JAN. 12 DISCUSSION NOTES
SECTION B05/B06, MATH 20D (WI21)

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1. Review

1.1. **Classification of Differential Equations.** An **ordinary differential equation (ODE)** involves derivatives with respect to a single independent variable. A **partial differential equation (PDE)** involves partial derivatives with respect to more than one independent variable. The **order** of a differential equation is the order of the highest-order derivative present in the equation. In a **linear differential equation** the dependent variable and its derivatives appear as additive combinations of their first powers. Hence a linear differential equation can be written as

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) \tag{1}$$

where $a_n(x), \ldots, a_0(x)$, and $F(x)$ depend only on x. If a differential equation is not linear it is *non-linear*.

Caution 1. *Note that in a linear equation, the restriction is only made on the dependent variable! Above, the $a_k(x)$ are still permitted to be arbitrary functions of x. However y and its derivatives must appear as linear terms.*

1.2. **Separable Equations.** A first-order equation is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y).$$

1.3. **Solving Separable Equations.** Suppose we are given the (separable) differential equation

$$\frac{dy}{dx} = g(x)p(y).$$

We solve this using the following steps:

(1) "Separate" the x and y parts by multiplying both sides by dx and dividing by $p(y)$ to obtain

$$\frac{1}{p(y)}dy = g(x)dx.$$

(2) Integrate both sides to get

$$P(y) = \int \frac{1}{p(y)}dy = \int g(x)dx = G(x) + C.$$

The equation $P(y) = G(x) + C$ gives an implicit solution to the differential equation. We may now solve for y in terms of x to get an explicit solution.

1.4. **Linear Equations.** Using the notation from (1), we can write a linear first-order equation as

$$a_1(x)\frac{dy}{dx} + a_0(x)y = F(x). \tag{2}$$

Here $a_1(x)$, $a_0(x)$, and $F(x)$ depend only on x. We may divide by $a_1(x)$ to obtain the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3}$$

of (2). Above,

$$P(x) = \frac{a_0(x)}{a_1(x)} \qquad \text{and} \qquad Q(x) = \frac{F(x)}{a_1(x)}.$$

1.5. **Solving Linear Equations.** We use the following steps to solve a first order linear equation:

(1) First bring the equation in standard form as in (3).
(2) Calculate the integrating factor using the formula

$$\mu(x) = \exp\left[\int P(x)dx\right] = e^{\int P(x)dx}.$$

(3) Multiply both sides of (3) by $\mu(x)$ to obtain

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)\frac{dy}{dx} + P(x)\mu(x)y = \mu(x)Q(x).$$

(4) Integrate both sides and then divide by $\mu(x)$ to get the **general solution**

$$\mu(x)y = \int \mu(x)Q(x)dx + C \implies y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

Caution 2. *Make sure you don't forget to take the exponential while computing the integrating factor!*

2. Problems

Problem 1. Identify the independent variables, dependent variable, and the order of the given equation. Classify the equation as ODE/PDE and linear/non-linear.

1. $i\hbar\frac{d}{dt}\Psi = H\Psi$
2. $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \frac{\partial^2}{\partial x^2}\psi - \frac{\partial^2}{\partial y^2}\psi - \frac{\partial^2}{\partial z^2}\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$
3. $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$
4. $\frac{d^2\theta}{dt^2} + \sin(\theta) = 0$
5. $\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + e^{\lambda u} = 0$

Solution.

1. The variable t is independent and Ψ is dependent. The given equation is a first-order linear ODE.

2. The variables t, x, y, z are independent and ψ is dependent. The given equation is a second-order linear PDE.
3. The variables t, S are independent and V is dependent. The given equation is a second-order linear PDE.
4. The variable t is independent and θ is independent. The given equation is a second-order non-linear ODE.
5. The variables x and y are independent and u is dependent. The given equation is a second-order non-linear PDE.

Problem 2. Suppose A and B are constants, show that $y = A\sin(2x) + B\cos(2x)$ solves the differential equation $y'' = -4y$.

Solution. Note that it suffices to compute $y'' + 4y$.

$$\begin{aligned} y'' + 4y &= \frac{d}{dx}\left[\frac{d}{dx}\left(A\sin(2x) + B\cos(2x)\right)\right] + 4A\sin(2x) + 4B\cos(2x) \\ &= \frac{d}{dx}\left[2A\cos(2x) - 2B\sin(2x)\right] + 4A\sin(2x) + 4B\cos(2x) \\ &= -4A\sin(2x) - 4B\cos(2x) + 4A\sin(2x) + 4B\cos(2x) \\ &= 0. \end{aligned}$$

Hence, the given value of y indeed solves the required equation.

Problem 3. Solve the differential equation $x' = 2 + 2x + t + tx$

Solution. Factoring the right hand side as $(2+t)(1+x)$, we see that the differential equation is separable. We can now compute

$$\begin{aligned} \frac{dx}{dt} = (2+t)(1+x) &\implies \frac{1}{1+x}dx = (2+t)dt \\ &\implies \int \frac{1}{1+x}dx = \int 2 + tdt \\ &\implies \ln(|x+1|) = 2t + \frac{t^2}{2} + C \\ &\implies x = K\exp\left[2t + \frac{t^2}{2}\right] - 1. \end{aligned}$$

Above $K = e^C$ is a constant.

Problem 4. Solve the differential equation $(x^2+1)y' = xy$.

Solution. Rewriting, we get

$$\frac{dy}{dx} = \frac{x}{x^2+1} \cdot y.$$

Since the equation is separable, we can now compute

$$\frac{dy}{y} = \frac{x}{x^2+1}\,dx \implies \int \frac{1}{y}dy = \int \frac{x}{x^2+1}dx \implies \ln(|y|) = \frac{1}{2}\ln(x^2+1) + C$$

where C is the constant of integration. To compute the integral with respect to x, we use u substitution with $u = x^2 + 1$. We can now obtain the explicit solution by taking the exponential on both sides. Hence

$$y = K\sqrt{x^2+1}$$

where $K = e^C$ is a constant.

Problem 5. Find an implicit solution to the initial value problem

$$y' = \frac{\sin(x)}{\sin(y)}, \quad y(0) = \frac{\pi}{2}.$$

Solution. Since the given differential equation is separable, we get

$$\frac{dy}{dx} = \frac{\sin(x)}{\sin(y)} \implies \sin(y)dy = \sin(x)dx \implies -\cos(y) = -\cos(x) + C.$$

From the given initial condition,

$$\cos(\pi/2) = \cos(0) - C \implies C = 1.$$

Hence an implicit solution is $\cos(y) = \cos(x) - 1$.

Problem 6. When a raindrop falls, it increases in size. Suppose its mass at time t is given by $m(t)$. The rate of growth of the mass is $km(t)$ for some $k > 0$. When we apply Newton's Law of Motion to the raindrop, we get $(mv)' = mg$, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The terminal velocity of the raindrop is $\lim_{t\to\infty} v(t)$. Find an expression of the terminal velocity in terms of g and k. You may assume that the raindrop begins at rest.

Solution. We first need to compute $v(t)$. Since the raindrop begins at rest, we have the initial condition $v(0) = 0$. Moreover, we are given

$$\frac{dm}{dt} = km \quad \text{and} \quad (mv)' = mv' + vm' = gm.$$

Hence

$$mv' + v(km) = gm \implies v' + vk = g.$$

Since the resulting differential equation is separable, we can compute $v(t)$.

$$\begin{aligned} \frac{dv}{dt} = g - kv &\implies \int \frac{1}{g - kv} dv = \int dt \\ &\implies -\frac{1}{k} \ln |g - kv| = t + C \\ &\implies v = \frac{g}{k} - \frac{A}{k} e^{-kt}. \end{aligned}$$

Using the initial conditions, we get $A = g$ and hence

$$v(t) = \left(1 - e^{-kt}\right) \frac{g}{k}.$$

Now the terminal velocity can be computed by taking the limit $t \to \infty$. Since $k > 0$, we get

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \left(1 - e^{-kt}\right) \frac{g}{k} = \left(1 - \lim_{t\to\infty} e^{-kt}\right) \frac{g}{k} = \frac{g}{k}.$$

Problem 7. Solve the differential equation

$$\frac{dy}{dx} + 3x^2 y = 6x^2.$$

Solution. Sinc the given equation is a first order linear equation and is already in the standard form, we can directly compute the integrating factor

$$\mu(x) = \exp\left[\int 3x^2 dx\right] = \exp(x^3).$$

Multiplying both sides by $\mu(x)$, we see that

$$\frac{d}{dx}\left[\exp(x^3)y\right] = 6x^2\exp(x^3).$$

Using $u = x^3$, integration yields

$$\exp(x^3)y = \int 6x^2\exp(x^3)dx = \int 2\exp(u)du = 2\exp(x^3) + C.$$

Hence

$$y = 2 + C\exp(-x^3)$$

is a solution to the given differential equation.

Problem 8. Solve the differential equation

$$y' + y = \sin(e^x).$$

Solution. Since the given equation is a first order equation and is already in the standard form, we can directly compute the integrating factor

$$\mu(x) = \exp\left[\int 1dx\right] = e^x.$$

Multiplying both sides by $\mu(x)$, we see that

$$\frac{d}{dx}\left[e^x y\right] = e^x\sin(e^x).$$

Using $u = e^x$, integration yields

$$e^x y = \int e^x\sin(e^x)dx = \int \sin(u)du = -\cos(e^x) + C.$$

Hence

$$y = Ce^{-x} - e^{-x}\cos(e^x)$$

is a solution to the given different differential equation.

Problem 9. Find a solution to the initial value problem

$$xy' + y = x\cos(x^2), \qquad y(\sqrt{\pi}) = 1.$$

Solution. Since the given equation is a first order linear equation, we first divide by x to bring the equation in standard form

$$\frac{dy}{dx} + \frac{1}{x}\cdot y = \cos(x^2).$$

The integrating factor is given by

$$\mu(x) = \exp\left[\int \frac{1}{x}dx\right] = \exp(\ln(|x|)) = x.$$

Multiplying both sides by $\mu(x)$, we see that

$$\frac{d}{dx}\left[xy\right] = x\cos(x^2)$$

Using $u = x^2$, integration yields

$$xy = \int x\cos(x^2)dx = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(x^2) + C.$$

Hence

$$y = \frac{\sin(x^2)}{2x} + \frac{C}{x}.$$

Using the initial conditions,

$$1 = \frac{\sin(\pi)}{2\sqrt{\pi}} + \frac{C}{\sqrt{\pi}} \implies C = \sqrt{\pi}.$$

Hence a solution is given by

$$y = \frac{\sin(x^2)}{2x} + \frac{\sqrt{\pi}}{x}.$$

Problem 10. Find a solution to the initial value problem

$$xy' = y + x^2\sin(x), \qquad y(\pi) = 0.$$

Solution. Since the given equation is a first order linear equation, we first bring it in the standard form

$$y' - \frac{1}{x}\cdot y = x\sin(x).$$

The integrating factor is given by

$$\mu(x) = \exp\left[-\int \frac{1}{x}dx\right] = \exp(-\ln(|x|)) = \frac{1}{x}.$$

Multiplying both sides by $\mu(x)$, we see that

$$\frac{d}{dx}\left[\frac{y}{x}\right] = \sin(x).$$

Integration yields

$$\frac{y}{x} = \int \sin(x)dx = -\cos(x) + C.$$

Hence

$$y = Cx - x\cos(x).$$

Using the initial conditions,

$$0 = C\pi - \pi\cos(\pi) \implies C = -1.$$

Hence a solution is given by

$$y = -x - x\cos(x).$$