends on y, then let

$$\mu(x,y) = \mu(y) = \exp\left[\int \left(\frac{M_y - N_x}{-M}\right) dy\right]$$

Caution 2. *If a non-exact equation had a solution, then an integrating factor must exist. This does not mean, however, that they are easy to compute! The above method does not help us find integrating factors in all cases.*

1.6. **Homogeneous Linear Equations.** Recall that a linear differential equation is a differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x).$$

The equation above is called **homogeneous** is $F(x) = 0$. The equation above has constant coefficients if each $a_k(x)$ is a constant function. Hence a **homogeneous differential equation with constant coefficients** has the form

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0 \qquad (1)$$

where $a_0, \cdots, a_n$ are real numbers.

Fact 1. *If $y_1(x), \cdots, y_n(x)$ are solutions to a homogeneous equation and $c_1, \ldots, c_n$ are arbitrary constants then $y(x) = C_1 y_1(x) + \cdots C_n y_n(x)$ is also a solution.*

1.7. **Solutions to Homogeneous Linear Equations with Constant Coefficients.** Start with an guess[1] $y = e^{rx}$. Plugging this into (1), we get

$$\begin{aligned} 0 &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} \\ &= (a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) e^{rx}. \end{aligned}$$

Since $e^{rx} \neq 0$ for all x, the above equation is only zero at the roots $r_1, \ldots, r_k$ of the polynomial above. A general solution to (1) is found by adding solutions corresponding to various r_i.

[1] Some books often call this initial guess an *Ansatz*.

2. Problems

Problem 1. Check whether the following equations are exact. Find the general solution for those which are exact.

1. $3x^2ydx + (x^3 + y^3)dy = 0$
2. $(x^2 - y^2)dx + (y^2 - x^2)dy = 0$
3. $ve^{uv}du + ue^{uv}dv = 0$
4. $2xydx - x^2dy = 0$

Solution.

1. We have $M(x, y) = 3x^2y$, $N(x, y) = (x^3 + y^3)$. Computing the partials we get
$$\frac{\partial M}{\partial y} = 3x^2, \qquad \frac{\partial N}{\partial x} = 3x^2.$$
Hence the equation is exact. We can now integrate to get
$$\int M(x, y)dx = yx^3 + g(y)$$
$$\int N(x, y)dy = yx^3 + \frac{y^4}{4} + h(x).$$
Combining, we get the solution
$$F(x, y) = yx^3 + \frac{y^4}{4} = C.$$
2. We have $M(x, y) = x^2 - y^2$, $N(x, y) = y^2 - x^2$. Computing the partials we get
$$\frac{\partial M}{\partial y} = -2y, \qquad \frac{\partial N}{\partial x} = -2x.$$
Hence the equation is not exact.
3. We have $M(u, v) = ve^{uv}$, $N(u, v) = ue^{uv}$. Computing the partials, we get
$$\frac{\partial M}{\partial v} = uve^{uv} + e^{uv}, \qquad \frac{\partial N}{\partial u} = uve^{uv} + e^{uv}.$$
Hence the equation is exact. We can now integrate (using substitution)
$$\int M(u, v)du = \int ve^{uv}du = e^{uv} + g(v)$$
$$\int N(u, v)dv = \int ue^{uv}dv = e^{uv} + h(u).$$
Combining, we get the solution
$$F(u, v) = e^{uv} = C.$$
4. We have $M(x, y) = 2xy$, $N(x, y) = -x^2dy$. Computing the partials we get
$$\frac{\partial M}{\partial y} = 2x, \qquad \frac{\partial N}{\partial x} = -2x^2.$$
Hence the equation is not exact.

Problem 2. Find a general solution to the differential equation

$$\frac{dy}{dx} + \frac{x^2 y}{x^3 + y} = 0.$$

Solution. Rearranging the equation, we obtain

$$(x^2 y)dx + (x^3 + y)dy = 0.$$

So $M(x, y) = x^2 y$ and $N(x, y) = x^3 + y$. Computing the partials

$$\frac{\partial M}{\partial y} = x^2, \qquad \frac{\partial N}{\partial x} = 3x^2$$

we conclude that the given equation is not exact. We compute

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M(x, y)} = \frac{x^2 - 3x^2}{-x^2 y} = \frac{2}{y}$$

to get a function that depends just on y. So we can compute the integrating factor

$$\mu(x, y) = \mu(y) = \exp\left[\int \frac{2}{y} dy\right] = \exp[2\ln(y)] = y^2.$$

Multiplying both sides of the original equation by $\mu(x, y)$, we get an exact equation

$$(x^2 y^3)dx + (x^3 y^2 + y^3)dy = 0.$$

Now,

$$\int x^2 y^3 dx = \frac{x^3 y^3}{3} + g(y), \qquad \int x^3 y^2 + y^3 dy = \frac{x^3 y^3}{3} + \frac{y^4}{4}.$$

Combining these terms, we see that

$$F(x, y) = \frac{1}{3} x^3 y^3 + \frac{1}{4} y^4 = C$$

is a general solution.

Problem 3. Find a general solution to the differential equation

$$y'' - 2y' - 3y = 0$$

Solution. We note that the given equation is a second order homogeneous equation. Hence start with the ansatz $y = e^{rx}$, then

$$(r^2 - 2r - 3)e^{rx} = 0 \implies (r + 1)(r - 3) = 0 \implies r = -1, r = 3.$$

Hence a general solution is given by

$$y(x) = C_1 e^{-x} + C_2 e^{3x}$$

where C_1 and C_2 are constants.

Problem 4. Find a second-order linear homogeneous ODE whose general solution is $y = c_1 e^x + c_2 e^{2x}$

Solution. An arbitrary second-order homogeneous ODE looks like

$$a_2 y'' + a_1 y' + a_0 y = 0.$$

If we start with an ansatz $y = e^{rx}$, we get

$$(a_2 r^2 + a_1 r + a_0)e^{rx} = 0.$$

From the general solution we conclude that $r = 1$ and $r = 2$ must be roots of the quadratic. Hence

$$a_2 r^2 + a_1 r + a_0 = (r-1)(r-2) = r^2 - 3r + 2.$$

Hence $a_2 = 1, a_1 = -3, a_0 = 2$ and the correct differential equation is

$$y'' - 3y' + 2y = 0.$$

Problem 5. Find the solution to the differential equation

$$y'' + 4y' + 3y = 0$$

such that $y(0) = 1, y'(0) = -\frac{5}{3}$.

Solution. The given equation is second-order linear homogeneous so we start with the ansatz $y = e^{rx}$. Then

$$r^2 + 4r + 3 = 0 \implies (r+3)(r+1) = 0 \implies r = -3, r = -1.$$

Hence the general solution is given by

$$y = c_1 e^{-x} + c_2 e^{-3x}.$$

Using the initial conditions, we get

$$c_1 + c_2 = 1$$
$$-c_1 - 3c_2 = -\frac{5}{3}.$$

Solving for c_1, c_2 we get $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. Hence the solution is given by

$$y(x) = \frac{2}{3}e^{-x} + \frac{1}{3}e^{-3x}$$