INVARIANT RADON MEASURES FOR UNIPOTENT FLOWS AND PRODUCTS OF KLEINIAN GROUPS

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ABSTRACT. Let $G = \mathrm{PSL}_2(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and consider the space $Z = (\Gamma_1 \times \Gamma_2)\backslash(G \times G)$ where $\Gamma_1 < G$ is a co-compact lattice and $\Gamma_2 < G$ is a geometrically finite discrete Zariski dense subgroup. For a horospherical subgroup N of G, we classify all ergodic, conservative, invariant Radon measures on Z for the diagonal N-action.

1. INTRODUCTION

The celebrated theorem of M. Ratner in 1992 classifies all *finite* invariant measures for unipotent flows on the quotient space of a connected Lie group by its discrete subgroup [16]. The problem of classifying invariant *locally finite* Borel measures (i.e., Radon measures) is far from being understood in general. Most of known classification results are restricted to the class of horospherical invariant measures on a quotient of a simple Lie group of rank one (e.g., [7, 17, 23, 2, 11, 12, 20, 15]). In this article, we obtain a classification of Radon measures invariant under unipotent flows in one of the most basic examples of the quotient of a higher rank semisimple Lie group by a discrete subgroup of infinite co-volume.

Let $G = \mathrm{PSL}_2(\mathbb{F})$ where the field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let Γ_1 and Γ_2 be geometrically finite Zariski dense, discrete subgroups of G. Consider the quotient space

$$Z := (\Gamma_1 \times \Gamma_2)\backslash(G \times G) = X_1 \times X_2$$

where $X_i = \Gamma_i\backslash G$ for $i = 1, 2$. For a subset $S \subset G$, $\Delta(S) := \{(s, s) : s \in S\}$ denotes the diagonal embedding of S into $G \times G$.

Theorem 1.1 ([5], Benoist-Quint). *Assume that $\Gamma_1 < G$ is co-compact. Then any ergodic $\Delta(G)$-invariant Radon measure μ on Z is, up to a constant multiple, one of the following:*

- *μ is the product $m^{\mathrm{Haar}} \times m^{\mathrm{Haar}}$ of Haar measures;*

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- *μ is the graph of the Haar measure, in the sense that for some $g_0 \in G$ with $[\Gamma_2 : g_0^{-1}\Gamma_1 g_0 \cap \Gamma_2] < \infty$, $\mu = \phi_* m^{\text{Haar}}_{(g_0^{-1}\Gamma_1 g_0 \cap \Gamma_2)}$, i.e., the push-forward of the* Haar-*measure on $(g_0^{-1}\Gamma_1 g_0 \cap \Gamma_2)\backslash G$ to the closed orbit $[(g_0, e)]\Delta(G)$ via the isomorphism ϕ given by $[g] \mapsto [(g_0 g, g)]$.*

Indeed, it is proved in [5] that any ergodic Γ_2-invariant Borel probability measure on X_1 is either a Haar measure or supported on a finite orbit of Γ_2. This result is equivalent to the above theorem, in view of the homeomorphism $\nu \mapsto \tilde{\nu}$ between the space of all Γ_2-invariant measures on X_1 and the space of all $\Delta(G)$-invariant measures on Z, given by

$$\tilde{\nu}(f) = \int_{X_2} \int_{X_1} f(\Gamma_1 hg, \Gamma_2 g)\, \mathrm{d}\nu(h)\, \mathrm{d}m^{\text{Haar}}(g).$$

Since the Haar measure m^{Haar} on X_1 is ergodic for any element of G which generates an unbounded subgroup, it follows that m^{Haar} is Γ_2-ergodic and hence the product $m^{\text{Haar}} \times m^{\text{Haar}}$ of the Haar measures in $X_1 \times X_2$ is $\Delta(G)$-ergodic. We also refer to [6] for the topological version of Theorem 1.1 (also see [4]).

In this paper, we consider the action of $\Delta(N)$ on Z where N is a horospherical subgroup of G, i.e., N is conjugate to the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{F} \right\}.$$

A $\Delta(N)$-invariant Radon measure on Z is said to be *conservative* if for any subset S of positive measure in Z, the measure of $\{n \in N : xn \in S\}$, with respect to the Haar measure of N, is infinite for almost all $x \in S$.

The aim of this paper is to classify all $\Delta(N)$-invariant ergodic conservative Radon measures on Z, assuming Γ_1 is cocompact. Since Ratner [16] classified all such finite measures, our focus lies on *infinite* Radon measures.

Note that if μ is a $\Delta(N)$-invariant measure, then the translate $w_*\mu$ is also $\Delta(N)$-invariant for any w in the centralizer of $\Delta(N)$. The centralizer of $\Delta(N)$ in $G \times G$ is equal to $N \times N$. Hence it suffices to classify $\Delta(N)$-invariant measures, up to a translation by an element of $N \times N$.

Let $m^{\text{BR}}_{\Gamma_2}$ denote the N-invariant Burger-Roblin measure on X_2. It is known that, up to a constant multiple, $m^{\text{BR}}_{\Gamma_2}$ is the unique N-invariant ergodic conservative measure on X_2, which is not supported on a closed N-orbit ([7], [17], [23]). When Γ_2 is of infinite co-volume, $m^{\text{BR}}_{\Gamma_2}$ is an infinite measure.

In the following two theorems, which are main results of this paper, we assume that $\Gamma_1 < G$ is cocompact and Γ_2 is a Zariski dense, geometrically finite subgroup of G with infinite co-volume.

Theorem 1.2. *The product measure $m^{\text{Haar}} \times m^{\text{BR}}_{\Gamma_2}$ on Z is a $\Delta(N)$-ergodic conservative infinite Radon measure.*

Theorem 1.3. *Any $\Delta(N)$-invariant, ergodic, conservative, infinite Radon measure μ on Z is one of the following, up to a translation by an element of $N \times N$ and up to a constant multiple:*

(1) *μ is the product measure $m^{\text{Haar}} \times m^{\text{BR}}_{\Gamma_2}$;*

(2) *μ is the graph of the BR-measure, in the sense that for some $g_0 \in \text{PSL}_2(\mathbb{F})$ with $[\Gamma_2 : g_0^{-1}\Gamma_1 g_0 \cap \Gamma_2] < \infty$,*

$$\mu = \phi_* m^{\text{BR}}_{(g_0^{-1}\Gamma_1 g_0 \cap \Gamma_2)},$$

i.e., the push-forward of the BR-measure on $(g_0^{-1}\Gamma_1 g_0 \cap \Gamma_2)\backslash G$ to the closed orbit $[(g_0, e)]\Delta(G)$ via the isomorphism ϕ given by $[g] \mapsto [(g_0 g, g)]$.

(3) *$\mathbb{F} = \mathbb{C}$ and there exists a closed orbit $x_2 N$ in X_2 homeomorphic to $\mathbb{R} \times \mathbb{S}^1$ such that μ is supported on $X_1 \times x_2 N$. To describe μ more precisely, let $U < N$ denote the one dimensional subgroup containing $\text{Stab}_N(x_2)$ and $\mathrm{d}n$ the N-invariant measure on $x_2 N$ in X_2. We then have one of the two possibilities:*

(a)
$$\mu = m^{\text{Haar}} \times \mathrm{d}n;$$

(b) *there exist a connected subgroup $L \simeq \text{SL}_2(\mathbb{R})$ with $L \cap N = U$, and a compact L orbit Y in X_1 such that*

$$\mu = \int_{x_2 N} \mu_x \, \mathrm{d}x$$

where $\mu_{x_2 n_0}$ is given by $\mu_{x_2 n_0}(\psi) = \int_Y \psi(y n_0, x_2 n_0)\, \mathrm{d}y$ with $\mathrm{d}y$ being the L-invariant probability measure on Y.

We deduce Theorem 1.2 as a consequence of Theorem 1.3 (see subsection 3.2).

Two main ingredients of the proof of Theorem 1.3 are Ratner's classification of probability measures on X_1 which are invariant and ergodic under a one parameter unipotent subgroup of G, and the classification of N-equivariant (set-valued) Borel maps $X_2 \to X_1$, established in our earlier work [14].

2. Recurrence and algebraic actions on measure spaces

In this section, let $G = \text{PSL}_2(\mathbb{C})$ and let $\Gamma < G$ be a Zariski dense geometrically finite discrete subgroup. Set $X = \Gamma\backslash G$. Let N be the horospherical subgroup

$$N = \left\{ n_t := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{C} \right\}$$

and let m^{BR} denote the Burger-Roblin measure on X invariant under N.

Recall that m^{BR} is the unique ergodic N-invariant Radon measure on X which is not supported on a closed N-orbit.

Let $U < N$ be a non-trivial connected subgroup of N. We denote by $\mathcal{P}(U\backslash N)$ the space of probability measures on $U\backslash N$. The natural action N on $U\backslash N$ induces an action of N on $\mathcal{P}(U\backslash N)$.

We will use the following lemma [24, B.5].

Lemma 2.1. *Let H be a locally compact and second countable group. Let X be a standard Borel H-space with quasi-invariant measure and that Y is a standard Borel H-space. Let $f : X \to Y$ be a Borel function so that for every $h \in H$, $f(hx) = hf(x)$ for almost all $x \in X$. Then there exists an H-invariant conull Borel subset $X_0 \subset X$ and a Borel H-map $\tilde{f} : X_0 \to Y$ such that $f = \tilde{f}$ almost all.*

The aim of this section is to prove the following technical result:

Theorem 2.2. *Let $\Gamma < G$ be a Zariski dense geometrically finite discrete subgroup and $U < N$ be a one-dimensional connected subgroup. Then there is no N-equivariant Borel map*

$$f : (X, m^{\mathrm{BR}}) \to \mathcal{P}(U\backslash N).$$

For the proof, we will first observe that the N action on $\mathcal{P}(U\backslash N)$ is smooth [24, Def. 2.1.9]. By the fact that m^{BR} is N-ergodic, it then follows that after possibly modifying f on a BR-null set, f is concentrated on a single N-orbit in $\mathcal{P}(U\backslash N)$. We will use a recurrence property of m^{BR}, which is stronger than the conservativity, to prove $U = N$.

We begin with the following lemma. The space $\mathcal{P}(\mathbb{R})$ is equipped with the weak-star topology: i.e., $\nu_n \to \nu$ if and only if $\nu_n(\psi) \to \nu(\psi)$ for all $\psi \in C_c(\mathbb{R})$.

Lemma 2.3. *If $\{t_n : n = 1, 2, \cdots\}$ is sequence in $\mathbb{R}$, so that $t_{n*}\sigma \to \sigma'$ for some $\sigma, \sigma' \in \mathcal{P}(\mathbb{R})$, then $\{t_n\}$ is bounded.*

Proof. Assume the contrary and after passing to a subsequence suppose $t_n \to \infty$. Since σ and σ' are probability measures on $\mathbb{R}$, there is some $M > 1$ such that

$$\sigma([-M, M]) > 0.9 \quad \text{and} \quad \sigma'([-M, M]) > 0.9.$$

Let $\psi \in C_c(\mathbb{R})$ be a continuous function so that $0 \le \psi \le 1$, $\psi|_{[-M,M]} = 1$ and $\psi|_{(-\infty,-M-1)\cup(M+1,\infty)} = 0$. Since $t_n \to \infty$ we have

$$([-M-1, M+1] - t_n) \cap [-M-1, M+1] = \emptyset \text{ for all large } n.$$

Therefore, $t_{n*}\sigma(\psi) < 0.1$ but $\sigma'(\psi) > 0.9$, which contradicts the assumption that $t_{n*}\sigma \to \sigma'$. □

As was mentioned above, we will need certain recurrence properties of the action of N on (X, m^{BR}). This will be deduced from recurrence properties of the Bowen-Margulis-Sullivan measure m^{BMS} on X with respect to the diagonal flow $\mathrm{diag}(e^{t/2}, e^{-t/2})$. We normalize so that m^{BMS} is the probability measure. These two measures m^{BMS} and m^{BR} are quasi-product measures and on weak-stable manifolds (i.e., locally transversal to N-orbits), they are absolutely continuous to each other.

Set $M = \{\mathrm{diag}(z, z^{-1}) : |z| = 1\}$. Then G/M can be identified with the unit tangent bundle of the hyperbolic 3-space $\mathbb{H}^3$. Hence for every $g \in G$, we can associate a point g^- in the boundary of $\mathbb{H}^3$ which is the backward end point of the geodesic determined by the tangent vector gM.

Now the set $X_{\mathrm{rad}} := \{\Gamma g \in X : g^- \text{ is a radial limit point of } \Gamma\}$ has a full BMS-measure as well as a full BR-measure. For $x \in X_{\mathrm{rad}}$, $n \mapsto xn$ is a bijection $N \to xN$, and μ_x^{PS} denotes the leafwise measure of m^{BMS}, considered as a measure on N (see [14, §2]).

We recall the following result of Rudolph [18, Theorem 17]: for any Borel set B of X and any $\eta > 0$, the set

$$\left\{ x \in X_{\mathrm{rad}} : \liminf_T \frac{1}{\mu_x^{\mathrm{PS}}(B_N(T))} \int_{B_N(T)} \chi_B(xn_{\mathbf{t}}) d\mu_x^{\mathrm{PS}}(\mathbf{t}) \geq (1-\eta) m^{\mathrm{BMS}}(B) \right\} \tag{2.1}$$

has full BMS measure m^{BMS}.

Lemma 2.4. *Let U be a one-dimensional connected subgroup of N. Then for every subset $B \subset X$ of positive* BMS*-measure, the set*

$$\{n \in U \backslash N : xn \in B\}$$

is unbounded for m^{BMS}-a.e. $x \in X$.

Proof. We denote by $\mathrm{Nbd}_{\mathrm{R}}(\mathrm{U})$ the R-neighborhood of U, i.e.,

$$\mathrm{Nbd}_R(U) = \{n_{\mathbf{t}} \in N : |\mathbf{t} - s| < R \text{ for some } n_s \in U\}.$$

We set $B_N(R) := \mathrm{Nbd}_R(\{e\})$ which is the R-neighborhood of e.

Let $B \subset X$ be any Borel set of positive BMS-measure. Then by (2.1), there is a BMS full measure set X' of X_{rad} with the following property: for all $x \in X'$, there is $T_x > 0$ such that if $T > T_x$, then

$$\mu_x^{\mathrm{PS}}\{n_{\mathbf{t}} \in B_N(T) : xn_{\mathbf{t}} \in B\} \geq 0.9\, \mu_x^{\mathrm{PS}}(B_N(T)) m^{\mathrm{BMS}}(B). \tag{2.2}$$

Let $x \in X'$. Since x is a radial limit point for Γ, there exists a sequence $T_i \to \infty$ so that $xa_{-\log T_i}$ converges to some $y \in \mathrm{supp}(m^{\mathrm{BMS}})$. Therefore, we have

$$\mu^{\mathrm{PS}}_{xa_{-\log T_i}} \to \mu_y^{\mathrm{PS}}, \tag{2.3}$$

in the space of regular Borel measures on N endowed with the weak-topology (see [14, Lemma 2.1]).

Moreover, since $\mu_y^{\mathrm{PS}}(U) = 0$, by [14, Lemma 4.3], for every $\epsilon > 0$, there exists $\rho_0 > 0$ such that for every $0 < \rho \leq \rho_0$ we have

$$\mu_y^{\mathrm{PS}}(B_N(1) \cap \mathrm{Nbd}_\rho(U)) \leq \epsilon \cdot \mu_y^{\mathrm{PS}}(B_N(1)) \tag{2.4}$$

Recall that $\mu_x^{\mathrm{PS}}(B_N(T)) = T^\delta \mu^{\mathrm{PS}}_{xa_{-\log T}}(B_N(1))$. Therefore,

$$\frac{\mu_x^{\mathrm{PS}}(B_N(T_i) \cap \mathrm{Nbd}_R(U))}{\mu_x^{\mathrm{PS}}(B_N(T_i))} = \frac{\mu^{\mathrm{PS}}_{xa_{-\log T_i}}(B_N(1) \cap \mathrm{Nbd}_{R/T_i}(U))}{\mu^{\mathrm{PS}}_{xa_{-\log T_i}}(B_N(1))}.$$

Hence, it follows from (2.3) and (2.4) that for every $\epsilon > 0$ and for all sufficiently large i such that $R/T_i < \rho$,

$$\mu_x^{\rm PS}(B_N(T_i) \cap {\rm Nbd}_R(U)) \le \epsilon \cdot \mu_x^{\rm PS}(B_N(T_i)). \tag{2.5}$$

Put $\epsilon = 1/10 \cdot m^{\rm BMS}(B)$. Given any j, there exists $i_j > j$ such that $T_{i_j} > T_x$ and

$$\mu_x^{\rm PS}(B_N(T_j)) \le \epsilon \cdot \mu_x^{\rm PS}(B_N(T_{i_j})).$$

Then for all sufficiently large $i > i_j$, we have

$$\mu_x^{\rm PS}(B_N(T_j)) + \mu_x^{\rm PS}(B_N(T_i) \cap {\rm Nbd}_R(U)) \le 2\epsilon \mu_x^{\rm PS}(B_N(T_i)).$$

Therefore it follows from (2.2) that for any j and for all $i > i_j$,

$$\mu_x^{\rm PS}\{n_t \in B_N(T_i) \setminus (B_N(T_j) \cup {\rm Nbd}_R(U)) : xn_t \in B\} \ge \\ 0.5\mu_x^{\rm PS}(B_N(T_i)) m^{\rm BMS}(B) > 0.$$

This implies that the set of x with $xn_t \in B$ cannot be contained in any bounded neighborhood of U, proving the claim. $\square$

Proof of Theorem 2.2. First by modifying f on a BR-null set, we may assume that for all $x \in X$, and for all $n \in N$,

$$f(xn) = n_* f(x).$$

Fix a compact subset $Q \subset X_{\rm rad}$ such that f is continuous on Q and $m^{\rm BR}(Q) > 0$. This is possible by Lusin's theorem. We claim that for some $y \in Q$, the set

$$\{n \in N : yn \in Q\}$$

is unbounded in the quotient space $U \backslash N$.

First note that there exists $\rho_0 > 0$ such that $QB_N(\rho_0)$ has a positive BMS-measure.

By Lemma 2.4 there is a BMS-full measure set X' so that for all $x \in X'$,

$$\{n \in N : xn \in QB_N(\rho_0)\} \text{ is unbounded in } U\backslash N.$$

Using the fact that N is abelian, the above implies that

$$\{n \in N : yn \in Q\} \text{ is unbounded in } U\backslash N \text{ for all } y \in X'N. \tag{2.6}$$

The set $X'N$ is a BR-conull set and $m^{\rm BR}(Q) > 0$. Therefore, there is some $y \in Q$ which satisfies (2.6), proving the claim. Now, there is a sequence $\{n_{t_i} \in N\}$ such that $n_{t_i} \to \infty$ in $U\backslash N$ and that $yn_{t_i} \in Q$ and $yn_{t_i} \to z \in Q$. The function f is continuous on Q. Therefore we get

$$(n_{t_i})_* f(y) \to f(z).$$

Since $f(y)$ and $f(z)$ are probability measures on $U\backslash N \simeq \mathbb{R}$, and $n_{t_i} \to \infty$ in $U\backslash N \simeq \mathbb{R}$, this contradicts Lemma 2.3. This yields that $U = N$, yielding a contradiction. $\square$

3. Proof of Theorems 1.2 and 1.3

We continue the notations set up in the introduction. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $G = \mathrm{PSL}_2(\mathbb{F})$. Let $\Gamma_1 < G$ be a cocompact lattice and $\Gamma_2 < G$ be a geometrically finite and Zariski dense subgroup. Set $X_i = \Gamma_i \backslash G$ for $i = 1, 2$. Let $Z = X_1 \times X_2$. Let $N = \{n_t : t \in \mathbb{F}\} < G$ be a horospherical subgroup.

We denote by $m^{\mathrm{BR}}_{\Gamma_2}$ the N-invariant Burger-Roblin measure on X_2; this is unique up to a constant multiple.

Let μ be a $\Delta(N)$-invariant, ergodic, conservative *infinite* Radon measure on Z. Let

$$\pi : Z \to X_2$$

be the canonical projection. Since X_1 is compact, the push-forward $\pi_*\mu$ defines an N-invariant ergodic conservative *infinite* Radon measure on X_2.

Theorem 3.1. *Up to a constant multiple,*

$$\pi_*\mu = m^{\mathrm{BR}}_{\Gamma_2} \quad \text{or} \quad \pi_*\mu = \mathrm{d}n$$

for the N-invariant measure $\mathrm{d}n$ *on a closed orbit x_2N homeomorphic to $\mathbb{R} \times \mathbb{S}^1$. The latter happens only when $\mathbb{F} = \mathbb{C}$ and Γ has a parabolic limit point of rank one.*

Proof. Since Γ_2 is assumed to be geometrically finite and Zariski dense, up to a proportionality, the measure $\pi_*\mu$ is either $m^{\mathrm{BR}}_{\Gamma_2}$ or it is the N-invariant measure supported on a closed N-orbit x_2N in X_2 ([17] and [23]). In the latter case, x_2N is homeomorphic to one of the following: $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{R} \times \mathbb{R}$, and $\mathbb{R} \times \mathbb{S}^1$. The first possibility cannot happen as that would mean that μ is a finite measure. The second possibility would contradict the assumption that μ is N-conservative. Hence x_2N must be $\mathbb{R} \times \mathbb{S}^1$, up to a homeomorphism. $\square$

The following is one of the main ingredients of our proof of Theorem 1.3, established in [14].

Theorem 3.2. *One of the following holds, up to a constant multiple:*

1. $\pi_*\mu = m^{\mathrm{BR}}_{\Gamma_2}$ *and μ is invariant under $U \times \{e\}$ for a non-trivial connected subgroup U of N;*
2. $\pi_*\mu = m^{\mathrm{BR}}_{\Gamma_2}$ *and the fibers of the map π are finite with the same cardinality almost surely. Moreover, in this case, μ is the graph of the* BR*-measure in the sense of Theorem 1.3(2);*
3. $\mathbb{F} = \mathbb{C}$ *and $\pi_*\mu = \mathrm{d}n$ for the N-invariant measure* $\mathrm{d}n$ *on a closed orbit x_2N homeomorphic to $\mathbb{R} \times \mathbb{S}^1$.*

Proof. For the case when $\pi_*\mu = m^{\mathrm{BR}}_{\Gamma_2}$, it follows from [14, Thm. 7.12 and Thm. 7.17] either that the fibers of the map π are finite with the same cardinality almost surely or that μ is invariant under a non-trivial connected subgroup of N, yielding the cases (1) and (2). Indeed [14, Thm. 7.12] states this under the assumption that μ is an N-joining, but all that is used in the proof is the fact that the projection of the measure onto one of the factors is the BR measure.

□

3.1. **Proof of Theorem 1.3.**

3.1.1. *The case of* $G = \mathrm{PSL}_2(\mathbb{R})$. In this case, $m^{\mathrm{BR}}_{\Gamma_2}$ is the unique infinite conservative N-invariant measure on X_2. Therefore we may assume, after the normalization of $m^{\mathrm{BR}}_{\Gamma_2}$ if necessary, that $\pi_*\mu = m^{\mathrm{BR}}_{\Gamma_2}$. By the standard disintegration theorem, see [1], we have

$$\mu = \int_{X_2} \mu_x \,\mathrm{d}m^{\mathrm{BR}}_{\Gamma_2}(x)$$

where μ_x is a probability measure on X_1 for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. x.

Suppose that Theorem 3.2(1) holds, i.e., μ is invariant under $N \times \{e\}$. Then, since every element in the σ-algebra

$$\{X_1 \times B : B \subset X_2 \text{ is a Borel set}\}$$

is invariant under $N \times \{e\}$, we get that μ_x is an N-invariant probability measure on X_1 for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. x.

By the unique ergodicity of N on the compact space X_1 [8], we have

$$\mu_x = m^{\mathrm{Haar}} \quad \text{for } m^{\mathrm{BR}}_{\Gamma_2}\text{-a.e. } x; \tag{3.1}$$

hence $\mu = m^{\mathrm{Haar}} \times m^{\mathrm{BR}}_{\Gamma_2}$.

If Theorem 3.2(2) holds, we obtain that μ is the graph of the BR-measure as desired in Theorem 1.3.

3.1.2. *The case of* $G = \mathrm{PSL}_2(\mathbb{C})$. In analyzing the three cases in Theorem 3.2, we use the following special case of Ratner's measure classification theorem [16]:

Theorem 3.3. *Let* $\Gamma_1 < G = \mathrm{PSL}_2(\mathbb{C})$ *be a cocompact lattice. Let* U *be a one-parameter unipotent subgroup of* G. *Let* $L \simeq \mathrm{PSL}_2(\mathbb{R})$ *be the connected subgroup generated by* U *and its transpose* U^t. *Then any ergodic* U*-invariant probability measure on* $\Gamma_1\backslash G$ *is either the Haar measure or a* $v^{-1}Lv$*-invariant measure supported on a compact orbit* $\Gamma_1\backslash\Gamma_1 gLv$ *for some* $g \in G$ *and* $v \in N$.

Indeed, since there are no compact U orbits in the compact space $\Gamma_1\backslash G$, the same conclusion holds for any ergodic u-invariant probability measure on $\Gamma_1\backslash G$ for any non-trivial element $u \in U$; see [16] and also [22].

Also note that in the second case of Theorem 3.3, the support of the measure is contained in yLN for some compact orbit yL.

We now investigate each case of Theorem 3.2 as follows:

Theorem 3.4. *For* $k = 1, 2, 3$, *Theorem 3.2(k) implies Theorem 1.3(k).*

Proof. Observe first that the case of $k = 2$ follows directly from Theorem 3.2.

Consider the case $k = 1$: suppose that μ is invariant under a subgroup $U \times \{e\}$ for a non-trivial connected subgroup U of N. We normalize $m^{\mathrm{BR}}_{\Gamma_2}$ so that $\pi_*\mu = m^{\mathrm{BR}}_{\Gamma_2}$. It follows from the standard disintegration theorem, see [1], that

$$\mu = \int_{X_2} \mu_x \, \mathrm{d}m^{\mathrm{BR}}_{\Gamma_2}(x). \tag{3.2}$$

Arguing as in §3.1.1, since μ is invariant under $U \times \{e\}$, we get that μ_x is a U-invariant probability measure on X_1 for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. x. We claim that

$$\mu_x = m^{\mathrm{Haar}} \quad \text{for } m^{\mathrm{BR}}_{\Gamma_2}\text{-a.e. } x; \tag{3.3}$$

this implies $\mu = m^{\mathrm{Haar}} \times m^{\mathrm{BR}}_{\Gamma_2}$ and finishes the proof in this case.

We apply Theorem 3.3 to U. Let $L \simeq \mathrm{PSL}_2(\mathbb{R})$ be defined as in Theorem 3.3. Compactness of $\Gamma_1\backslash\Gamma_1 gL$ implies that $g^{-1}\Gamma_1 g \cap L$ is a cocompact lattice of L. In particular, $g^{-1}\Gamma_1 g \cap L$ is finitely generated and Zariski dense in L. This implies there are only countably many compact L orbits in X_1.

Let $\{y_i L : i = 0, 1, 2, \ldots\}$ be the collection of all compact L-orbits in X_1. Then for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. $x \in X_2$, we have

$$\mu_x = c_x m^{\mathrm{Haar}} + \sum_i \mu_{x,i} \tag{3.4}$$

where $c_x \geq 0$ and $\mu_{x,i}$ is a U-invariant finite measure supported in $y_i LN$.

The set $\{(x_1, x_2) : c_{x_2} > 0\}$ is a $\Delta(N)$-invariant Borel measurable set. Therefore, (3.3) follows if this set has positive measure.

In view of this, we assume from now that $c_x = 0$ for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. x. Then the support of μ is contained in a countable union

$$\bigcup_i (y_i LN \times X_2).$$

Hence for some i,

$$\mu(y_i LN \times X_2) > 0. \tag{3.5}$$

Without loss of generality, we may assume $i = 0$.

Since $y_0 LN \times X_2$ is $\Delta(N)$-invariant and μ is $\Delta(N)$-ergodic, (3.5) implies that $y_0 LN \times X_2$ is μ-conull. Therefore, μ_x is supported on $y_0 LN$ for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. $x \in X_2$.

For each $n \in N$, let η_n be the probability measure supported on $y_0 Ln$, invariant under $n^{-1}Ln$. Noting that $y_0 Ln = y_0 Ln'$ if $n \in Un'$, the map $n \mapsto \eta_n$ factors through $U\backslash N$. We also have

$$n_{0*}\eta_n = \eta_{nn_0} \quad \text{for any } n, n_0 \in N. \tag{3.6}$$

By Theorem 3.3, the collection $\{\eta_n : n \in U\backslash N\}$ provides all U-invariant ergodic probability measures on X_1 whose supports are contained in $y_0 LN$.

Hence the U-ergodic decomposition of μ_x gives that for a.e. $x \in X_2$, there is a probability measure σ_x on $U\backslash N$ such that

$$\mu_x = \int_{U\backslash N} \eta_n \,\mathrm{d}\sigma_x(n).$$

Since μ is $\Delta(N)$-invariant, we have

$$\mu_{xn_0} = n_{0*}\mu_x \qquad \text{for } m^{\mathrm{BR}}_{\Gamma_2}\text{-a.e. } x \in X_2 \text{ and all } n_0 \in N. \tag{3.7}$$

Observe that

$$\mu_{xn_0} = \int_{U\backslash N} \eta_n \,\mathrm{d}\sigma_{xn_0}(n), \tag{3.8}$$

and that

$$n_{0*}\mu_x = \int_{X_1} n_{0*}\eta_n \,\mathrm{d}\sigma_x(n) = \int_{X_1} \eta_{nn_0} \,\mathrm{d}\sigma_x(n) = \int_{X_1} \eta_n \,\mathrm{d}(n_0\sigma_x)(n).$$

Therefore (3.7) implies that for $m^{\mathrm{BR}}_{\Gamma_2}$-a.e. $x \in X_2$ and for a.e. $n_0 \in N$,

$$n_{0*}\sigma_x = \sigma_{xn_0}. \tag{3.9}$$

It follows that the Borel map $f : (X_2, m^{\mathrm{BR}}_{\Gamma_2}) \to \mathcal{P}(U\backslash N)$ defined by

$$f(x) := \sigma_x$$

is essentially N-equivariant for the natural action of N on $\mathcal{P}(U\backslash N)$.

As U is one dimensional, this yields a contradiction to Proposition 2.2 and hence completes the proof of case $k = 1$.

We now turn to the proof of the case $k = 3$. The argument is similar to the above case. Let x_2N be a closed orbit as in the statement of Theorem 3.2(3). We disintegrate μ as follows:

$$\mu = \int_{x_2N} \mu_x \,\mathrm{d}n \tag{3.10}$$

where μ_x is a probability measure on X_1 for a.e. $x \in x_2N$. As x_2N is homeomorphic to $\mathbb{R} \times \mathbb{S}^1$, the stabilizer of x_2 in N is generated by a unipotent element, say, u. Note that u acts trivially on x_2N and $\Delta(u)$ leaves μ invariant. Hence again we have

$$\mu_x \text{ is } u\text{-invariant almost surely.} \tag{3.11}$$

We apply (3.4) for u-invariant measures μ_x. Let $L \simeq \mathrm{PSL}_2(\mathbb{R})$ denoted the connected closed subgroup containing u and u^t and let $\{y_iL : i = 0, 1, \dots\}$ be the collection of all compact L-orbits. Then for almost every $x \in x_2N$ we write

$$\mu_x = c_x m^{\mathrm{Haar}} + \sum_i \mu_{x,i},$$

where $\mu_{x,i}$ is a u-invariant finite measure supported in y_iLN. As before, if $c_x > 0$ on a positive measure subset of x_2N, then $c_x = 1$ almost surely by the $\Delta(N)$ ergodicity of μ. Then $\mu = m^{\mathrm{Haar}} \times dn$; note that this measure is $\Delta(N)$ ergodic since m^{Haar} is u-ergodic. This is the case of Theorem 1.3(3)(a).

Lastly we consider the case when $c_x = 0$ almost surely. As before,

$$\mu(y_i LN \times x_2 N) > 0$$

for some i, and hence almost all μ_x is supported on one $y_i LN$ by the ergodicity of μ. We assume $i = 0$ without loss of generality.

Set $U = L \cap N$. Then $\{\eta_n : n \in U\backslash N\}$ (with η_n defined as in the previous case) is the set of all u-ergodic probability measures on X_1 whose supports are contained in $y_0 LN$ by Theorem 3.3 and the remark following it. Therefore, we get a probability measure $\sigma_x \in \mathcal{P}(U\backslash N)$ such that

$$\mu_x = \int_{n \in U\backslash N} \eta_n \, \mathrm{d}\sigma_x(n).$$

Moreover, $n_* \sigma_x = \sigma_{xn}$ for a.e. x and all $n \in N$.

Put $\sigma := \sigma_x$ for some fixed x. Without loss of generality, we assume $x = x_2$. Then for $\psi \in C_c(Z)$,

$$\mu(\psi) = \int_{n \in U\backslash N} \int_{x_2 n_0 \in x_2 N} \int_Y \psi(y n_0 n, x_2 n_0) dy \, \mathrm{d}n_0 \, \mathrm{d}\sigma(n)$$

where $Y = y_0 L$ and dy is the probability Haar measure on Y.

However for each $n \in U\backslash N$, $\psi \mapsto \int_{x_2 n_0 \in x_2 N} \int_Y \psi(y n_0 n, x_2 n_0) dy \, \mathrm{d}n_0$ defines a $\Delta(N)$-invariant measure, and hence by the $\Delta(N)$-ergodicity assumption on μ, σ must be a delta measure at a point, say $n \in U\backslash N$. Therefore we arrive at Theorem 1.3(3)(b). □

3.2. **Proof of Theorem 1.2.** Suppose that the product measure

$$\mu := m^{\mathrm{Haar}} \times m^{\mathrm{BR}}_{\Gamma_2}$$

is not ergodic for the action of $\Delta(N)$. Let Ω be the support of μ. We consider the decomposition $\Omega = \Omega_d \cup \Omega_c$ where Ω_d and Ω_c are maximal $\Delta(N)$-invariant dissipative and conservative subsets respectively. That is, for any positive measure $S \subset \Omega_d$ (resp. $S \subset \Omega_c$), the Haar measure of $\{n \in N : xn \in S\}$ is finite (resp. infinite) for almost all $s \in S$ (see [9]).

Consider the ergodic decomposition of μ. By Theorem 1.3, any ergodic conservative component in the ergodic decomposition, see [1], of μ should be one of the measures as described in Theorem 1.3(2) and 1.3(3).

Now μ gives measure zero to sets of the form

$$(x_1, x_2)\Delta(G)(N \times \{e\})$$

where $(x_1, x_2)\Delta(G)$ is a closed orbit. Moreover, there are only countably many closed $\Delta(G)$ orbits in Z.

Also, any closed N orbit $x_2 N$ gives rise to the family $x_2 NA$ of closed N-orbits where A is the diagonal subgroup. There are only finitely many such AN-orbits in X_2, as Γ_2 is geometrically finite and hence there are only finitely many Γ orbits of parabolic limit points. Therefore $m^{\mathrm{BR}}_{\Gamma_2}$ gives zero measure to the set of all closed N-orbits in X_2.

It follows that Ω_c is trivial and hence the product measure $m^{\text{Haar}} \times m^{\text{BR}}_{\Gamma_2}$ is completely dissipative. This is a contradiction since X_1 is compact and $m^{\text{BR}}_{\Gamma_2}$ is N-conservative. This proves Theorem 1.2. □

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