0.1 Basic definitions of Lie algebra

Remark 0.1 (Irreducible $\neq$ Indecomposable when G is infinite). *Recall in finite group representation, every representation is a direct sum of irreducible representations (prop 1.5 in Fulton-Harris). However, it is not true in the infinite case. (note: decomposable $\implies$ reducible works for all (linear) representation).*

Consider a representation of $G = \mathbb{C}$ *by* $\rho : G \to GL_2(V) = GL_2(\mathbb{C})$ *by* $\rho(r) \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. *Then, for* $V_0 = \begin{pmatrix} a \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} V_0 = \begin{pmatrix} a \\ 0 \end{pmatrix}$. *So since* V_0 *is* G*-invariant,* V_0 *is a subrepresentation of* V*, however,* V *is indecomposable because if* V *is decomposable,* $V = W_0 \oplus W_1$ *for some* G*-invariant linear subspace* W_0, W_1 *of* V*. There exists* λ *such that* $\rho(g)u = \lambda u$ *for all* $u = (u_0, u_1) \in W_0, g \in G$*, (because* $G = \mathbb{C}$*). Then, it implies that* $u_0 + r_g u_1 = \lambda u_0, u_1 = \lambda u_1 \implies \lambda = 1$ *and* $r_g u_1 = 0$ *for all* $g \in G$*. Therefore,* $u_1 = 0$*. So,* $W_0 = \mathbb{C}(1, 0)$*. Similarly, we get* $W_1 = \mathbb{C}(1, 0)$*, which is a contradiction.*

However, if a representation is semisimple, which is a concept introduced later, gives complete reducibility (indecomposable $\iff$ *irreducible) to it (by Weyl's theorem).*

Definition 0.2 (Lie algebra). *A Lie algebra is a vector space* $\mathfrak{g}$ *with an extra operation called bracket* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that following property holds: for all* $X, Y, Z \in \mathfrak{g}$*,*
(i) $[\cdot, \cdot]$ *is bilinear,*
(ii) (alternating property) $[X, X] = 0$ *for all* $X \in \mathfrak{g}$ *(* $\iff [X, Y] = -[Y, X]$*), and*
(iii) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$*.*

Definition 0.3 (Lie subalgebra). *Let* $\mathfrak{g}$ *be a Lie algebra. A linear subspace* $\mathfrak{h} \subseteq \mathfrak{g}$ *is a Lie subalgebra if for all* $X, Y \in \mathfrak{h}$*,* $[X, Y] \in \mathfrak{h}$*.*

Definition 0.4 (Lie algebra homomorphism). *Let* $\mathfrak{g}, \mathfrak{h}$ *be Lie algebras. A linear map* $f : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism if it preserves Lie bracket i.e.,* $f([X, Y]) = [f(X), f(Y)]$ *for all* $X, Y \in \mathfrak{g}$*.*

Definition 0.5 ($\mathfrak{g}$-module). *Let* $\mathfrak{g}$ *be a Lie algebra. A vector space* V *over a field* F*, endowed with an operation* $L \times V \to V, (x, v) \mapsto xv$ *is called an* $\mathfrak{g}$*-module if for all* $a, b \in F, x, y \in \mathfrak{g}, v, w \in V$*:*
(a) $(ax + by)v = a(xv) + b(yv)$*,*
(b) $x(av + bw) = a(xv) + b(xw)$*, and*
(c) $[x, y]v = xyv - yxv$*.*

Definition 0.6 (Adjoint representation of a **Lie group**). *Let* G *be a Lie group,* T_eG *be a tangent space of* G *at the identity* e*. Let* $\Psi : G \to Aut(G), g \mapsto \Psi_g$ *where* $\Psi_g : G \to G, h \mapsto ghg^{-1}$*. Let* $\mathrm{Ad} : G \to \mathrm{Aut}(T_eG), g \mapsto Ad(g)$ *where* $Ad(g) = (d\Psi_g)_e : T_eG \to T_eG$*.* Ad *is indeed a Lie group homomorphism.*

From this definition, we define a adjoint representation for a Lie algebra associated to Lie group by taking a differential of Ad. So we define:

Definition 0.7 (Adjoint representation of a **Lie algebra**). *Let* $ad : \mathfrak{g} = T_eG \to \mathrm{End}(T_eG), X \mapsto d(Ad)_e(X)$*. We define* $[X, Y] = ad(X)(Y)$*.*

Proposition 0.8 (Bracket operation in $\mathfrak{gl}(n)$ is commutator). *As we saw in Stillwell's book, when a Lie group* $G = GL_n(\mathbb{R})$*, by looking at tangent vector at the identity* $e \in G$*,*

$$
\begin{aligned}
[X, Y] &= \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}(\gamma(t))(Y)) && (1) \\
&= \left.\frac{d}{dt}\right|_{t=0} \gamma(t) Y \gamma(t)^{-1} && (2) \\
&= \gamma'(0) Y \gamma(0) + \gamma(0) Y (-\gamma(0)^{-1} \gamma'(0)) && (3) \\
&= XY - YX && (4)
\end{aligned}
$$

where $\gamma : [0, 1] \to G$ *is a (differentiable) path such that* $\gamma(0) = e, \gamma'(0) = X$*.*

Definition 0.9 (Adjoint Representation). *Let V be a $\mathfrak{g}$-module. A map* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(\mathfrak{g}), X \to \mathrm{ad}(X)$ *where* $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ *by* $\mathrm{ad}(X)(Y) = [X, Y]$ *is called adjoint representation. Indeed,* ad *is a Lie algebra homomorphism (linearity is easy), i.e.,*

$$
\begin{aligned}
\mathrm{ad}([X,Y])(Z) &= [[X,Y],Z] && (5)\\
&= -[Z,[X,Y]] \quad (\because \textit{alternating property}) && (6)\\
&= [X,[Y,v]] + [Y,[Z,X]] \quad (\because \textit{Jacobi's identity}) && (7)\\
&= [X,[Y,Z]] - [Y,[X,Z]] && (8)\\
&= \mathrm{ad}(X)(\mathrm{ad}(Y)(Z)) - \mathrm{ad}(Y)(\mathrm{ad}(X)(Z)) && (9)\\
&= [\mathrm{ad}(X), \mathrm{ad}(Y)](Z) && (10)
\end{aligned}
$$

since this is for all $Z \in \mathfrak{g}$, $\mathrm{ad}([X,Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$.

Definition 0.10 (Ideal of a Lie algebra). *A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ of Lie algebra $\mathfrak{g}$ is said to be an ideal if $[X,Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}, Y \in \mathfrak{g}$.*

Definition 0.11 (Quotient algebra). *Let $\mathfrak{g}$ be a Lie algebra, I be an ideal of $\mathfrak{g}$. A quotient algebra of a $\mathfrak{g}/I$ is a quotient (vector) space of $\mathfrak{g}$ with Lie bracket $[\bar{X}, \bar{Y}] := \overline{[X,Y]}$.*

This is operation is indeed well-defined because for $X \sim X', Y \sim Y'$, $X' = X + u, Y' = Y + v$ for some $u, v \in I$, then

$$
\begin{aligned}
[\bar{X'}, \bar{Y'}] = \overline{[X', Y']} &= \overline{[X+u, Y+v]} && (11)\\
&= \overline{[X+u, Y] + [X+u, v]} && (12)\\
&= \overline{[X,Y] + [u,Y] + [X,v] + [u,v]} && (13)\\
&= \overline{[X,Y]} = [\bar{X}, \bar{Y}]. && (14)
\end{aligned}
$$

Definition 0.12 (Center of Lie algebra). *The center $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ such that for all $X \in Z(\mathfrak{g})$, $[X,Y] = 0$ for all $Y \in \mathfrak{g}$.*

Definition 0.13 (Simple Lie algebra). *A Lie algebra $\mathfrak{g}$ is simple if* $\dim \mathfrak{g} > 1$ *(as vector space) and it contains no nontrivial ideals, i.e., only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.*

Definition 0.14 (Lower central series). *Let $\mathfrak{g}$ be a Lie algebra. A lower central series of subalgebras $D_k\mathfrak{g}$ is defined inductively by $D_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $D_k\mathfrak{g} = [\mathfrak{g}, D_{k-1}\mathfrak{g}]$.*

Remark 0.15. *$D_k\mathfrak{g}$ is indeed an ideal.*

Proof. Since for all $h \in D_1\mathfrak{g}$, $h = [X,Y]$ for some $X, Y \in \mathfrak{g}$, and for all $Z \in \mathfrak{g}$, $[h,Z] = [[X,Y],Z] = -[Z,[X,Y]] = [X,[Y,Z]] + [Y,[Z,X]] \in [\mathfrak{g},\mathfrak{g}] = D_1\mathfrak{g}$. Suppose $D_k\mathfrak{g}$ is an ideal. Then, for all $h \in D_{k+1}\mathfrak{g}$, $h = [X,Y]$ for some $X \in \mathfrak{g}, Y \in D_k\mathfrak{g}$, and $[h,Z] = [[X,Y],Z] = -[Z,[X,Y]] = [X,[Y,Z]] + [Y,[Z,X]] \in [\mathfrak{g}, D_k\mathfrak{g}] = D_{k+1}\mathfrak{g}$. By induction, $D_k\mathfrak{g}$ is an ideal for all $k \in \mathbb{Z}_{\geq 1}$. □

Definition 0.16 (Derived series). *Let $\mathfrak{g}$ be a Lie algebra. A derived series of subalgebras $D^k\mathfrak{g}$ is defined inductively by $D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$.*

Remark 0.17. *$D^k\mathfrak{g}$ is indeed an ideal.*

Proof. Base case is the same as lower central series. Suppose $D^k\mathfrak{g}$ is an ideal. Then for all $h \in D^{k+1}$, $h = [X,Y]$ for some $X, Y \in D^k\mathfrak{g}$. For all $Z \in \mathfrak{g}$, $[h,Z] = [[X,Y],Z] = -[Z,[X,Y]] = [X,[Y,Z]] + [Y,[Z,X]] \in [D^k\mathfrak{g}, D^k\mathfrak{g}] = D^{k+1}\mathfrak{g}$. Therefore, by induction, $D^k\mathfrak{g}$ is an ideal for all $k \in \mathbb{Z}_{\geq 1}$. □

Definition 0.18 (Nilpotent). *A Lie algebra $\mathfrak{g}$ is said to be nilpotent if $D_k\mathfrak{g} = 0$ for some k.*

Definition 0.19 (Solvable). *A Lie algebra $\mathfrak{g}$ is said to be solvable if $D^k\mathfrak{g} = 0$ for some k.*

Example 0.20 (Solvable Lie algebra). *An abelian Lie algebra* $\mathfrak{g}$ *i.e.,* $[X,Y]=0$ *for all* $X,Y\in\mathfrak{g}$ *is a solvable Lie algebra.*

Remark 0.21 (Simple implies NOT solvable). *If a Lie algebra* $\mathfrak{g}$ *is simple, then* $\mathfrak{g}$ *is not solvable.*

Assume it is solvable. Then, since $D^k\mathfrak{g}$ *is an ideal, it has to be* $D^k\mathfrak{g}=0$ *for some* k*. Since derived series is a descending chain,* $k=1$*. And this implies that* $\mathfrak{g}$ *is abelian, which contradicts to* $\dim\mathfrak{g}>1$*.*

Example 0.22 (Nilpotent Lie algebra). *Strictly upper triangular matrix.*

Proposition 0.23 (Equivalent condition to solvability). *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{g}$ *has a sequence of Lie subalgebras* $\mathfrak{g}=\mathfrak{g}_o\supset\mathfrak{g}_1\supset\cdots\supset\mathfrak{g}_k=\{0\}$ *such that* $\mathfrak{g}_{i+1}$ *is an ideal in* $\mathfrak{g}_i$ *and* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian.*

Proof. ($\Longrightarrow$) Since $\mathfrak{g}$ is solvable, there exists $D^k\mathfrak{g}=0$. So by taking $\mathfrak{g}_k=D^k$ and $\mathfrak{g}_0=\mathfrak{g}$, we are done.

($\Longleftarrow$) Since $\mathfrak{g}_0/\mathfrak{g}_1$ is abelian, for all $h\in D^1\mathfrak{g}, h=[X,Y]$ for some $X,Y\in\mathfrak{g}$. Since $[X,Y]\in\mathfrak{g}_0$ and $[X,Y]=0$ in $\mathfrak{g}_0/\mathfrak{g}_1$, $[X,Y]\in\mathfrak{g}_1$. Suppose $D^k\mathfrak{g}\subseteq\mathfrak{g}_k$. For all $h\in D^{k+1}\mathfrak{g}$, there exist $X,Y\in D^k\mathfrak{g}\subseteq\mathfrak{g}_k$ s.t. $h=[X,Y]$. Since $[X,Y]\in\}_k$ because $\mathfrak{g}_k$ is an ideal, and also $[X,Y]=0$ because $\mathfrak{g}_k/\mathfrak{g}_{k+1}$ is abelian, it implies that $[X,Y]\in\mathfrak{g}_{k+1}$. □

Proposition 0.24. *Let* L *be a Lie algebra. Then, (a) If* L *is solvable, then so are all every subalgebra* $h\in L$ *and homomorphic images of* L*.*
(b) If I *is a solvable ideal of* L *such that* L/I *is solvable, then* L *is solvable.*
(c) If I,J *are solvable ideals of* L*, then so is* $I+J$*.*

Proof. (a) Since $h\subseteq L$, its derived series $D^kh\subseteq D^kL=0$ for some k.

If a Lie algebra homomorphism $f:L\to\operatorname{Im}f$, then $0=f(D^kL)=D^kf(L)=D^k\operatorname{Im}f$.

(b) Since L/I is solvable, $D^kL/I=0$ for some k. Let $\pi:L\to L/I$ be a quotient map. Then $f(D^kL/I)=0\implies D^kL\subseteq I$. If $D^mI=0$ for some m, then $D^mD^kL=D^{m+k}L\subseteq D^mI=0$.

(c) Let $f:I\to I+J/J$ be a Lie group epimorphism (surjective homomorphism). By isomorphism theorem, $I/(I\cap J)\cong(I+J)/J$. By (a), $(I+J)/J$ is solvable and since J is solvable, by (b), $I+J$ is solvable. □

Definition 0.25 (Semisimple). *A Lie algebra* $\mathfrak{g}$ *is said to be semisimple if* $\mathfrak{g}$ *has no nonzero solvable ideals.*

Remark 0.26. *Let* V *be finite-dimensional vector space over a field* F*. If* $X\in\operatorname{End}(V)$*,* x *is semisimple if the roots of the characteristic polynomial over* F *are all distinct* $\Longleftrightarrow$ X *is diagonalizable.*

Definition 0.27 (Radical of a Lie algebra). *Let* $\mathfrak{g}$ *be a Lie algebra. Then the radical* $\operatorname{Rad}(\mathfrak{g})$ *of* $\mathfrak{g}$ *is the maximal solvable ideal.*

Remark 0.28. $\operatorname{Rad}(\mathfrak{g})$ *is unique.*

Proof. If I,J are maximal solvable ideals, then $I+J$ is also solvable and $I\subseteq I+J$. By maximality of I, $J\subseteq I$. Similarly, $I\subseteq J$. Therefore, $I=J$. □

Proposition 0.29. *Simple* $\implies$ *semisimple.*

Proof. If a Lie algebra $\mathfrak{g}$ is simple, then it has no nontrivial ideal, therefore, $\operatorname{Rad}(\mathfrak{g})=0$ or $\operatorname{Rad}(\mathfrak{g})=\mathfrak{g}$. If $\operatorname{Rad}(\mathfrak{g})=\mathfrak{g}$, then it contradicts to remark 0.21. Thus, $\operatorname{Rad}(\mathfrak{g})=0$. □

Proposition 0.30. $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ *is semisimple.*

Proof. Since $\operatorname{Rad}(\mathfrak{g})$ is maximal solvable ideal, for every solvable ideal I, $I\subseteq\operatorname{Rad}(\mathfrak{g})$ because $I+\operatorname{Rad}(\mathfrak{g})=\operatorname{Rad}(\mathfrak{g})$. And by quotient map $\pi:\mathfrak{g}\to\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$, $\pi(I)=0$. Therefore, $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ has no nonzero solvable ideal. □

A short exact sequence

$$0\to\operatorname{Rad}(\mathfrak{g})\to\mathfrak{g}\to\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})\to 0$$

is true for all Lie algebra $\mathfrak{g}$. Therefore, it is extremely important to understand solvable ideals and semisimple Lie algebras ($\mathfrak{g}/\operatorname{Rad}$).

0.2 $\mathfrak{sl}_2\mathbb{C}$

Slogan: $\mathfrak{sl}_2\mathbb{C}$ plays a crucial role for understanding a general semisimple Lie algebra because any semisimple Lie algebra contains copies of $\mathfrak{sl}_2\mathbb{C}$ as a Lie subalgebra. Also, surprisingly, a representation of any semisimple Lie algebra can be well-understood through $\mathfrak{sl}_2\mathbb{C}$ and a finite group called Weyl group, so it is important to first look at the representation of $\mathfrak{sl}_2\mathbb{C}$,

Proposition 0.31. *$\mathfrak{sl}_n\mathbb{C}$ has trace zero.*

Proof. Since $SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \mid \det A = 1\}$, for all $X \in \mathfrak{sl}_n\mathbb{C}$, consider a path $\gamma : [0,1] \to SL_n(\mathbb{C})$ by $\gamma(t) = e^{tX}$ then $\gamma(0) = 1, \gamma'(0) = X$. Then, since $\det e^{tX} = 1$ for all t, $1 = \mathrm{e}^{\mathrm{tr\,tX}} = \mathrm{e}^{\mathrm{t\,tr\,X}}$. Therefore, $\operatorname{tr} X = 0$. ☐

Proposition 0.32. *$\mathfrak{sl}_2(\mathbb{C})$ is simple, thus semisimple.*

Proof. Since $\mathfrak{sl}_2\mathbb{C}$ has trace zero, every element of $\mathfrak{sl}_2\mathbb{C}$ is of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Therefore, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a basis for $\mathfrak{sl}_2\mathbb{C}$. and $[H, X] = HX - XH = 2X, [H, Y] = -2Y, [X, Y] = H$. Let I be a nonzero ideal of $\mathfrak{sl}_2(\mathbb{C})$. Then, there exists an element in I of the form $aX + bY + cH \neq 0$ for some $a, b, c \in \mathbb{C}$. Then, $\operatorname{ad}(X)^2(aX + bY + cH) = \operatorname{ad}(X)(bH - 2cX) = -2bX$ and $\operatorname{ad}(Y)^2(aX + bY + cH) = \operatorname{ad}(Y)(-aH + 2cY) = -2aY$. Therefore, $X, Y \in I \implies [X, Y] = H \in I \implies I = \mathfrak{sl}_2\mathbb{C}$. Thus, $\mathfrak{sl}_2\mathbb{C}$ is simple. ☐

Theorem 0.33 (Jordan-Chevalley decomposition)**.** *Let V be a finite dimensional vector space over $\mathbb{C}$, $x \in \operatorname{End}(V)$. There exist unique $x_s, x_n \in \operatorname{End}(V)$ satisfying the conditions: $x = x_s + x_n$, where x_s is semisimple and x_n is nilpotent and $x = x_n + x_s$.*

Theorem 0.34 (Weyl's theorem)**.** *Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional representation of a semisimple Lie algebra. Then ϕ is completely reducible.*

Theorem 0.35 (Preservation of Jordan Decomposition)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. For any element $X \in \mathfrak{g}$, there exist X_s and $X_n \in \mathfrak{g}$ such that for any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ we have*

$$\rho(X)_s = \rho(X_s) \text{ and } \rho(X)_n = \rho(X_n)$$

By the preceding theorem, since H is a diagonal matrix, the action of H on V is diagonalizable, i.e., $\rho(H)$ is semisimple (diagonalizable) where ρ is a representation. Therefore, an irreducible representation V of $\mathfrak{sl}_2\mathbb{C}$ can be written as a direct sum of eigenspaces of the representation of H i.e., $V = \oplus_{\alpha \in \Lambda} V_\alpha$ where Λ is the set of eigenvalues of $\rho(H)$. Then $H(v) = \alpha v$ for all $v \in V_\alpha$.

(More precisely, if $\rho : \mathfrak{sl}_2\mathbb{C} \to GL(V)$ is a adjoint representation, then $\Lambda = \{\alpha \in \mathbb{C} \mid \rho(H)(v) = \alpha v\}$.)

Proposition 0.36 (Fundamental Calculation of $\mathfrak{sl}_2\mathbb{C}$)**.** *$H(v) = \alpha v$, $H(X(v)) = (\alpha + 2)X(v), H(Y(v)) = (\alpha - 2)X(v)$.*

Proof. Since

$$\begin{aligned} H(X(v)) &= [H, X](v) + X(H(v)) && (15) \\ &= 2X(v) + X(\alpha v) && (16) \\ &= (\alpha + 2)X(v), && (17) \end{aligned}$$

this implies that if v is an eigenvector for H with eigenvalue α, then $X(v)$ is also eigenvector for H, with eigenvalue $\alpha + 2$. In other words, $X : V_\alpha \to V_{\alpha+2}$.

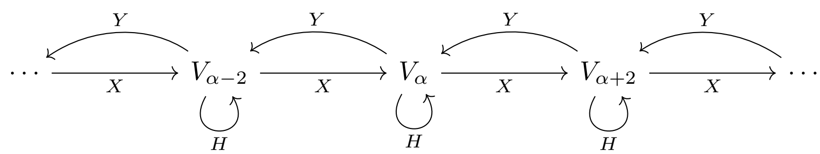
Moreover, since

$$H(Y(v)) = [H,Y](v) + Y(H(v)) \tag{18}$$
$$= -2Y(v) + Y(\alpha v) \tag{19}$$
$$= (\alpha - 2)Y(v), \tag{20}$$

so $Y : V_\alpha \to V_{\alpha-2}$. □

Now that both $X(v)$ and $Y(v)$ are eigenvector for H with eigenvalues $(\alpha+2), (\alpha-2)$ respectively. So X sends eigenvector $v \in V_\alpha$ to $X(v) \in V_{\alpha+2}$, $Y(v) \in V_{\alpha-2}$.

We call that each α a weight and V_α a weight space. If a weight m_0 has $V_{m_0} \neq 0$ and $V_{m_0+2} = 0$, then m_0 is called highest weight (this is well-defined because $\dim V < \infty$) and elements of V_{m_0} are called maximal vector. The result can be interpreted as this picture:



Lemma 0.37. *Let $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, V be irreducible $\mathfrak{g}$-module. Choose a maximal vector $v_0 \in V_\alpha$; set $v_{-1} = 0, v_i = \frac{1}{i!}Y^i v_0 (i \geq 0)$. Then the following statements hold:*
(a) $Hv_i = (\alpha - 2i)v_i$,
(b) $Yv_i = (i+1)v_{i+1}$, and
(c) $Xv_i = (\alpha - i + 1)v_{i-1}$.

Proof. (a) Since $Y^i v_0 \in V_{\alpha-2i}$, so for $v_i \in V_{\alpha-2i}$, $Hv_i = (\alpha-2i)v_i$. (b) $Yv_i = \frac{1}{i!}Y^{i+1}v_0 = (i+1)\frac{1}{i+1!}Y^{i+1}v_0 = (i+1)v_{i+1}$.

(c) Use induction. When $i = 0$ is clear because $Xv_0 = 0$ and $v_{-1} = 0$ by definition. Suppose (c) is true up to $i-1$, then

$$\begin{aligned} iXv_i &= iXY(1/i!)Y^{i-1}v_0 = XYv_{i-1} \\ &= [X,Y]v_{i-1} + YXv_{i-1} \\ &= Hv_{i-1} + YXv_{i-1} \\ &= (\alpha - 2(i-1))v_{i-1} + (\alpha - i + 2)Yv_{i-2} \\ &= (\alpha - 2i + 2))v_{i-1} + (i-1)(\alpha - i + 2)v_{i-1} \\ &= i(\alpha - i + 1)v_{i-1}. \end{aligned}$$

□

Corollary 0.38. *The highest weight α of a given representation is an integer.*

Proof. Following the same notation with the preceding lemma, let m be the smallest integer such that $v_m \neq 0$ and $v_{m+1} = 0$. When $i = m+1$, since by (c) $0 = Xv_i = (\alpha - i + 1)v_{i-1} = (\alpha - m)v_m$ and $v_m \neq 0$, $\alpha = m$. □

Theorem 0.39. *Let V be an irreducible $(m+1)$ dimensional $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$ module and m_0 be the highest weight. Then,*
(a) $V = \oplus_{i=0}^m V_{m-2i} = V_m \oplus V_{m-2} \oplus \cdots \oplus V_{-m+2} \oplus V_{-m}$, in particular, $m = m_0$.
(b) For each weight μ, $\dim V_\mu = 1$ if $V_\mu \neq 0$.

(c) The matrix representations of H, X, Y with regard to the basis $\langle v_0, v_1, \ldots, v_{m_0}\rangle$ are as follows:

$$H = \begin{bmatrix} m & 0 & \cdots & 0 \\ 0 & m-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -m \end{bmatrix}, X = \begin{bmatrix} 0 & m & 0 & \cdots & 0 \\ 0 & 0 & m-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & m & 0 \end{bmatrix}.$$

Let $\phi_m : \mathrm{End}(V) \to M_{m+1}(\mathbb{C})$ be this matrix representation.

Proof. (a) Since by using the same notation as preceding lemma, $\langle v_0, v_1, \ldots, v_{m_0}\rangle \neq 0$ is a linear subspace of V and in fact, a $\mathfrak{g}$-invariant subspace because by the preceding lemma. Because also $v_0, v_1, \ldots, v_{m_0}$ are eigenvector that has different eigenvalues, $v_0, v_1, \ldots, v_{m_0}$ are linearly independent. By irreducibility of V, we have $V = \langle v_0, v_1, \ldots, v_{m_0}\rangle$. This implies that $m_0 + 1 = \dim\langle v_0, v_1, \ldots, v_{m_0}\rangle = \dim V = m+1 \implies m_0 = m$. Since $v_i \in V_{m_0-2i}$ by lemma, $V = \langle v_0, v_1, \ldots, v_{m_0}\rangle = \oplus_{i=1}^{m}\langle v_i\rangle \subseteq \oplus_{i=1}^{m} V_{m-2i} = V$. And this tower of inclusion also implies that V_{m-2i} is generated by a single element $\langle v_i\rangle$, which proves (b).

(c) follows from the preceding lemma. □

Corollary 0.40. *Let $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, V be any finite dimensional $\mathfrak{g}$-module. Then by Weyl's theorem $V = \oplus_{i=1}^{r} W_i$ for some $r \in \mathbb{Z}_{\geq 0}$ and W_i's are irreducible $\mathfrak{g}$-invariant subspace of V. Then $r = \dim V_0 + \dim V_1$ where V_0, V_1 are the eigenspaces of eigenvalue 0 and 1 respectively. In particular, V is irreducible $\iff \dim V_0 + \dim V_1 = 1$. Therefore, ϕ_m is irreducible representation.*

Proof. Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. By the preceding theorem, for all $1 \leq i \leq r$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $\phi_{|W_i} \equiv \phi_m$. In W_i, since $\phi(H) = \begin{bmatrix} m & 0 & \cdots & 0 \\ 0 & m-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -m \end{bmatrix}$, the eigenvalue of $\phi(H)$ are $m, m-2, \ldots, -m$. Therefore, for each W_i, we have either 0 or 1 as an eigenvalue and by the preceding theorem, its dimension is 1. Therefore, $\dim(W_i)_0 + \dim(W_i)_1 = 1$. Thus,

$$r = \sum_{i=1}^{r} \dim(W_i)_0 + \dim(W_i)_1 \tag{21}$$
$$= \dim \oplus_{i=1}^{r}(W_i)_0 + \dim \oplus_{i=1}^{r}(W_i)_1 \tag{22}$$
$$= \dim V_0 + \dim V_1. \tag{23}$$

□

Remark 0.41. *By the corollary, there exists unique representation $V^{(n)}$ for each $n \in \mathbb{Z}_{\geq 0}$. $V^{(n)}$ is $(n+1)$-dimensional vector space with eigenvalues $n, n-2, \ldots, -n+2, -n$. And this is indeed irreducible representation as we shall see later.*

So an irreducible representation V of $\mathfrak{sl}_2\mathbb{C}$ is completely determined by its highest weight.

$$\{\textit{representation } V\} \iff \{\textit{highest weight } m\}.$$

Remark 0.42. *Any representation V of $\mathfrak{sl}_2\mathbb{C}$ such that the eigenvalues of H all have the same parity (odd or even) with multiplicity one $\implies$ irreducible by the preceding corollary.*

Example 0.43. *A trivial representation $V^{(0)}$ has $\dim V^{(0)} = 1$. So it is irreducible by the corollary.*

Example 0.44. *Let V be the standard representation of $\mathbb{C}^2$ i.e., $V = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2 \mid a+b=0\} = \{\begin{pmatrix} a \\ -a \end{pmatrix} \mid a \in \mathbb{C}\}$. Then, for standard basis $x = (1,0), y = (0,1)$, $H(x) = x, H(y) = -y$ by matrix calculation. Therefore, $V = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y = V_{-1} \oplus V_1$. Thus, V is irreducible and in fact, $V = V^{(1)}$.*

Example 0.45. *With the same notation with the previous example, let* $W = \operatorname{Sym}^2 V = \operatorname{Sym}^2 \mathbb{C}^2 = \langle x^2, xy, y^2 \rangle$*. By (8.12) in Fulton-Harris, we have* $H(x \cdot x) \coloneqq x \cdot H(x) + H(x) \cdot x = 2x \cdot x, H(x \cdot y) = x \cdot H(y) + H(x) \cdot y = 0, H(y \cdot y) = yH(y) + H(y)y = -2y \cdot y$*. So the representation* $W = \mathbb{C} \cdot x^2 \oplus \mathbb{C} \cdot xy \oplus \mathbb{C} \cdot y^2 = W_{-2} \oplus W_0 \oplus W_2$*. And this representation is indeed irreducible and* $\operatorname{Sym}^2 V = V^{(2)}$*.*

Theorem 0.46. *Any irreducible representation of* $\mathfrak{sl}_2\mathbb{C}$ *is a symmetric power of the standard representation* $V \cong \mathbb{C}^2$*.*

Proof. $\operatorname{Sym}^n V$ of V has a basis $\langle x^n, x^{n-1}y, \ldots, y^n \rangle$ and we have $H(x^{n-k}y^k) = (n-k) \cdot H(x) \cdot x^{n-k-1}y^k + k \cdot H(y) \cdot x^{n-k}y^{k-1} = (n-2k)x^{n-k}y^k$. Therefore, eigenvalues of H on $\operatorname{Sym}^n V$ are $n, n-2, \ldots, -n$ and by the corollary, $\operatorname{Sym}^n V$ is irreducible $\implies V^{(n)} = \operatorname{Sym}^n V$. $\square$