MEASURES INVARIANT UNDER HOROSPHERICAL SUBGROUPS IN POSITIVE CHARACTERISTIC

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ABSTRACT. Measure rigidity for the action of maximal horospherical subgroups on homogeneous spaces over a field of positive characteristic is proved. In the case when the lattice is uniform we prove the action of any horospherical subgroup is uniquely ergodic.

1. INTRODUCTION

The study of unipotent flows in homogeneous spaces and their applications has been a very active field for more four decades now. M. S. Raghunathan formulated a conjecture describing the closure of unipotent orbits in homogeneous spaces of Lie groups. A special case of this conjecture was proved by G. Margulis in his celebrated proof of the Oppenheim conjecture [Mar86, Mar87a]. Some other special cases were also proved by S. Dani and G. Margulis [DM89, DM90]. The conjecture in its full generality was settled affirmatively by M. Ratner in a series of seminal papers [R90a, R90b, R91, R92]. Ratner actually proved a conjecture of S. Dani classifying ergodic invariant probability measures for the action of the unipotent groups on homogeneous spaces and then obtained an equidistribution result for unipotent orbits, this equidistribution result was first conjectured by N. Shah.

Both measure classification and equdistribution conjectures have natural generalization to homogeneous spaces over other local fields. The case of local fields of characteristic zero was proved by Margulis and Tomanov [MT94] and M. Ratner [R95] independently. The case of positive characteristic local field however is far from understood. Recently there has been some development on this problem. See [EG10, EM, M] for partial results on this problem.

Classifying probability measures which are invariant under the action of horspherical subgroups is indeed an extreme special case of measures invariant under unipotent subgroups and was considered well before Ratner's work. Indeed Hedlund (topological) and Furstenberg (measure theoretical) studied horocycle flows on surfaces. S. Dani [D79, D81] classified all ergodic invariant measure for the action of a maximal horospherical subgroup of semisimple Lie groups. He also classified closure of the orbits of (not necessarily maximal) horospherical subgroups in [D86]. His method relies ultimately on Furstenberg's idea for the proof of the unique ergodicity of horocycle flow on compact surfaces. Another possible approach to this problem would be to use representation theory, see the work of M. Burger in [Bu90] where these lines of ideas are utilized.

In this paper we use mixing properties of semisimple elements to obtain measure classification for the action of the corresponding horospherical subgroups. These lines of ideas go back to G. A. Margulis' PhD thesis. They have been extensively utilized by many others ever since, see [R83] where similar ideas are present.

The problem in the horospherical case is much simpler than the general case. The main reason for this can be described as follows; There is a direction, namely the direction where the horosphere is expanded, such that if we flow a "nice" neighborhood of some piece of the horosphere in that direction the points in the neighborhood only diverge along the horosphere. This is to say expanding translates of pieces of horospheres are approximated by push forward of a nice open set. Now, as a result of mixing, translates of an open set become independent of any other set. Thus one gets that expanding translates of pieces of horospheres are equidistibuted shows that long closed horospheres become equidistributed, see [KM96] and also Proposition 2.5 below.

When the lattice Γ is uniform this can be done quite successfully and one gets a unique ergodicity statement and no maximality assumption is required on the horospherical subgroup.

When Γ is not uniform however, there are possible intermediate measures which are naturally associated with certain parabolic subgroups. In order to capture these measures one proceeds inductively. We use quantitative non-divergence of unipotent orbits to get the "correct" parabolic subgroup. These non-divergence results which go back to Dani and Margulis are presented in section 3. The maximality assumption then implies invariance under the unipotent radical of the corresponding parabolic subgroup which reduces our problem to a problem about a semisimple group with smaller dimension. This maximality assumption seems inevitable in both Dani's and Burger's approaches as well.

Let us fix some notation in order to state the main theorem. Let K be a global function field and let S be a finite set of places in K. Let $\mathcal{O}_S$ be the ring of S-integers. For each $\nu \in S$ let K_ν be the completion of K with respect to ν and let $K_S = \prod_{\nu \in S} K_\nu$. Let $\mathbb{G}$ be a connected semisimple group defined over K. Let $G = \prod_{\nu \in S} \mathbb{G}(K_\nu)$. The locally compact topology of K_S induces a locally compact topology on G which we will refer to as the Hausdorff topology. Let Γ be a congruence lattice in $\mathbb{G}(\mathcal{O}_S)$. We let $X = G/\Gamma$. By a horospherical subgroup we mean the unipotent radical of a K_S-parabolic subgroup of G. Let $\mathbb{P}$ be a K-parabolic of G, let $\mathbb{P} = \mathbb{L}\mathbb{W}$ be the Levi decomposition of $\mathbb{P}$ and let $\mathbb{M} = [\mathbb{L}, \mathbb{L}]$. Let $^{\circ}P = MW$ where $M = \mathbb{M}(K_S)$ and $W = \mathbb{W}(K_S)$. We let P^+ denote the group generated by all K_S-split unipotent subgroups (see [B91, Chapter 5] for the definition of split unipotent groups) of $P = \mathbb{P}(K_S)$, see section 2 for more details. In this paper we prove

Theorem 1.1. *Let notation be as above. Assume U is a horospherical subgroup of G. Let μ be a U-invariant ergodic probability measure on $X = G/\Gamma$.*

(i) *If Γ is a uniform lattice, then the action of U is uniquely ergodic. Furthermore μ is the probability Haar measure on the closed orbit $\overline{G^+\Gamma}/\Gamma$ where the closure is with respect to the Hausdorff topology.*

(ii) *If Γ is non-uniform, then assume that there exists some $\nu \in S$ such that U contains a maximal horospherical subgroup of $\mathbb{G}(K_\nu)$. There exists some K-parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$ and some $g \in G$ such that $g^{-1}Ug \subset P = \mathbb{P}(K_S)$ and μ is the Σ-invariant measure on the closed orbit $\Sigma\, g\Gamma$ where $\Sigma = g\overline{P^+(^\circ P \cap \Gamma)}g^{-1}$ and the closure is with respect to the Hausdorff topology.*

Using this theorem and the Linearization techniques, see [DM93], we conclude, as a corollary, equidistribution of orbits of subgroups satisfying the assumptions of Theorem 1.1. The statement of this corollary needs some notation which will be fixed in section 2 hence we postpone the discussion to the end of the paper, see Corollary 4.2. It is worth mentioning that the method which is used here in order to prove Theorem 1.1 seems to actually be enough to get the equidistribution result without appealing to linearization techniques, indeed in the course of the proof of Theorem 1.1 we prove equidistribution in case (i) and "generic" case of (ii), see the proof in section 4 for more details.

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2. Notation and Preliminaries

Let G be as above. By the Zariski topology on G we mean the product of the Zariski topologies on $\mathbb{G}(K_\nu)$. Similarly we define K_S algebraic subvarieties, K_S pints etc. We also will use the locally compact topology which is induced from the locally compact topology of K_S. We will refer to the locally compact topology also as the Hausdorff topology.

Let $\mathcal{K}$ be a good maximal compact subgroup of G in the sense of [Ti79, Section 3], see also [Oh02, Proposition 2.1]. In particular the Iwasawa decomposition i.e. $G = \mathcal{K}P$ for any minimal parabolic subgroup P, and the Cartan decomposition for G holds with $\mathcal{K}$, as the maximal compact subgroup.

Let ϖ_ν be the uniformizer of K_ν and $\mathfrak{o}_\nu$ be the valuation ring in K_ν for all $\nu \in S$. We may an will assume that $\mathbb{G}$ is a subgroup of some $\mathbb{SL}_n$ and hence G is a subgroup of $\mathrm{SL}_n(K_S)$ we further may and will assume that $\mathcal{K} \subset \prod_{\nu \in S} \mathrm{SL}_n(\mathfrak{o}_\nu)$. The ℓ's congruence subgroup of $\mathcal{K}^\ell$ is then defined to be the kernel in $\mathcal{K}$ of reduction mod $(\varpi_\nu^\ell)_{\nu \in S}$. We will refer to ℓ as the depth of $\mathcal{K}^\ell$.

We fix a left G-invariant bi-$\mathcal{K}$-invariant metric d on G and let $\|g\| = d(1, g)$. We let $C^\infty(X)$ denote the space of functions $\phi : X \to \mathbb{R}$ such that there exists a compact open subgroup $\mathcal{K}_\phi$ of G which leaves ϕ invariant. Note that G acts on $C^\infty(X)$ indeed if $g \in G$ and $\phi \in C^\infty(X)$ we have $(g\,\phi)(x) = \phi(g^{-1}x)$ and $g\,\phi$ in invariant under $g\mathcal{K}_\phi g^{-1}$. Let $\phi \in C^\infty(X)$ then since $\mathcal{K}_\phi$ is finite index in $\mathcal{K}$ we get that the span of $\mathcal{K} \cdot \phi$ is finite dimensional.

Let $s \in G$ be a diagonalizable element over K_S such that the ν-component s_ν of s has eigenvalues which are integer powers of the uniformizer ϖ_ν, for all $\nu \in S$.

According to Margulis-Tomanov [MT94] we call such s "an element from class $\mathcal{A}$," see also [MT94, Lemma 2.1] where equivalent definitions are provided. Define

$$W_G^+(s) = \{x \in G \mid s^n x s^{-n} \to e \ \text{ as } \ n \to -\infty\}$$

$$W_G^-(s) = \{x \in G \mid s^n x s^{-n} \to e \ \text{ as } \ n \to +\infty\}$$
$$Z_G(s) = \{x \in G \mid s x s^{-1} = x\}$$

If it is clear from the context we sometimes omit the subscript G above. Note that $W_G^\pm(s)$ and $Z_G(s)$ are the groups of K_S-points of K_S-algebraic subgroups $\mathbb{W}_G^\pm(s)$ and $\mathbb{Z}_G(s)$ respectively. The product map is a K_S-isomorphism of K_S-varieties between $\mathbb{W}^-(s) \times \mathbb{Z}_G(s) \times \mathbb{W}^+(s)$ and a Zariski open dense subset of $\mathbb{G}$ containing the identity, see for example [B91, chapter V] for these statements. Hence if $\mathcal{K}^\ell$ is a deep enough congruence subgroup then

$$(1) \quad \mathcal{K}^\ell = (\mathcal{K}^\ell \cap W^-(s))((\mathcal{K}^\ell \cap Z_G(s)))((\mathcal{K}^\ell \cap W^+(s))) = (\mathcal{K}^\ell)^-(\mathcal{K}^\ell)^0(\mathcal{K}^\ell)^+$$

In the sequel $\bullet_\nu$ denotes the ν-component of $\bullet$. Let s be an element from class $\mathcal{A}$ and let $U \subset W^+(s)$ be a unipotent subgroup which is normalized by s. Let $U_0 = U \cap \mathcal{K}$. For any $m > 0$ define $U_m = s^m U_0 s^{-m}$ and $(U_\nu)_m = s^m (U_\nu)_0 s^{-m}$. These form a filtration of U. We let θ_ν be a Haar measure on $(U_\nu)_0$ normalized such that $\theta_\nu((U_\nu)_0) = 1$ and let $\theta = \prod_\nu \theta_\nu$. Note that $\{U_m\}$ and $\{(U_\nu)_m\}$ form an averaging sequence for U and U_ν respectively.

Let $\mathbb{H}$ be a group defined over K_S and let $H = \mathbb{H}(K_S)$. We will let H^+ denote the group generated by all K_S-points of all the K_S-split unipotent subgroups of H. Note that H^+ is a normal unimodular subgroup of H. Thanks to Theorem C.3.8 and Proposition C.3.11 in [CGP] we have that H^+ is the group generated by the K_S-points of all the K_S-split unipotent radicals of minimal K_S-pseudo parabolic subgroups of $\mathbb{H}$. We refer to [CGP, Definition 2.2.1] for the definition of pseudo parabolic subgroup it is however worth mentioning that if $\mathbb{H}$ is reductive then pseudo parabolic subgroups are parabolic subgroups of $\mathbb{H}$.

Let $\mathbb{H}$ be a K_S-algebraic group acting K_S-rationally on a K_S-algebraic variety $\mathbb{M}$. Assume that $H = \mathbb{H}(K_S)$ is generated by one parameter K_S-split unipotent algebraic subgroups and elements from class $\mathcal{A}$. The following is proved in [MT94].

Lemma 2.1. *(cf. [MT94, Lemma 3.1]) Let μ be an H-invariant Borel probability measure on $M = \mathbb{M}(K_S)$. Then μ is concentrated on the set of H-fixed points in M. In particular if μ is H-ergodic then μ is concentrated at one point.*

It is worth mentioning that the proof in [MT94] follows from the fact that the orbit space in this situation is Hausdorff which is a consequence of [BZ76, appendix A]).

We also need the following general and simple statement.

Lemma 2.2. *(cf. [MT94, Lemma 10.1]) Let A be a locally compact second countable group and Λ a discrete subgroup of A. If B is a normal unimodular subgroup of A and μ is a B-invariant ergodic measure on A/Λ. Let $\Sigma = \overline{B\Lambda}$, where the closure is with respect to the Hausdorff topology. There exists $x \in A/\Lambda$ such that Σx is closed and μ is the Σ-invariant Haar measure on Σx.*

Let m denote the Haar measure on X. For simplicity we will let $L^2(X) = L^2(X, m)$. In this unitary representation of G we let $L^2_{00}(X)$ denote the orthogonal complement of one dimensional subrepresentations of G on $L^2(X)$. Hence we have a decomposition $L^2(X) = L^2_{00}(X) \oplus (L^2_{00}(X))^\perp$ as G-representations. Indeed $L^2_{00}(X)$ can also be characterized as the orthogonal complement of G^+-fixed vectors in $L^2(X)$, this characterization will be used in this paper. We let $C^\infty_{00}(X) = C^\infty(X) \cap L^2_{00}(X)$. We recall Howe-Moore Theorem which is a consequence of Mautner phenomena, see [BO07, Theorem 2.5] and references there.

Theorem 2.3. *(Decay of Matrix Coefficients) Let G and Γ be as above. Then for every $\phi, \psi \in L^2_{00}(X)$ we have*

$$\lim_{g \to \infty} \langle g\phi, \psi \rangle = 0 \tag{2}$$

Remark 2.4. It is worth mentioning that one believes that an exponentially effective version of the above theorem is true. To be more precise, let G and Γ be as above. Then there exist positive constants κ and E depending on G and Γ such that for all $\phi, \psi \in C^\infty_{00}(X)$

$$|\langle g\phi, \psi \rangle| \leq E e^{-\kappa \|g\|} (\dim \langle \mathcal{K}\phi \rangle \dim \langle \mathcal{K}\psi \rangle)^{\frac{1}{2}} \tag{3}$$

where $\langle \mathcal{K} \bullet \rangle$ denotes the $\mathcal{K}$-span of $\bullet$.

If for all $\nu \in S$ the K_ν-rank of all of the K_ν-simple factors of $\mathbb{G}$ is at least 2, then the above is a result of Kazhdan property (T). In this case more is true in particular in this case the constant κ is independent of Γ, see [Oh02]. In general this theorem follows from [Cl03] and [BS91] in the number field case. It is known to experts that the function field case works out much the same way, however we were not able to find an account on this in the literature.

We will use the qualitative version as the reference for it in the literature is available. The following is an important consequence of this decay of matrix coefficients. This argument was used in G. Margulis thesis.

Proposition 2.5. *Let $s \in G^+$ be an element from class $\mathcal{A}$ and let $U = W^+_G(s)$. Then for any $f \in C^\infty_c(U)$, compactly supported locally constant function on U, for any $\phi \in C^\infty_{00}(X)$, any compact set $L \subset X$ and for any $\varepsilon > 0$ there exits $n_0 > 0$ such that for all $x \in L$ and $n \geq n_0$ we have*

$$\left| \int_U f(u)\phi(s^n u x) d\theta(u) \right| < \varepsilon \tag{4}$$

Proof. This is proved in [KM96] in the real case and the proof works in S-arithmetic setting also, we recall the proof in the positive characteristic case for the sake of completeness. Note that there is a compact open subgroup U_f (resp. $\mathcal{K}_\phi$) of U, (resp. G) which leaves f (resp. ϕ) invariant.

As was mentioned above if $g \in G$ is close enough to the identity then $g = u^- u^0 u^+$ where $u^\pm \in W^\pm_G(s)$ and $u^0 \in Z_G(s)$. For ℓ large enough let $\mathcal{K}^\ell$ denote the ℓ-th congruence subgroup of $\mathcal{K}$ and let $(\mathcal{K}^\ell)^\pm = W^\pm_G(s) \cap \mathcal{K}^\ell$ and $(\mathcal{K}^\ell)^0 = Z_G(s) \cap \mathcal{K}^\ell$.

Note that it suffices to prove the statement for f which is the characteristic function of U_f, we assume this is the case from now. Now since L and $\mathcal{K}_\phi$ are compact by taking ℓ large, we may assume that the map from the $\ell = \ell(L, \mathcal{K}_\phi)$-th congruence

subgroup $\mathcal{K}^\ell$ of $\mathcal{K}$ into X is injective at x for all $x \in L$ and that $\mathcal{K}^\ell \subset \mathcal{K}_\phi$. Replacing U_f by $U_f \cap (\mathcal{K})^\ell$ we may and will assume that $U_f = (\mathcal{K}^\ell)^+$. Hence the map from supp(f) to X is injective at all $x \in L$.

Let $f^- = \frac{1}{|(\mathcal{K}^\ell)^-|}\chi_{(\mathcal{K}^\ell)^-}$ and $f^0 = \frac{1}{|(\mathcal{K}^\ell)^0|}\chi_{(\mathcal{K}^\ell)^0}$, be the normalized characteristic functions of $(\mathcal{K}^\ell)^-$ and $(\mathcal{K}^\ell)^0$ respectively. Define now

$$\tilde{f}(u^- u^0) = f^-(u^-)f^0(u^0)$$

Let n be a large positive number. Note that the restriction of the module function of the corresponding parabolic to $(\mathcal{K}^\ell)^0$ is trivial, thus we have

$$(5) \qquad \left| \int_U f(u)\phi(s^n ux)d\theta(u) - \int_G \tilde{f}(u^- u^0)f(u)\phi(s^n u^- u^0 ux)d(u^- u^0 u) \right| =$$

$$\left| \int_G f^-(u^-)f^0(u^0)f(u)(\phi(s^n ux) - \phi(s^n u^- u^0 ux))d(u^- u^0 u) \right| = 0$$

The last equality follows from the fact that $\mathcal{K}^\ell \subset \mathcal{K}_\phi$. Define now $\psi = f^- f^0 f$ this is a function supported in a small neighborhood of the identity and thanks to the fact that $\mathcal{K}^\ell$ maps injectively to a neighborhood of $x \in L$ for any $x \in L$, we may also define $\psi_x(gx) = \psi(g)$. This is a (compactly supported) function in $C^\infty(X)$. Let $(\psi_x)_{00}$ be the component of ψ_x in $L^2_{00}(X)$. We now use (5) and get

$$\left| \int_U f(u)\phi(s^n ux)d\theta(u) \right| = |\langle s^n \phi, \psi_x \rangle| = |\langle s^n \phi, (\psi_x)_{00} \rangle|$$

Let now n_0 be as in Theorem 2.3 applied to ϕ and $(\psi_x)_{00}$ for the given ε and let $n > n_0$ we then get $\left| \int_U f(u)\phi(s^n ux)d\theta(u) \right| < \varepsilon$ as we wanted to show. $\qquad \square$

3. Non-divergence of unipotent flows

In the proof of Theorem 1.1 we will need results concerning non-divergence of unipotent orbits in X. Results of this sort were first established by Margulis in the course of the proof of arithmeticity of non-uniform lattices. Later Dani, Dani-Margulis and Kleinbock-Margulis proved quantitative versions of these non-divergence result. We need an S-arithmetic version of [DM91]. In characteristic zero such S-arithmetic statements are proved in [GO, Section 7]. Their proof, which is modeled on [DM91], essentially works in our setting as well. However there are several points which require some explanation. For the sake of completeness we will reproduce the proof in positive characteristic in this section.

Let the notation be as in the introduction. Let $\mathbb{A}$ be a maximal K-split torus of $\mathbb{G}$ and choose a system $\{\alpha_1, \ldots, \alpha_r\}$ of simple K-roots for $(\mathbb{G}, \mathbb{A})$. For each $1 \le i \le r$ let $\mathbb{P}_i$ be the standard maximal parabolic subgroup of $\mathbb{G}$ corresponding to α_i. The subgroup $\mathbb{P} = \cap_i \mathbb{P}_i$ is a minimal K-parabolic subgroup of $\mathbb{G}$.

We need some results from reduction theory, see [Sp94] and [Ha69] for general results concerning reduction theory in the function field case. In particular we need the following which is Theorem D in [Be87], we thank A. Salehi Golsefidy for pointing out this reference to us. It is worth mentioning that this could also be proved using [Sp94, Proposition 2.6].

$$(6) \qquad \text{There exists a finite set } F \subset \mathbb{G}(K) \text{ such that } \mathbb{G}(K) = \mathbb{P}(K)F\Gamma$$

Through out this section $U = U_\nu$ will denote a K_ν-split unipotent subgroup of G. We may and will assume $U \subset W_G^+(s)$ where s is an element from class $\mathcal{A}$. We will further assume that U is normalized by s. As before let $U_0 = U \cap \mathcal{K}$ and define $U_m = s^m U_0 s^{-m}$. We will assume that U is an N-dimensional K_ν-group, thus there is an s-equivariant isomorphism of K_ν-varieties between the Lie algebra of $\mathfrak{u}$ of U onto U, see [BS91, Corollary, 9.12]. We will let $\mathfrak{B}$ be the image of U_0 under this isomorphism. As in section 2 let θ denote the Haar measure on U normalized so that $\theta(U_0) = 1$. Abusing the notation we will let θ also denote the Haar measure on K_ν^N normalized so that $\theta(\mathfrak{B}) = 1$. Let

$$\mathcal{P}_{N,d} = \{\Theta : \mathfrak{B} \to G : \Theta \text{ is a polynomial map with degree at most } d\}$$

Note that there exists some d such that for any $g \in G$ the map $u \mapsto ug$ is in $\mathcal{P}_{N,d}$.

The following is the main result of this section, see Theorem 2 [DM91] and Theorem 7.1 [GO].

Theorem 3.1. *Let the notation be as above and let $\varepsilon > 0$. There exists a compact subset $L \subset X$ such that for all $x = g\Gamma \in X$ one of the following holds*

(1) *for all large m, depending on x,*

$$\theta(\{u \in U_m \ : \ ux \in L\}) \geq (1 - \varepsilon)\theta(U_m)$$

(2) *there is some $\lambda \in F\Gamma$, such that $g^{-1}Ug \subset \lambda \mathbb{P}_i \lambda^{-1}$.*

We will first make several reductions. Note that if we let $p : \mathbb{G} \to \overline{\mathbb{G}}$ be the K-central isogeny from $\mathbb{G}$ to the adjoint form $\overline{\mathbb{G}}$ of $\mathbb{G}$, then the natural map from G/Γ to $\overline{\mathbb{G}}(K_S)/\overline{\mathbb{G}}(\mathcal{O}_S)$ is a proper map. Thus the statement and the conclusion of Theorem 3.1 are not changed if we replace $\mathbb{G}$ by the adjoint form. Hence for the rest of this section we will assume $\mathbb{G}$ is adjoint form.

We need some more notation. Recall that $\{\alpha_1, \ldots, \alpha_r\}$ is the set of simple simple K-roots for $(\mathbb{G}, \mathbb{A})$. Since $\mathbb{G}$ is adjoint form they coincide with the fundamental weights. Note now that α_i's uniquely extend to a character of $\mathbb{P}_i$ which we will also denote by α_i. For any $1 \leq i \leq r$ we fix, once and for all, representations $(\rho_i, \mathbb{V}_i, v_i)$ of $\mathbb{G}$ defined over K and vectors $v_i \in \mathbb{V}_i(K)$ such that there are K-rational characters χ_i's so that

(7) $$\mathbb{P}_i = \{g \in \mathbb{G} : \ gv_i = \chi_i(g)v_i\}$$

We further assume $\chi_i = \alpha_i^{n_i}$ where n_i is a positive integer. We fix $\mathcal{K}_\nu$-invariant norms $\| \ \|_\nu$ on $\mathbb{V}_i(K_S)$ and assume $\|v_i\|_\nu = 1$ for all $\nu \in S$ and all $1 \leq i \leq r$. Define $\Phi_i : G \to \mathbb{R}^\times$ by

(8) $$\Phi_i((g_\nu)) = \prod_{\nu \in S} \|g_\nu v_i\|_\nu$$

Since $\Phi_i()$ is left $\mathcal{K}$-invariant for all i, it only depends on P_i component of g in the Iwasawa decomposition. Similar to [Sp94], representations ρ_i's and Φ_i's will be used as 'the standard representation of $\mathbb{G}$ corresponding to $\mathbb{P}_i$" which are used in reduction theory and are also extensively used in [DM91] and [GO].

We have the following

Theorem 3.2. *For any $\varepsilon > 0$ and $\alpha > 0$ there exists a compact subset L of G/Γ such that for any $\Theta \in \mathcal{P}_{N,d}$ one of the following holds*

(1) $\theta(\{t \in \mathfrak{B} \ : \ \Theta(t)\Gamma/\Gamma \in L\}) \geq (1-\varepsilon)$

(2) *there exist* $1 \leq i \leq r$ *and* $\lambda \in F\Gamma$ *such that* $\Phi_i(\Theta(t)\lambda) < \alpha$ *for all* $t \in \mathfrak{B}$.

Proof of Theorem 3.1 module Theorem 3.2. Let ε and α be given and let L be the compact set obtained as in the Theorem 3.2 for this ε and α. Assume now that (1) in the Theorem 3.1 does not hold, thus there exists a sequence $m_j \to \infty$ such that loc. cit. fails for U_{m_j}. As we mentioned before there exists some d such that the map $u \mapsto ug$ is in $\mathcal{P}_{N,d}$ for all U_{m_j}. We denote these maps by Θ_j. Now Theorem 3.2 implies that there exist some $1 \leq i \leq r$ and $\lambda_j \in F\Gamma$ such that $\Phi_i(\Theta_j(t)\lambda_j) < \alpha$ for all $t \in \mathfrak{B}$ and all j i.e. if $u \in U_{m_j}$ then $\Phi_i(ug\lambda_j) < \alpha$. In particular we have $\Phi_i(g\lambda_j) < \alpha$ for all j. Note that $gF\Gamma v_i$ is a discrete set. Hence λ's such that $\Phi_i(ug\lambda) < \alpha$ is finite module the stabilizer of v_i. Since U_{m_j}'s exhaust U the above discussion implies that there exists some $\lambda \in F\Gamma$ such that $\Phi_i(ug\lambda) < \alpha$ for all $u \in U$. If we utilize the isomorphism f between U and $K_\nu{}^N$, then the map $\rho_i(ug\lambda)v_i$ is a polynomial map from K_ν^N into V_i. Hence either it is unbounded or constant. Since $\Phi_i(ug\lambda) < \alpha$ we get $\rho_i(ug\lambda)v_i$ is constant which using (7) implies that $\lambda^{-1}g^{-1}Ug\lambda \subset \mathbb{P}_i$.

Let us recall the following quantitative non-divergence result.

Theorem 3.3. *(cf. [Gh05, Theorem, 4.3]) For any given compact subset* $L \subset G/\Gamma$ *and* $\varepsilon > 0$ *there exists a compact subset* $L' \subset G/\Gamma$ *such that for any* $\Theta \in \mathcal{P}_{N,d}$ *and any* $y \in G/\Gamma$ *such that* $\Theta(\mathfrak{B})y \cap L \neq \emptyset$ *we have*

$$\theta(\{t \in \mathfrak{B} : \ \Theta(t)y \in L'\}) \geq (1-\varepsilon)$$

Using Theorem 3.3 the proof of Theorem 3.2 reduces to the following

Theorem 3.4. *For any* $\alpha > 0$ *there exists a compact subset* L *of* G/Γ *such that for any* $\Theta \in \mathcal{P}_{N,d}$ *one of the following holds*

(1) $\Theta(\mathfrak{B})\Gamma/\Gamma \cap L \neq \emptyset$

(2) *there exist* $1 \leq i \leq r$ *and* $\lambda \in F\Gamma$ *such that* $\Phi_i(\Theta(t)\lambda) < \alpha$ *for all* $t \in \mathfrak{B}$.

The proof of Theorem 3.4 will occupy the rest of this section. Some more notation is needed. For any $1 \leq i \leq r$ we let $\mathbb{Q}_i = \{p \in \mathbb{P}_i : \ \alpha_i(p) = 1\}$ and let $\mathbb{A}_i = \{a \in \mathbb{A} : \ \alpha_j(a) = 1, \forall j \neq i\}$. For any subset $I \subset \{1, 2\ldots, r\}$ we let

$$\mathbb{P}_I = \cap_{i \in I}\mathbb{P}_i, \ \ \mathbb{Q}_I = \cap_{i \in I}\mathbb{Q}_i, \ \ \mathbb{A}_I = \prod_{i \in I}\mathbb{A}_i$$

$\mathbb{Q}_I$ is a normal K-subgroup of $\mathbb{P}_I$, $\mathbb{A}_I$ is a K-split torus and $\mathbb{P}_I = \mathbb{A}_I\mathbb{Q}_I$. Let $\mathbb{U}_I$ be the unipotent radical of $\mathbb{P}_I$ and let $\mathbb{H}_I$ be the centralizer of $\mathbb{A}_I$ in $\mathbb{Q}_I$. We have $\mathbb{Q}_I = \mathbb{H}_I\mathbb{U}_I$ and $\mathbb{H}_I$ and $\mathbb{U}_I$ are defined over K and $\mathbb{U}_I$ is K-split. As usual we let $P_I = \mathbb{P}_I(K_S)$, $Q_I = \mathbb{Q}_I(K_S)$, $A_I = \mathbb{A}_I(K_S)$ $H_I = \mathbb{H}_I(K_S)$, $U_I = \mathbb{U}_I(K_S)$.

Since $\mathbb{G}$ is adjoint for any subset $I \subset \{1, 2\ldots, r\}$ we have

(9)   The map $(\alpha_i)_{i=1}^r : \mathbb{A}_I \to \mathbb{G}_m^{|I|}$ is an isomorphism of K-varieties

For any $\nu \in S$ fix a uniformizer $\varpi_\nu \in K \cap \mathcal{O}_\nu$ and let

$$A_\nu^0 = \{a \in \mathbb{A}(K_\nu) : \ \alpha_i(a) \in \varpi_\nu^{\mathbb{Z}}, \ \text{for all } i = 1, \ldots, r\}$$

Note that by our choice of ϖ_ν and from (9) we get that $A^0_\nu \subset \mathbb{A}(K)$ for all $\nu \in S$. Let $A^0 = \prod_{\nu \in S} A^0_\nu$. For any subset $I \subset \{1, 2 \ldots, r\}$ we let $A^0_I = A_I \cap A^0$. We may and will assume that $\mathcal{K}$ is chosen so that $A = \mathcal{K}A^0_I$. We let

$$A^{(1)} = \left\{ a \in A^0 : \ \Phi_i(a) = 1, \ \text{for all } i = 1, \ldots, r \right\}$$

and let $A^{(1)}_I = A^{(1)} \cap A_I$.

Recall that Γ is a congruence subgroup of $\mathbb{G}(\mathcal{O}_S)$. Hence the choice of ϖ_ν, the fact that $\chi_i = \alpha^{n_i}_i$ and (9) imply that

(10)   There is a compact subset $Y \subset A_I$ such that $A^{(1)}_I \subset Y(A_I \cap \Gamma)$

We have the following

Lemma 3.5. *Let* $I \subset \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, r\} \setminus I$ *and* $0 < a < b$ *be given. Then there exists a compact subset* M_0 *of* Q_I *such that*

$$\{g \in Q_I : \ \Phi_j(g) \in [a, b]\} \subset M_0 Q_{I \cup \{j\}}(A_j \cap \Gamma)$$

Proof. Note that $H_I = (\mathcal{K} \cap H_I)(P_j \cap H_I)$. Hence we have $Q_I = H_I U_I = (\mathcal{K} \cap H_I)(P_j \cap H_I)U_I \subset (\mathcal{K} \cap Q_I)(P_j \cap Q_I)$. Note also that $P_j \cap Q_I = A_j Q_{j \cup I}$. Thus we have $Q_I = (\mathcal{K} \cap Q_I)A_j Q_{j \cup I}$. Now let $g \in Q_I$ such that $\Phi_j(g) \in [a, b]$ then we have $q = kaq$ where $k \in \mathcal{K} \cap Q_I$, $a \in A_j$, $q \in Q_{j \cup I}$. Hence we have $\Phi_j(g) = \Phi_j(a) \in [a, b]$ which is to say $g \in (\mathcal{K} \cap Q_I)\{a \in A_j : \ \Phi_j(a) \in [a, b]\}Q_{j \cup I}$. Note that there exists a compact subset $Y' \subset A_j$ such that $\{a \in A_j : \ \Phi_j(a) \in [a, b]\} \subset Y'A^{(1)}_j$. Hence using (10) we have $\{a \in A_j : \ \Phi_j(a) \in [a, b]\} \subset Y'Y(A_j \cap \Gamma)$ where Y is as in (10). Since A_j normalizes $Q_{j \cup I}$ the lemma thus follows with $M_0 = (\mathcal{K} \cap Q_I)Y'Y$. $\square$

For $I \subset \{1, \ldots, r\}$ there is a finite subset $\tilde{F}_I \subset \mathbb{Q}_I(K)$ such that

(11) $$\mathbb{Q}_I(K) = (\mathbb{P} \cap \mathbb{Q}_I)(K)\tilde{F}_I(Q_I \cap \Gamma)$$

Recall that $\mathbb{A}_I$ normalizes $\mathbb{Q}_I$ and elements with bounded denominators in $\mathbb{Q}_I(K)$ lie in finitely many cosets of $Q_I \cap \Gamma$. Thus we may find a finite subset $F_I \subset \mathbb{Q}_I(K)$ such that

(12) $(A_I \cap \Gamma)(Q_I \cap \Gamma)\tilde{F}^{-1}_I \subset (Q_I \cap \Gamma)(A_I \cap \Gamma)\tilde{F}^{-1}_I \subset (Q_I \cap \Gamma)F^{-1}_I(A_I \cap \Gamma)$

Let

(13) $$\Lambda(I) = (Q_I \cap \Gamma)F^{-1}_I \subset \mathbb{Q}_I(K)$$

Note that $\mathbb{P}_\emptyset = \mathbb{Q}_\emptyset = \mathbb{G}$, $\mathbb{A}_\emptyset = \mathbb{A}$, $\tilde{F}_\emptyset = F = F_\emptyset$ and $\Lambda(\emptyset) = \Gamma F^{-1}$.

Lemma 3.6. *For* $j \in \{1, \ldots, r\}$ *and* $I \subset \{1, \ldots, r\} \setminus \{j\}$ *there exists a finite subset* $E \subset \mathbb{P}(K)$ *such that* $\Lambda(I)\Lambda(I \cup \{j\}) \subset \Lambda(I)E$.

Proof. This is a consequence of the definition together with (11). $\square$

We now construct certain compact subsets of G/Γ using the above. These will eventually give us the set L which we need in the proof of Theorem 3.4.

An l-tuple $((i_1, \lambda_1), \ldots, (i_l, \lambda_l))$ where $l \geq 1$, $i_1 \ldots, i_l \in \{1, \ldots, r\}$ and $\lambda_1, \ldots \lambda_l \in \mathbb{G}(K)$ is called an called an *admissible* sequence of length l if $i_1 \ldots, i_l$ are distinct and $\lambda^{-1}_{j-1}\lambda_j \in \Lambda(\{i_1, \ldots, i_{j-1}\})$ for all $j = 1, \ldots, l$ and we let λ_0 be the identity element. The empty sequence is called an admissible sequence of length 0. If ξ and

η are two admissible sequences of length l and l' with $l \leq l'$ we say η extends ξ if the first l terms of η coincide with ξ. For an admissible sequence ξ of length $l \geq 0$ we let $\mathcal{C}(\xi)$ be the set of all pairs (i, λ) with $1 \leq i \leq r$ and $\lambda \in \mathbb{G}(K)$ for which there exists an admissible sequence η of length $l+1$ extending ξ, for such η the pair (i, λ) is necessarily the last term. Note that if $l = 0$ then $\mathcal{C}(\xi)$ consists of all pairs (i, λ) with $1 \leq i \leq r$ and $\lambda \in \Lambda(\emptyset)$.

For any admissible sequence ξ of length l we define the support of ξ, which we will denote by $\mathrm{supp}(\xi)$, to be the empty set of $l = 0$ and the set $\{(i_1, \lambda_1), \ldots, (i_l, \lambda_l)\}$ if $\xi = ((i_1, \lambda_1), \ldots, (i_l, \lambda_l))$.

Let ξ be an admissible sequence of length $l \geq 0$. Let α and $a < b$ be positive real numbers define

$$W_{\alpha,a,b}(\xi) = \{g \in G : \Phi_j(g\lambda) \geq \alpha, \forall (j, \lambda) \in \mathcal{C}(\xi)$$
$$\text{and } a \leq \Phi_i(g\lambda) \leq b, \forall (i, \lambda) \in \mathrm{supp}(\xi)\}$$

We have the following

Proposition 3.7. *Let ξ be an admissible sequence of length $l \geq 0$. Let α and $a < b$ be positive real numbers. Then $W_{\alpha,a,b}(\xi)\Gamma/\Gamma$ is a relatively compact subset of G/Γ.*

Proof. The same proof as in [GO, Proposition, 7.14] goes through.   □

We need a few standard facts about polynomial maps. For N, M, d positive integers let us denote by $\mathcal{P}^*_{N,M,d}$ the space of polynomials from $K_\nu^N \to K_\nu^M$ with degree bounded by d. As before we let ϖ_ν denote the uniformizer of K_ν and we assume $|\varpi_\nu|_\nu = 1/q_\nu$. In non-archimedean metrics any point of a ball can be considered as the center of the ball thus if $B \subset K_\nu^N$ is a ball and $t \in B$ we have $B = B(t, r)$ for some $r > 0$ depending on B, we let $\frac{1}{q_\nu}B(t)$ denote the ball centered at t with radius r/q_ν. We have

Lemma 3.8. *Let B be a ball in K_ν^N. Then for any $F \in \mathcal{P}^*_{N,M,d}$ there exists $t_0 \in B$ such that $|F(t)|_\nu = \sup_B |F|_\nu$ for all $t \in \frac{1}{q_\nu}B(t_0)$.*

Proof. Let f and B be given and let $t_0 \in B$ such that $|F(t_0)|_\nu = \sup_B |F|_\nu$. Expanding F about t_0 we may and will assume $t_0 = 0$. Now let $F = (F_1, \ldots, F_M)$ we may and will assume $\sup_B |F|_\nu = |F_1(0)|_\nu$. We write $f = F_1$ and let

$$f = a_0 + \sum a_\beta t_1^{i_1} \cdots t_N^{i_N} \quad \text{where} \quad a_0 = f(0)$$

The assumption that $|f(0)|_\nu = \sup_B |f|_\nu$ implies that $|a_\beta t_1^{i_1} \cdots t_N^{i_N}|_\nu \leq |a_0|_\nu$ for all $t = (t_1, \ldots, t_N) \in B$. Thus $|a_\beta t_1^{i_1} \cdots t_N^{i_N}|_\nu < |a_0|_\nu$ for all $t \in \frac{1}{q_\nu}B$. Using the ultra metric property, we get $|f(t)|_\nu = |a_0|_\nu$ for all $t \in \frac{1}{q_\nu}B$ as we wanted to show.   □

Recall from [KT07, Lemma, 2.4] that for any $F \in \mathcal{P}^*_{N,M,d}$, any ball $B \subset K_\nu^N$ and any $\varepsilon > 0$ we have

$$(14) \qquad \theta(\{t \in B : |F(t)|_\nu < \varepsilon \sup_B |F|_\nu\}) \leq C\varepsilon^{1/Nd}\theta(B)$$

where $C > 0$ is a constant depending only on N and d. As a corollary we get

Lemma 3.9. *Given $\eta \in (0, 1)$, there exists $R > 1$ such that if $B_0 \subset B$ are two balls with radius r_0 and r respectively such that $r_0 \geq \eta r$, then $\sup_B |F|_\nu \leq R \cdot \sup_{B_0} |F|_\nu$ for any $F \in \mathcal{P}^*_{N,M,d}$.*

Using Lemma 3.8 and Lemma 3.9 the same argument as in [GO, Proposition 7.12] gives the following

Proposition 3.10. *There exists $R > 1$ such that for any $\alpha > 0$, any ball B and any subfamily $\mathcal{F} \subset \mathcal{P}^*_{N,M,d}$ satisfying*

  (i) *for any $t_0 \in B$, $\#\{F \in \mathcal{F} : |F(t_0)|_\nu < \alpha\} < \infty$*
  (ii) *for any $f \in \mathcal{F}$, $\sup_B |F|_\nu \geq \alpha$*

one of the following holds

  (a) *there exists $t_0 \in B$ such that $|F(t_0)|_\nu \geq \alpha$ for all $F \in \mathcal{F}$,*
  (b) *there exists a ball $B_0 \subset B$ and $F_0 \in \mathcal{F}$ such that*

$$|F_0(B_0)|_\nu \subset [\alpha/R, R\alpha] \;\; and \;\; \sup_{B_0} |F|_\nu \geq \alpha/R \;\; for \; all \;\; F \in \mathcal{F}$$

The proof of Theorem 3.4 now goes through the same lines as the proof of Theorem 7.3 in [GO], see page 55 in [GO].

4. PROOF OF THEOREM 1.1 AND EQUIDISTRIBUTION

In this section we will prove Theorem 1.1. Here we give the proof assuming Theorem 2.3. Before starting the proof we need the following fact which is essentially a restatement of the fact that $L^2_{00}(X)$ is the orthogonal complement of the space of G^+-invariant vectors.

Lemma 4.1. *Let μ be a probability measure on X. Suppose that for any $\phi \in C^\infty_{00}(X)$ we have*

$$\int_X \phi(x) d\mu(x) = 0$$

Then μ is G^+-invariant

Proof. Recall that $L^2_{00}(X)$ is the orthogonal complement of the space of all G^+-fixed vectors in $L^2(X)$ and we have $L^2(X) = L^2_{00}(X) \oplus (L^2_{00}(X))^\perp$ as G-representation. Let $g \in G^+$ be arbitrary. For any $\psi \in L^2(X) \cap C^\infty(X)$ we let $\psi^g(x) = \psi(gx)$. We need to show $\int_X \psi^g(x) d\mu(x) = \int_X \psi(x) d\mu$. Let $\psi = \psi_0 + \psi_1$ where $\psi_0 \in L^2_{00}(X)$ and $\psi_1 \in (L^2_{00}(X))^\perp$. Since the decomposition is G-equivariant we have (i) $\psi^g = \psi_0^g + \psi_1^g$ is corresponding decomposition of ψ^g (ii) from the definition of $C^\infty(X)$ we have $\psi_0, \psi_0^g \in C^\infty_{00}(X)$. Since $g \in G^+$ we have $\psi_1^g = \psi_1$. Our hypothesis applied to ψ_0 and ψ_0^g implies that

$$\int_X \psi d\mu = \int_X \psi_1 d\mu = \int_X \psi_1^g d\mu = \int_X \psi^g d\mu$$

as we wanted to show. $\qquad\square$

Let us remark that the converse of the above lemma also holds but we do not need the converse here. We now start the

Proof of Theorem 1.1. In the proof we will denote $du = d\theta(u)$ for simplicity.

Proof of (i). By our assumption G/Γ is compact hence we may take $L = X$ in the statement of Proposition 2.5. Let $\phi \in C_{00}^\infty(X)$. Recall that loc. cit. that for any $\varepsilon > 0$ there exists n_0 such that if $n > n_0$

$$\left| \int_{U_0} \phi(s^n u z) \, du \right| < \varepsilon$$

where z is any point in X. If one takes $z = s^{-n}x$ one then gets

$$(15) \qquad \left| \int_{U_0} \phi(s^n u s^{-n} x) du \right| = \left| \frac{1}{\theta(U_n)} \int_{U_n} \phi(ux) du \right| < \varepsilon$$

As we mentioned in section 2 the filtration $\{U_m = s^m U_0 s^{-m} : m \geq 0\}$ is an averaging sequence for U. Since μ is U-ergodic (15) implies

$$(16) \qquad \int_X \phi d\mu = 0$$

This and Lemma 4.1 imply that μ is G^+-invariant. Now since G^+ is a normal, unimodular subgroup of G, Lemma 2.2 implies that μ is the $\overline{G^+\Gamma}$-invariant probability Haar measure on the closed orbit $\overline{G^+\Gamma}/\Gamma$ as we wanted to show.

Proof of (ii). We first show that the assertion holds for $U = U_\nu$, a maximal horospherical subgroup of G_ν.

Let $x \in X$ be μ-generic for the action of U i.e. for any bounded function $\phi \in C^\infty(X)$ and any $\varepsilon > 0$ there exists some n_0 such that if $n \geq n_0$ then we have

$$(17) \qquad \left| \int_{U_0} \phi(s^n u s^{-n} x) \, du - \int_X \phi \, d\mu \right| < \varepsilon$$

Let $\varepsilon > 0$ be given fix some n_0 as above and let L be the compact set obtained in Theorem 3.1 for this choice of ε. Assume now that x is such that (1) in the Theorem 3.1 holds. Since s normalizes U this implies that (1) in the Theorem 3.1 also holds for $s^{-n}x$ for any $n \in \mathbb{Z}$. Hence there exists a positive integer $m_0 = m_0(n)$, depending on $s^{-n}x$, such that if $m > m_0$, then

$$(18) \qquad \theta(\{u \in U_m : us^{-n}x \in L\}) \geq (1 - \epsilon)\theta(U_m)$$

For the rest of the argument we may and will assume that $\phi \in C_{00}^\infty(X)$. Let $\mathcal{K}^\ell$ be a deep congruence subgroup such that (i) (1) holds for $\mathcal{K}^\ell$ (ii) ϕ is fixed by $\mathcal{K}^\ell$ (iii) for any $x \in L$ the map from $k \mapsto kx$ is injective on $\mathcal{K}^\ell$. Fix a large enough $n > n_0$ such that (4) holds for $f = \frac{1}{\theta(\mathcal{K}^\ell \cap U)} \chi_{\mathcal{K}^\ell \cap U}$ in the formulation of Proposition 2.5. Now let $m_0 = m_0(n)$ be such that (18) holds for all $m > m_0$. We have

$$\int_{U_0} \phi(s^n s^m u s^{-m} s^{-n} x) du = \int_{U_0} \int_{U_0} \phi(s^n s^m u s^{-m} s^{-n} x) du f(v) dv =$$

$$\iint_{U_0} \phi(s^n s^m (s^{-m} v s^m) u s^{-m} s^{-n} x) f(v) du dv = \iint_{U_0} \phi(s^n v s^m u s^{-m} s^{-n} x) f(v) du dv$$

Note that in the above we used the fact that $s^{-m}vs^m \in U_0$ and that U_0 is a group to get the second identity. Let $B \subset U_0$ be such that $u \in B$ implies that $s^m u s^{-m} s^{-n} x \in L$ and let $A = U_0 \setminus B$. By (18) we have $|A| < \varepsilon$. We have

$$\int_{U_0} \int_{U_0} \phi(s^n vs^m u s^{-m} s^{-n} x) f(v) dv du = \int_A \int_{U_0} \phi(s^n vs^m u s^{-m} s^{-n} x) f(v) dv du$$
$$+ \int_B \int_{U_0} \phi(s^n vs^m u s^{-m} s^{-n} x) \psi(v) dv du$$

The first term is indeed bounded by $\varepsilon \cdot \sup(|\phi|)$. As for the second term recall that since $u \in B$ we may utilize Proposition 2.5 and get

$$\left| \int_{U_0} \phi(s^n vs^m u s^{-m} s^{-n} x) f(v) dv \right| < \varepsilon$$

So there is constant C' depending on ϕ such that

$$\left| \int_{U_0} \phi(s^{n+m} u s^{-n-m} x) du \right| < C\varepsilon$$

We now use (17) and get

$$\left| \int_X \phi d\mu \right| < C\varepsilon$$

which thanks to Lemma 4.1 says μ is invariant under G^+-invariant. Hence again by the Lemma 2.2 μ is the $\overline{G^+\Gamma}$-invariant invariant probability Haar measure on the closed orbit $\overline{G^+\Gamma}/\Gamma$ which finishes the proof if x satisfies (1) in Theorem 3.1.

We will now proceed by induction on the dimension of $\mathbb{G}$. Thanks to the previous paragraph we may and will assume that (2) in Theorem 3.1 holds i.e. if we let $x = g\Gamma$, then there exists $\mathbb{P}_i$ and $\lambda \in F\Gamma$ such that $g^{-1} U g \subset \lambda P_i \lambda^{-1}$ where $P_i = \mathbb{P}_i(K_S)$, with the notation as in the Theorem 3.1. Let $\mathbb{P}_i = \mathbb{L} \cdot \mathbb{W}$ be the Levi decomposition of $\mathbb{P}_i$ and let $W = \mathbb{W}(K_S)$. Let $\mathbb{M} = [\mathbb{L}, \mathbb{L}]$ be the derived group of $\mathbb{L}$. The group $\mathbb{M}$ is semisimple and is defined over K. We let $M = \mathbb{M}(K_S)$. Define $^\circ P_i = MW$.

Note also that $g^{-1} U g \subset \lambda {}^\circ P_i \lambda^{-1}$. So replacing U by $g^{-1} U g$ and $^\circ P_i$ by $\lambda {}^\circ P_i \lambda^{-1}$ we may and will assume that $U \subset {}^\circ P_i$. Since $\lambda \in \mathbb{G}(K)$ we have $\Delta = \Gamma \cap {}^\circ P_i$ is a lattice in $^\circ P_i$ and also $W \cap \Gamma$ is a (uniform) lattice in W hence $^\circ P_i/\Delta$ is a closed subset of X. These reductions also imply that $e\Gamma$ is generic for μ with respect to the action of U. As a consequence we may and will consider μ as a measure on $^\circ P_i/\Delta$. We write $U = VW$ where V is a maximal unipotent subgroup of M. Using the previous notation we let W_ν, V_ν and M_ν be the corresponding ν-components.

Let $\mu = \int_Y \mu_y d\sigma$ be the ergodic decomposition of μ with respect to W_ν. Recall that W_ν is a normal and unimodular subgroup of $^\circ P_i$. Hence by Lemma 2.2 we have that for σ-a.e. $y \in Y$ the measure μ_y is the $F = \overline{W_\nu \Delta}$-invariant measure on a closed F-orbit where the closure is taken with respect to the Hausdorff topology. Since Γ is a congruence lattice and $\mathbb{W}$ is a K-split unipotent group we deduce that $F = W\Delta$. Hence for σ-a.e. $y \in Y$ the measure μ_y is invariant under W. Since μ is U_ν-invariant ergodic measure we have σ is V_ν-invariant and ergodic measure on Y. Recall that μ is a measure on $^\circ P_i/\Delta$. Now since σ-a.e. μ_y is W-invariant and W is the unipotent radical of $^\circ P_i = MW$ we may and will identify Y with $M/\Delta \cap M$ and hence σ is a measure on $M/\Delta \cap M$.

Let us recall that $\mathbb{M}$ is a semisimple group defined over K and $\Delta \cap M$ is a congruence lattice in $\mathbb{M}(\mathcal{O}_S)$, furthermore σ is V_ν-invariant ergodic measure on $M/\Delta \cap M$. Hence we may apply the induction hypothesis and get: There exist some $g \in M$ and a connected K-parabolic group $\mathbb{Q}$ of $\mathbb{M}$ such that if we let $^\circ Q$ and Q^+ be defined as in the introduction then σ is the Haar measure on the closed orbit of $g\Sigma/^\circ Q \cap \Delta$ where $\Sigma = \overline{Q^+(^\circ Q \cap \Delta)}$ and the closure is taken with respect to the Hausdorff topology. Hence the measure μ is the Haar measure on the closed orbit $g\Sigma W/(Q \cap \Delta)(W \cap \Delta)$. Now let $\mathbb{H} = \mathbb{Q}\mathbb{W}$, this is a K-parabolic subgroup of $\mathbb{M}\mathbb{W}$. We let $H = \mathbb{H}(K_S)$. Since W is a subgroup of the split unipotent radical of all minimal K_S-pseudo parabolic subgroups of $\mathbb{H}$ we get $H^+ = Q^+W$ and that $H^+ = P^+$ for a K-parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$. Note also that since W is the unipotent radical and in particular a normal subgroup we have $\Sigma W = \overline{Q^+W(^\circ Q \cap \Delta)} = \overline{H^+(^\circ P \cap \Delta)}$. This finishes the proof in the case $U = U_\nu$.

We now turn to the general case. Hence $U_\nu \subset U$ and μ is U-invariant ergodic measure on X. Let $\mu = \int_Y \mu_y d\sigma$ be the ergodic decomposition of μ with respect to U_ν. For $x = g\Gamma \in X$ we let $y(x)$ denote the corresponding point from (Y, σ). Since $\mu_{y(x)}$'s are U_ν-ergodic the above argument says almost all $\mu_{y(x)}$'s are the $\Sigma(x)$-invariant measure on a closed orbit of $\Sigma(x) \cdot x$ where $\Sigma(x) = g\overline{P^+(^\circ P \cap \Gamma)}g^{-1}$, and $\mathbb{P}$ is a K-subgroup. We will say P is associated to x. For any K-subgroup, $\mathbb{P}$, of $\mathbb{G}$ we let $P = \mathbb{P}(K_S)$ and let

(19) $\mathcal{S}(P) = \{x \in Y : P \text{ is associated to } x\}$

Since there are only countably many such subgroups we have; there exists some P such that $\sigma(\mathcal{S}(P)) > 0$. Note also that U normalizes U_ν, so for every $u \in U$ the equality $u\mu_y(x) = \mu_{y(ux)}$ is true for μ-almost all $x \in X$. Furthermore we have $\Sigma(ux) = u\Sigma(x)u^{-1}$. Thus $\mathcal{S}(P)$ is U-invariant. This and the fact that μ is U-ergodic imply that $\sigma(\mathcal{S}(P)) = 1$. Observe that $P^+(^\circ P \cap \Delta)$ is Zariski dense in $^\circ P$ hence the map $x \mapsto \overline{\Sigma(x)}^z$, the Zariski closure of $\Sigma(x)$, is a U-equivariant Borel map from $\mathcal{S}(P)$ to $G/N_G(^\circ P)$. Lemma 2.1 now implies that this map is constant almost everywhere. Hence μ is the $g\overline{P^+(^\circ P \cap \Gamma)}g^{-1}$-invariant measure on a closed orbit $g\overline{P^+(^\circ P \cap \Gamma)}\Gamma/\Gamma$ for some $g \in G$. This finishes the proof of Theorem 1.1.

Equidistribution of orbits of horospherical subgroups. As we mentioned in the introduction one application of Theorem 1.1 is to prove equidistribution of orbits of U which satisfy the conditions of Theorem 1.1. It is possible to refine the above proof and get equidistribution statement from the proof. In here however we use the linearization technique in order to get equidistribution result. This seems to be shorter and it is a "standard" method by now. The linearization technique was developed by Dani and Margulis [DM93] in the Lie group case. Similar linearization statements in the S-arithmetic setting in characteristic zero were proved by Tomanov [To00]. In the function field setting such results were proved in [EM, section 5]. The linearization technique is an avoidance principle. Roughly speaking it states that unipotent orbits of algebraically generic points do not spend a "long" time "too close" to proper algebraic varieties. We have the following

Corollary 4.2. *Let the notation be as before and assume that either (i) or (ii) of Theorem 1.1 is satisfied. Then for any $x = g\Gamma \in X$ and $f \in C_c(X)$ we have*

$$\lim_{m \to \infty} \frac{1}{U_m} \int_{U_m} f(ux)d\theta(u) = \int_X fd\mu$$

where μ is the $g\overline{P^+(^\circ P \cap \Gamma)}g^{-1}$-invariant probability measure on the closed orbit $g\overline{P^+(^\circ P \cap \Gamma)}g^{-1} \cdot x$, and $\mathbb{P}(K_S)$ is a K-parabolic subgroup $\mathbb{P}$.

Proof. The proof is standard. Let $x = g\Gamma$ assume P is K-parabolic subgroup of G minimal with the property that $g^{-1}Ug \subset {}^\circ P$. We will show that the theorem holds with this ${}^\circ P$. Let $\widetilde{X}$ be the one-point compactification of X if X is not compact and be X if X is compact. For any natural number m define the probability measure μ_m on X by

$$\int_X f(y)d\mu_m(y) = \frac{1}{U_m} \int_{U_m} f(ux)d\theta(u)$$

where f is a bounded continuous function on X. As $\widetilde{X}$ is compact the space of probability measures on $\widetilde{X}$ is weak* compact. Let μ be a limit point of $\{\mu_n\}$. By identifying μ we show that there is only one limit point which in return gives convergence. Note that since $Y = {}^\circ P/({}^\circ P \cap \Gamma)$ is closed we may and will assume that μ_m's and also μ are supported on $Y \cap \{\infty\}$. It follows from nondivergence of unipotent trajectories, see Theorem 3.3, that μ is concentrated on Y. Note that μ is U-invariant. We let $\mu = \int_Y \mu_y d\sigma(y)$ be a decomposition of μ into U-ergodic components. For any proper K-parabolic subgroup F of P let

$$\mathcal{S}(F) = \{p \in {}^\circ P : p^{-1}Up \subset F\}$$

By Theorem 1.1 each ergodic component is supported on some $\mathcal{S}(F)$. Minimality of P together with theorem 5.5 in [EM] implies $\mu(\mathcal{S}(F)) = 0$ for all proper subgroups F. Since there are only countably many such F we get that σ-almost all $\mu_{y(z)}$'s are the P^+-invariant measure on the closed orbit $g_z\overline{P^+(^\circ P \cap \Gamma)}g_z^{-1} \cdot z$. Now since $g^{-1}Ug \subset P^+$ we see that the support of μ is in $g\overline{P^+(^\circ P \cap \Gamma)}g^{-1} \cdot x$ which then implies the theorem. $\square$

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