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HOROCYCLES IN HYPERBOLIC 3-MANIFOLDS

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1 Introduction

Let $M = \Gamma\backslash\mathbb{H}^3$ be a complete hyperbolic 3-manifold. A *horocycle* $\chi \subset M$ is an isometrically immersed copy of $\mathbb{R}$ with zero torsion and geodesic curvature 1. The torsion condition means that χ lies in an immersed totally geodesic plane.

One can regard χ as a limit of planar circles whose centers have moved off to infinity. It is natural to ask what the possibilities are for its closure,

$$\overline{\chi} \subset M.$$

When M has finite volume, it is well–known that strong rigidity properties hold; e.g. $\overline{\chi}$ is always a properly immersed, homogeneous submanifold of M [Sh], [Rn]. Continuing the investigation begun in [MMO], this paper shows that rigidity persists for horocycles in certain *infinite volume* 3-manifolds. These are the first examples of Zariski dense discrete groups $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$, other than lattices, where the topological behavior of horocycles in $\Gamma\backslash\mathbb{H}^3$ has been fully described.

Horocycles in acylindrical manifolds. To state the main results, recall that the *convex core* of M is given by:

$$\mathrm{core}(M) = \Gamma\backslash\,\mathrm{hull}(\Lambda) \subset M,$$

where $\Lambda \subset \widehat{\mathbb{C}}$ is the limit set of Γ, and $\mathrm{hull}(\Lambda) \subset \mathbb{H}^3$ is its convex hull. We say M is a *rigid acylindrical manifold* if its convex core is a compact submanifold of M with nonempty, totally geodesic boundary. Our first result describes the behavior of horocycles in M.

Theorem 1.1. *Let $\chi \subset M = \Gamma\backslash\mathbb{H}^3$ be a horocycle in a rigid acylindrical 3-manifold. Then either:*

1. *$\chi \subset M$ is a properly immersed 1-manifold; or*
2. *$\overline{\chi} \subset M$ is a properly immersed 2-manifold, parallel to a totally geodesic surface $S \subset M$; or*
3. *$\overline{\chi}$ is the entire 3-manifold M.*

Corollary 1.2. *The closure of any horocycle is a properly immersed submanifold of M.*

Similar results for geodesic planes in M are obtained in [MMO].

Homogeneous dynamics. To make Theorem 1.1 more precise, we reformulate it in terms of the frame bundle $\mathrm{F}M \to M$.

Let G denote the simple, connected Lie group $\mathrm{PGL}_2(\mathbb{C})$. Within G, we have the following subgroups:

$$\begin{aligned}
H &= \mathrm{PSL}_2(\mathbb{R}),\\
A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,:\, a \in \mathbb{R}_+ \right\},\\
K &= \mathrm{SU}(2)/(\pm I),\\
N &= \left\{ n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \,:\, s \in \mathbb{C} \right\},\\
U &= \{ n_s \,:\, s \in \mathbb{R} \}, \quad \text{and}\\
V &= \{ n_s \,:\, s \in i\mathbb{R} \}.
\end{aligned}$$

Upon identifying $\mathbb{H}^3$ with G/K, we obtain the natural identifications

$$\mathrm{F}M \cong \Gamma\backslash G \quad \text{and} \quad M \cong \Gamma\backslash G/K.$$

Every (oriented) horocycle χ in M lifts to a unique unipotent orbit xU in the frame bundle $\mathrm{F}M$. Let $A_+ = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \,:\, a \geq 1\}$ be the positive semigroup in A, and let

$$\mathrm{RF}_+M = \{ x \in \mathrm{F}M \,:\, \overline{xA_+} \subset \mathrm{F}M \text{ is compact} \}.$$

This locus is closed and invariant under AN.

Our main result may now be stated as follows (see Sect. 6).

Theorem 1.3. *Let $M = \Gamma\backslash\mathbb{H}^3$ be a rigid acylindrical 3-manifold. Then for any $x \in \mathrm{F}M$, either*

1. *xU is closed;*
2. *$\overline{xU} = xvHv^{-1} \cap \mathrm{RF}_+M$ for some $v \in V$; or*
3. *$\overline{xU} = \mathrm{RF}_+M$.*

It is readily verified that these three alternatives give the three cases in Theorem 1.1, using the fact that the map $\mathrm{F}M \to M$ is proper and its restriction to RF_+M is surjective.

Corollary 1.4. *The closure of any U–orbit in RF_+M is homogeneous, in the sense that*

$$\overline{xU} = xS \cap \mathrm{RF}_+M$$

for some closed subgroup $S \subset G$ with $U \subset S$.

Indeed, we can take $S = U$, vHv^{-1} or G. As we will see in Sect. 7, the classification of AU–orbits follows from Theorem 1.3 as well:

Corollary 1.5. *For any $x \in \mathrm{RF}_+M$, we have $\overline{xAU} = \overline{xH} \cap \mathrm{RF}_+M$.*

The possibilities for $\overline{xH}$ are recalled in Theorem 2.3 below. (For $x \notin \mathrm{RF}_+M$, it is easy to see that the orbit xAU is closed.)

Strategy. The mechanism behind the proof of Theorem 1.3 is the following dichotomy. Suppose a horocycle $\chi \subset M$ limits on a properly embedded, totally geodesic surface S (such as one of the boundary components of the convex core of M). If χ is contained in S then χ is *trapped* and $\overline{\chi} = S$; otherwise, χ is *scattered* by S, and $\overline{\chi} = M$. In both cases the behavior of χ is strongly influenced by the behavior of the horocycle flow on S. To complete the proof we show that, up to the action of V, *every* recurrent horocycle accumulates on such a surface S. This step uses the classification of H–orbits from [MMO].

We remark that any connected subgroup of G generated by unipotent elements is conjugate to N, H or U. Theorem 1.3 completes the description of the topological dynamics of these groups acting on $\mathrm{F}M$, since the behavior of H and N was previously known (see Sect. 2).

Outline of the paper. The remainder of the paper is devoted to the proof of Theorem 1.3. In Sect. 2 we review existing results about dynamics on $\mathrm{F}M$. In Sect. 3 we establish a general lemma about the double coset space $U\backslash G/H$, and in Sect. 4 we prove an approximation theorem for U–orbits. The space of exceptional frames is introduced in Sect. 5, and the proof of Theorem 1.3 is completed in Sect. 6. Corollary 1.5 is deduced in Sect. 7.

Remark. General acylindrical manifolds. When M is a convex cocompact, acylindrical manifold that is *not* rigid, the behavior of horocycles can be radically different from the rigid case. For example, a horocycle orthogonal to a closed leaf of the bending lamination of $\partial \operatorname{core}(M)$ can be properly embedded, giving rise to a frame $x \in \mathrm{F}M$ with a compact A–orbit and a nonrecurrent U–orbit. The scattering argument also breaks down, due to the lack of totally geodesic surfaces in M. It is an open problem to develop a rigidity theory for these and other infinite-volume hyperbolic 3-manifolds.

2 Background

In this section we introduce notation and recall known results regarding topological dynamics on $\mathrm{F}M$.

Geometry on $\mathbb{H}^3$. Notation for G and its subgroups was introduced in Sect. 1. We also let $A_{\mathbb{C}} = \{\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1}\end{smallmatrix}\right) \ : \ z \in \mathbb{C}^*\}$. The action of G on $\mathbb{H}^3 = G/K$ extends continuously to a conformal action of G by Möbius transformations on the Riemann sphere,

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong G/A_{\mathbb{C}}N,$$

and the union $\mathbb{H}^3 \cup \widehat{\mathbb{C}} \cong B^3$ is compact. We let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ denote the standard circle on $\widehat{\mathbb{C}}$. Its orientation–preserving stabilizer in G is H.

Let $M = \Gamma\backslash\mathbb{H}^3$ be a hyperbolic 3-manifold. The natural covering map

$$\mathrm{F}\mathbb{H}^3 \cong G \to \mathrm{F}M \cong \Gamma\backslash G$$

will be denoted by $g \mapsto [g]$. The *limit set* of Γ is characterized by $\Lambda = \widehat{\mathbb{C}} \cap \overline{\Gamma p}$, for any $p \in \mathbb{H}^3$; the *domain of discontinuity* is its complement, $\Omega = \widehat{\mathbb{C}} - \Lambda$. The convex hull of Λ is the smallest convex subset of $\mathbb{H}^3$ containing all geodesics with both endpoints in the limit set; and its quotient gives the convex core of M:

$$\mathrm{core}(M) = \Gamma\backslash \operatorname{hull}(\Lambda) \subset M.$$

A discrete group is *elementary* if it contains an abelian subgroup with finite index. We will always assume that $\Gamma \cong \pi_1(M)$ is a nonelementary group.

Surfaces in M. There is a natural correspondence between

(i) Closed H–orbits $[xH] \subset \mathrm{F}M$,
(ii) Properly immersed, totally geodesic surfaces $S \subset M$, and
(iii) Circles $C \subset \widehat{\mathbb{C}}$ such that $[\Gamma C]$ is discrete in the space of all circles, $\mathcal{C} \cong G/H$.

This correspondence is given, with suitable orientation conventions, by $S =$ the projection of $\operatorname{hull}(C) \subset \mathbb{H}^3$ to M, $xH = \mathrm{T}S$, the bundle of frames tangent to S, and $[\Gamma C] = [xH]$ in $\Gamma\backslash G/H$.

Convex cocompact manifolds. Now assume that the convex core of M is compact. The *renormalized frame bundle* of M is defined by

$$\mathrm{RF}M = \{x \in \mathrm{F}M \ : \ \overline{xA} \subset \mathrm{F}M \text{ is compact}\}.$$

Replacing A with A_+ in the definition above, we obtain the locus RF_+M. Note that $\mathrm{RF}M$ is invariant under A and RF_+M is invariant under AN.

In terms of the universal cover, we have $[g] \in \mathrm{RF}_+M$ if and only if $g(\infty) \in \Lambda$, while $[g] \in \mathrm{RF}M$ if and only if $\{g(0), g(\infty)\} \subset \Lambda$.

Minimality. We now turn to some dynamical results. Let L be a closed subgroup of G. We say $X \subset \mathrm{F}M$ is an *L–minimal set* if $\overline{xL} = X$ for all $x \in X$.

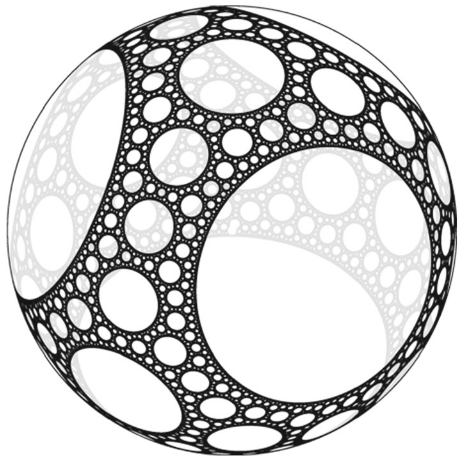

Figure 1: Limit set of a rigid acylindrical manifold

Theorem 2.1. (Ferte) *If M is convex cocompact, then the locus* RF_+M *is an N–minimal set.*

See [Fer, Cor.C(iii)]; a generalization appears in [Win]. We also record the following result from [Da]:

Theorem 2.2. (Dal'bo) *If $\Gamma \subset H$ is a nonelementary convex cocompact Fuchsian group, then* $(\Gamma\backslash H) \cap \mathrm{RF}_+M$ *is a U–minimal set.*

Rigid acylindrical manifolds. Recall that M is a *rigid acylindrical* manifold if M is convex cocompact, of infinite volume, and $\partial\,\mathrm{core}(M)$ is totally geodesic. In this case $\Omega \subset \widehat{\mathbb{C}}$ is the union of a dense set of round disks with disjoint closures, and Λ is a Sierpiǹski curve; see Fig. 1.

Theorem 2.3. *Let M be a rigid acylindrical manifold. Then for any $x \in \mathrm{RF}M$, either xH is closed or $\overline{xH} = (\mathrm{RF}_+M)H$.*

Proof. Since Ω is a union of round disks, any circle that meets Λ in just one point can be approximated by a circle meeting Λ in two or more points; thus

$$\overline{(\mathrm{RF}M)H} = (\mathrm{RF}_+M)H. \tag{2.1}$$

Let $H' = \mathrm{PGL}_2(\mathbb{R}) = H \cup jH$, where $j = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Note that $Aj = jA$ and hence $(\mathrm{RF}M)j = \mathrm{RF}M$.

With H' in place of H, Theorem 2.3 is proved in [MMO, Cor. 1.7]. Using the H' version, we can conclude that either xH is closed or $\overline{xH'} = (\mathrm{RF}_+M)H'$. In the latter case, $\mathrm{RF}M$ is contained in $\overline{xH} \cup \overline{xH}j$. But $\mathrm{RF}M$ has a dense A–orbit [MMO, Thm. 4.3], so $\mathrm{RF}M$ is contained in $\overline{xH}$ or $\overline{xH}j$. In either case, we have

$$\mathrm{RF}M = (\mathrm{RF}M)j \subset \overline{xH}j^2 = \overline{xH}.$$

Hence $\overline{xH} = (\mathrm{RF}_+M)H$ by equation (2.1) above. $\square$

3 Configuration Spaces and Double Cosets

This section and the next present two self–contained results that will be used in Sect. 6 below. In this section we will prove:

Theorem 3.1. *Suppose* $g_n \to \mathrm{id}$ *in* $G - VH$, *and* $T_n \subset U$ *is a sequence of* K*–thick sets. Then there is a* K'*–thick set* $V_0 \subset V$ *such that*

$$\limsup T_n g_n H \supset V_0.$$

Double cosets. As motivation for the theorem, we remark that the double coset space $U\backslash G/H$ is the moduli space of pairs $(\chi, P) \subset \mathbb{H}^3$, where χ is a horocycle and $P \cong \mathbb{H}^2$ is a hyperplane. This moduli space is highly nonseparated near the identity coset, where $\chi \subset P$. This means that as χ approaches P, the pair (χ, P) can have many different limiting configurations, depending on how we choose coordinates. The Theorem above describes, more precisely, the different limiting configurations that arise. The appearance of multiple configurations is a basic mechanism at work in homogeneous dynamics.

Limits of sets. We recall that the limsup of a sequence of sets $X_n \subset G$ consists of all limits of the form $g = \lim x_{n_k}$, where $n_k \to \infty$ and $x_{n_k} \in X_{n_k}$.
Thick sets and polynomials. We say $T \subset \mathbb{R}$ is K*-thick* if

$$[1, K] \cdot |T| = [0, \infty).$$

This notion also makes sense for T inside any Lie group isomorphic to $\mathbb{R}$, such as U or V. A basic fact about thick sets, which will be used below, is the following. Let $p \in \mathbb{R}[x]$ be a polynomial of degree d, and let $T \subset \mathbb{R}$ be K–thick. Then for any symmetric interval $I = [-a, a] \subset \mathbb{R}$, we have

$$\max_{x \in T \cap I} |p(x)| \geq k \max_{x \in I} |p(x)|, \tag{3.1}$$

where $k > 0$ depends only on K and d. For more details, see [MMO, §8].

Proof of Theorem 3.1. Fix $y > 0$. We will first show that $\limsup U g_n H$ contains v or v^{-1}, where $v(z) = z + iy$.

Let $C_n = g_n(\widehat{\mathbb{R}})$. Since $g_n \to \mathrm{id}$, we have $C_n \to \widehat{\mathbb{R}}$ in the Hausdorff topology on closed subsets of $\widehat{\mathbb{C}}$. Note that for $n \gg 0$, $C_n \cap \mathbb{C}$ is either a circle of large radius or a straight line of nonzero slope (since $g_n \notin VH$). Thus C_n meets the locus $L = \{z : |\Im(z)| = y\}$ for all $n \gg 0$. Passing to a subsequence, we can assume that $C_n \cap L \neq \emptyset$ for all n, and that the point of $C_n \cap L$ closest to the origin has the form $x_n + \epsilon y$ for a fixed $\epsilon = \pm 1$ (see Fig. 2). Let $u_n(z) = z - x_n$; then

$$u_n g_n(\widehat{\mathbb{R}}) \to \widehat{\mathbb{R}} + i\epsilon y$$

as $n \to \infty$. It follows that $u_n g_n h_n(z) \to z + i\epsilon y$ for suitable $h_n \in H$, since the latter group can be used to reparameterize $\widehat{\mathbb{R}}$. Equivalently, v or v^{-1} belongs to $\limsup U g_n H$.

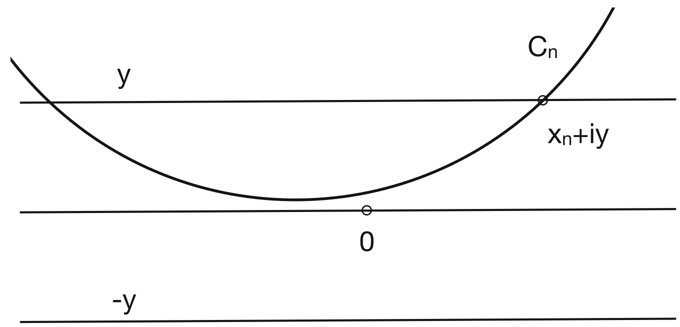

Figure 2: The circles $C_n \to \widehat{\mathbb{R}}$ eventually meet the locus $|\Im(z)| = y$

We now take into account the thick sets T_n. Note that at the scale $|x_n|$, the arc of $g_n(\widehat{\mathbb{R}})$ close to $\mathbb{R}$ is well–modeled by a parabola, i.e. the graph of a quadratic polynomial. Applying equation (3.1) to this polynomial, we find there is a K' depending only on K, and a sequence $x'_n + iy'_n \in C_n$, such that $u'_n(z) = z - x'_n \in T_n$, and $1 \leq |y/y'_n| \leq K'$. Passing to a subsequence and arguing as above, we conclude that $v(z) = z + iy'$ belongs to $\limsup T_n g_n H$ for some y' with $1 \leq |y/y'| \leq K'$. Since $y > 0$ was arbitrary, this shows that $V \cap (\limsup T_n g_n H)$ is a K'–thick subset of V. $\square$

Remark. Theorem 3.1 is a strengthening of [MMO, Lem. 8.2]; the proof here is more geometric.

4 Moving to the Renormalized Frame Bundle

In this section we describe how to use U to move points *close to* $\mathrm{RF}M$ *into* $\mathrm{RF}M$. The boundary of the convex core of M gives rise to an exceptional case.

Theorem 4.1. *Suppose $x_n \in (\mathrm{RF}M)U$ and $x_n \to y \in \mathrm{RF}M$. Then there exists a sequence $u_n \in U$ such that $x_n u_n \in \mathrm{RF}M$ and*

1. *We have $u_n \to \mathrm{id}$, and hence $x_n u_n \to y$; or*
2. *There is a component S of $\partial\,\mathrm{core}(M)$ such that $yH = \mathrm{T}S$, and $x_n u_n$ accumulates on $\mathrm{T}S$ as $n \to \infty$.*

The proof relies on the following fact from planar hyperbolic geometry.

Lemma 4.2. *Let $\gamma, \chi \subset \mathbb{H}^2$ be a geodesic and a horocycle respectively, let δ be a geodesic joining the base of χ to one of the endpoints of γ, and let $\{p\} = \delta \cap \chi$. Then for all $R \gg 0$, if $d(\chi, \gamma) < R - 1$, then $d(p, \gamma) < R$.*

The proof is indicated in Fig. 3, where the endpoint in common to γ and δ is at infinity. Note that an R–neighborhood of $\gamma \subset \mathbb{H}^2$, for $R \gg 0$, is bounded by a pair of rays meeting at an angle of nearly $180°$.

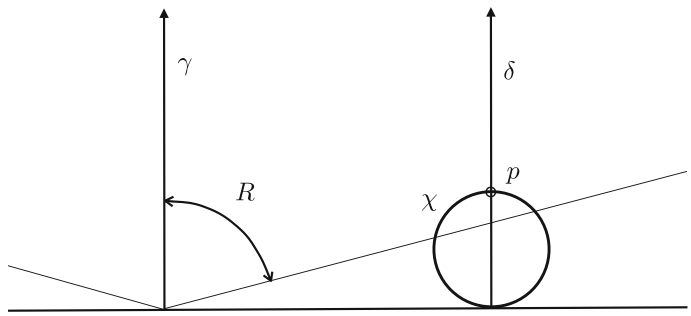

Figure 3: If $d(\gamma, \chi) \le R - 1$, with $R \gg 0$, then $d(p, \gamma) < R$

Proof of Theorem 4.1. Choose $g_n \to g_0$ in G such that $[g_n] = x_n$ and $[g_0] = y$, and let $C_n = g_n(\widehat{\mathbb{R}})$.

Recall that $[g] \in \mathrm{RF}M$ if and only if $\{g(0), g(\infty)\} \subset \Lambda$. By assumption, $g_n(\infty) \in \Lambda$ for all n, and $g_0(0) \in \Lambda$. Moreover, since $x_n \in (\mathrm{RF}M)U$, there exist $s_n \in \mathbb{R}$ such that $g_n(s_n) \in \Lambda$. Let us arrange that $|s_n|$ is as small as possible; then $g_n(I_n) \subset \Omega$, where $I_n = (-s_n, s_n)$. Setting $u_n(z) = z - s_n$, we then have $[g_n u_n] = x u_n \in \mathrm{RF}M$.

It remains to verify that (1) or (2) is true. If $|s_n| \to 0$, then clearly we are in case (1), so let us assume that $s = \limsup |s_n| > 0$. In this case, we claim C_0 bounds a component Ω_0 of Ω. To see this, recall that Ω is a union of round disks with disjoint closures. The arc $J = g_0(-s, s)$ is the limit, along a subsequence, of arcs $g_n(I_n) \subset \Omega$; since Ω has only finitely many components with diameter greater than $\mathrm{diam}(J)/2$, there is a unique component Ω_0 of Ω such that $g_0(0) \in \partial\Omega_0$. In fact the entire circular arc J must lie in $\overline{\Omega_0}$, and hence $C_0 \subset \overline{\Omega_0}$. Since $|C_0 \cap \Lambda| \ge 2$, we have $C_0 = \partial\Omega_0$.

Consequently the plane $H_0 = \mathrm{hull}(C_0) \subset \mathbb{H}^3$ covers a component S of $\partial\,\mathrm{hull}(M)$. In particular, we have $y \in \mathrm{T}S$.

Now even in this case, we have $s_n \to 0$ along the subsequence where $g_n(0) \notin \Omega_0$. Thus to complete the proof, it suffices to show that (2) holds under the assumption that $g_n(0) \in \Omega_0$ for all n. Under this assumption, $C_n \cap \Omega_0$ is a circular arc with two distinct endpoints, one of which is $g_n(s_n)$. Equivalently, $H_n = \mathrm{hull}(C_n)$ meets H_0 along a geodesic $\gamma_n \subset \mathbb{H}^3$, with one end converging to $g_n(s_n)$.

Let $\chi_n \subset H_n$ denote the horocycle resting on $g_n(\infty)$ whose natural lift to $\mathrm{F}\mathbb{H}^3$ gives the orbit $g_n U$. Let δ_n denote the geodesics in $\mathbb{H}^3$ connecting $g_n(\infty)$ to $g_n(s_n)$. Note that δ_n and χ_n both lie in the plane H_n, and cross at a unique point p_n.

We claim that $d(p_n, H_0) \to 0$. To see this, fix $\epsilon > 0$. It is easy to see that the set of points in H_n that are within hyperbolic distance ϵ of H_0 is convex and invariant under translation along γ_n; thus

$$H_n(\epsilon) = \{p \in H_n \,:\, d(p, H_0) < \epsilon\} = \{p \in H_n \,:\, d(\gamma_n, p) < R_n\}$$

for some $R_n > 0$. Since $x_n \to y$, we have $H_n \to H_0$ and hence $R_n \to \infty$; moreover, χ_n converges to a horocycle in H_0, so eventually $d(\gamma, \chi_n) < R_n - 1$. By Lemma 4.2, this implies that $d(p_n, \gamma) < R_n$, and hence $d(p_n, H_0) < \epsilon$ for all $n \gg 0$.

By construction we have $g_n u_n \in \mathrm{F}_{p_n}\mathbb{H}^3$. Since the frame $g_n u_n$ is tangent to the geodesic δ_n, whose endpoints lie in the limit set, we have $[g_n u_n] \in \mathrm{RF}M$; and since $d(p_n, H_0) \to 0$ (and indeed H_n and H_0 are nearly parallel near p_n), the frames $g_n u_n$ accumulate on $\mathrm{T}H_0$ and hence the frames $x_n u_n = [g_n u_n]$ accumulate on $\mathrm{T}S$. $\square$

5 Exceptional Frames

Let M be a rigid acylindrical manifold. We define the locus of *exceptional frames* in $\mathrm{F}M$ by

$$\mathrm{E}M = \bigcup \{xHV \;:\; x \in \mathrm{RF}M \text{ and } xH \subset \mathrm{F}M \text{ is closed}\}.$$

In this section we develop some basic properties of the exceptional locus.

Immersed surfaces. As we remarked in Sect. 1, when $x \in \mathrm{RF}M$ and xH is closed, its projection to M is a properly immersed, totally geodesic surface S passing through the convex core of M. For $v \in V$, the projection of xHv to M is a surface equidistant from S. The exceptional locus accounts for all the horocycles that lie on such surfaces.

Like RF_+M, the locus $\mathrm{E}M$ is invariant under the action of AN. In terms of the universal cover, we have $[g] \in \mathrm{E}M$ iff $g(\widehat{\mathbb{R}})$ is tangent, at $g(\infty)$, to a circle C such that $|C \cap \Lambda| \geq 2$ and ΓC is discrete. Note that

$$\mathrm{E}M \cap \mathrm{RF}M \neq \emptyset, \tag{5.1}$$

since $\mathrm{E}M$ contains the compact H–orbits coming from the totally geodesic boundary components of the convex core of M.

Lemma 5.1. *If $x \in \mathrm{RF}M$, then $\overline{xAU}$ meets $\mathrm{E}M$.*

Proof. If xH is closed, then we have $x \in \mathrm{E}M$ already. Otherwise, we have $\overline{xH} = (\mathrm{RF}_+M)H$ by Theorem 2.3, and $\overline{xAU}H = \overline{xH}$, since $AU\backslash H$ is compact. Thus $\overline{xAU}H = (\mathrm{RF}M_+)H$ contains one of the compact orbits $yH \subset \mathrm{E}M$ coming from the boundary of the convex core of M, so $\overline{xAU}$ must meet this orbit as well. $\square$

Lemma 5.2. *For any $x \in \mathrm{E}M \cap \mathrm{RF}_+M$, the locus $Y = \overline{xU}$ is a U–minimal set, and $Y = xvHv^{-1} \cap \mathrm{RF}_+M$ for some $v \in V$.*

Proof. Since U commutes with the action of V, it suffices to treat the case where xH is closed in $\mathrm{F}M$. In this case, $xH = \mathrm{T}S$ for some properly immersed, totally geodesic surface $S \subset M$. The subgroup $\pi_1(S) \subset \pi_1(M)$ determines a covering space $M' \to M$, which we can normalize so that $M' = \Gamma'\backslash\mathbb{H}^3$ with $\Gamma' \subset H$. (If S happens to be nonorientable, we pass to the orientation–preserving subgroup of index two.)

Since S is properly immersed, M' is convex cocompact; and since M is acylindrical, M' is nonelementary. It is now easy to check that the covering map $\mathrm{F}M' \to \mathrm{F}M$ sends $(\Gamma'\backslash H) \cap \mathrm{RF}_+M'$ isomorphically to $Y = (xH) \cap \mathrm{RF}_+M$, respecting the action of U (cf. [MMO, Thm. 6.2, Prop. 7.2]). The result then follows from Dal'bo's minimality Theorem 2.2. $\square$

Lemma 5.3. *For any $x \in \mathrm{RF}_+M - \mathrm{E}M$, the orbit xU meets $\mathrm{RF}M$.*

Proof. Suppose $x \in \mathrm{RF}_+M$ but xU does *not* meet $\mathrm{RF}M$. Then $x = [g]$ where $C = g(\widehat{\mathbb{R}})$ meets Λ in just one point. Therefore C is tangent to $D = \partial\Omega_0$ for some component $\Omega_0 \subset \Omega$, and ΓD is discrete, so $x \in \mathrm{E}M$. □

6 Classification of U–Orbit Closures

We can now complete the proof of Theorem 1.3. The interaction between $\overline{xU}$ and the exceptional locus $\mathrm{E}M$ plays a leading role in the proof.

Lemma 6.1. *For any $x \in \mathrm{RF}_+M$, the orbit closure $X = \overline{xU}$ meets $\mathrm{E}M$.*

Proof. Note that the result holds for x if and only if it holds for some $x' \in xAN$. Thus we are free to adjust x by elements of AN in the course of the proof.

Suppose X is disjoint from $\mathrm{E}M$. By Lemma 5.3, after replacing x with an element of xU, we may assume $x \in \mathrm{RF}M$. Then X contains a closed, U–invariant set Y such that $YL_+ \subset Y$ for some 1-parameter semigroup $L_+ \subset AV$, by [MMO, Prop. 9.3 and Thm. 9.4]. Let $L \subset AV$ be the group generated by L_+. Note that either $L = V$ or $L = vAv^{-1}$ for some $v \in V$.

Choose $\ell_n \to \infty$ in $L_+ \cong \mathbb{R}_+$. Then $L = \bigcup \ell_n^{-1} L_+$. The locus $Y\ell_n \subset X$ is U–invariant, so by Lemma 5.3 again we can find $y_n \in \mathrm{RF}M \cap Y\ell_n$. Pass to a subsequence such that $y_n \to z \in \mathrm{RF}M$. We have $y_n \ell_n^{-1} L_+ \subset X$ for all n, so in the limit we obtain $zL \subset X$.

If $L = V$, then we have $zN \subset X$, so $X = \mathrm{RF}_+M$ by Theorem 2.1, and thus X meets $\mathrm{E}M$ by equation (5.1). Otherwise, $L = vAv^{-1}$ for some $v \in V$. Therefore

$$X \supset \overline{zvAU}v^{-1}.$$

Again, we can find $u \in U$ such that $y = zuv \in \mathrm{RF}M$. Then $\overline{yAU} = \overline{zvAU}$. By Lemma 5.1, $\overline{yAU}$ meets $\mathrm{E}M$, so X meets $\mathrm{E}M$ as well. □

Typical orbits. Using the results of Sects. 3 and 4, we can now finally describe the behavior of U–orbits outside of the exceptional locus.

Theorem 6.2. *Suppose $x \in \mathrm{RF}_+M - \mathrm{E}M$. Then $\overline{xU} = \mathrm{RF}_+M$.*

Proof. Let $X = \overline{xU}$. Choose $y \in X \cap \mathrm{E}M$, using Lemma 6.1. By Lemma 5.2, there is a $v \in V$ such that $Z = yvHv^{-1}$ is closed, we have

$$X \supset Y = \overline{yU} = Z \cap \mathrm{RF}_+M,$$

and Y is a U–minimal set. Replacing x with xv, we can assume that $v = \mathrm{id}$, and hence $Z = yH$. Then $Y \cap \mathrm{RF}M \neq \emptyset$, so we can also assume that $y \in \mathrm{RF}M$. By Lemma 5.3, after replacing x with xu for some $u \in U$, we can further assume that $x \in \mathrm{RF}M$.

Let $X^* = X \cap \mathrm{RF}M$, and let

$$G_0 = \{g \in G \,:\, Zg \cap X^* \neq \emptyset\}.$$

We claim there is a sequence $g_n \to \mathrm{id}$ in $G_0 - HV$. To see this, first note that since $y \in X$, we can find $u_n \in U$ and $g_n \to \mathrm{id}$ in G such that $xu_n = yg_n$. In particular, we have $xu_n \to y$. We now apply Theorem 4.1. This Theorem implies that after changing our choice of $u_n \in U$, we can assume that $xu_n \in X^*$ and either (i) $xu_n \to y$, or (ii) $Z = Y$ is compact, and xu_n accumulates on Y. In either case, after passing to a subsequence and (in case (ii)) possibly changing our choice of $y \in Y$, we still have $xu_n = yg_n$. Then clearly $g_n \in G_0$, we have $g_n \to \mathrm{id}$, and $g_n \notin HN = HV$ because $yH \subset \mathrm{E}M$ while $x \notin \mathrm{E}M$.

Since Z is H–invariant, we have $HG_0 = G_0$. By [MMO, Lem. 9.2], there is also a $K > 1$ and a sequence of K–thick sets T_n such that $g_nT_n \subset G_0$ for all n. Applying Theorem 3.1 (with the order of factors reversed) to the sequence $Hg_nT_n \subset G_0$, we find that G_0 contains a thick subset $V_0 \subset V$. In particular, we can choose $v_n \to \infty$ in $V \cap G_0$. Then Zv_n meets X^* by the definition of G_0. But $Zv_n \cap \mathrm{RF}_+M = Yv_n$, so the U–minimal set Yv_n also meets X^*, and thus $Yv_n \subset X$ for all n. Now Yv_n is invariant under the closed subgroup $v_n^{-1}AUv_n$ of AN, which converges to N as $n \to \infty$. By compactness of X^*, we conclude that X contains the N–orbit of a point in X^*, and hence $X = \mathrm{RF}_+M$ by Theorem 2.1. □

Proof of Theorem 1.3. Let x be an element of $\mathrm{F}M$.

(1) If $x \notin \mathrm{RF}_+M$, then xU is closed. Indeed, in this case xU corresponds to a horocycle $\chi \subset \mathbb{H}^3$ resting on a point of Ω, and the projection of χ to M is a proper immersion.

(2) If $x \in \mathrm{E}M \cap \mathrm{RF}_+M$, then we $\overline{xU} = xvHv^{-1} \cap \mathrm{RF}_+M$ for some $v \in V$, by Lemma 5.2.

(3) Finally, if $x \in \mathrm{RF}_+M - \mathrm{E}M$, then $\overline{xU} = \mathrm{RF}_+M$ by Theorem 6.2. □

7 Classification of AU–Orbit Closures

In this final section we use the classification of U–orbits to show that

$$\overline{xAU} = \overline{xH} \cap \mathrm{RF}_+M \tag{7.1}$$

for all $x \in \mathrm{RF}_+M$, as stated in Corollary 1.5.

Generic circles. Let $M = \Gamma\backslash\mathbb{H}^3$ be a rigid acylindrical manifold. Let $\mathcal{C} = G/H$ be the space of oriented circles in $\widehat{\mathbb{C}}$, let

$$\mathcal{C}_0 = \{C \in \mathcal{C} \,:\, |C \cap \Lambda| \geq 2\},$$

and let

$$\mathcal{C}_1 = \{C \in \mathcal{C}_0 \,:\, \Gamma C \text{ is discrete in } \mathcal{C}\}.$$

Lemma 7.1. *The set $\mathcal{C}_1$ is countable.*

Proof. A circle $C \in \mathcal{C}_1$ corresponds to a properly immersed, totally geodesic surface S with fundamental group $\pi_1(S) \cong \Gamma^C$. Thus Γ^C is a finitely generated, nonelementary group and C is the unique circle containing $\Lambda(\Gamma^C)$. Since Γ is countable, there are only countably many possibilities for Γ^C, and hence only countably many possibilities for C. □

Lemma 7.2. *There is a circle $C \in \mathcal{C}_0$ that is not tangent to any circle in $\mathcal{C}_1$.*

Proof. It is easy to see that $\mathcal{C}_0$ has nonempty interior, while the set of circles tangent to a given $C \in \mathcal{C}_1$ is nowhere dense. Since $\mathcal{C}_1$ is countable, the result follows from the Baire category theorem. □

Rephrased in terms of $\Gamma\backslash G$, this shows:

Corollary 7.3. *There is an orbit $yH \subset \mathrm{F}M - \mathrm{E}M$ that meets $\mathrm{RF}M$.*

Proof of Corollary 1.5*.* The argument is similar to the proof of Lemma 5.1. Consider $x \in \mathrm{RF}_+M$. We always have $\overline{xAU} \subset \mathrm{RF}_+M$, since the latter set is closed and AU invariant.

If xU meets $\mathrm{RF}M$, then we can reduce to the case where $x \in \mathrm{RF}M$. Under this assumption, if xH is closed, then $\overline{xU} = \overline{xH} \cap \mathrm{RF}_+M$ by Theorem 1.3; since the latter set is A–invariant, it also coincides with $\overline{xAU}$. Otherwise, by Theorem 2.3 and compactness of $AU\backslash H$, we have

$$\overline{xAU}H = \overline{xH} = (\mathrm{RF}M_+)H.$$

In particular, by Corollary 7.3, $\overline{xAU}$ meets $\mathrm{RF}_+M - \mathrm{E}M$. Let y denote a point in their intersection. Then we have

$$\mathrm{RF}_+M = \overline{yU} \subset \overline{xAU}$$

by Theorem 6.2, so equation (7.1) holds in this case as well.

Finally, if $x \in \mathrm{RF}_+M$ but xU does not meet $\mathrm{RF}M$, then xH corresponds to a circle tangent to Λ in just one point, and (7.1) is easily verified using minimality of the horocycle flow on a compact hyperbolic surface (cf. [MMO, Thm. 1.5]). □

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