# $\mathbf{SL_n}(\mathbb{Z}[t])$ is not $FP_{n-1}$

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**Abstract**

We prove the result from the title using the geometry of Euclidean buildings.

## 1 Introduction

Little is known about the finiteness properties of $\mathbf{SL_n}(\mathbb{Z}[t])$ for arbitrary $n$.

In 1959 Nagao proved that if $k$ is a field then $\mathbf{SL_2}(k[t])$ is a free product with amalgamation [Na]. It follows from his description that $\mathbf{SL_2}(\mathbb{Z}[t])$ and its abelianization are not finitely generated.

In 1977 Suslin proved that when $n \geq 3$, $\mathbf{SL_n}(\mathbb{Z}[t])$ is finitely generated by elementary matrices [Su]. It follows that $H_1(\mathbf{SL_n}(\mathbb{Z}[t]), \mathbb{Z})$ is trivial when $n \geq 3$.

More recent, Krstić-McCool proved that $\mathbf{SL_3}(\mathbb{Z}[t])$ is not finitely presented [Kr-Mc].

In this paper we provide a generalization of the results of Nagao and Krstić-McCool mentioned above for the groups $\mathbf{SL_n}(\mathbb{Z}[t])$.

**Theorem 1.** *If $n \geq 2$, then $\mathbf{SL_n}(\mathbb{Z}[t])$ is not of type $FP_{n-1}$.*

Recall that a group $\Gamma$ is *of type $FP_m$* if if there exists a projective resolution of $\mathbb{Z}$ as the trivial $\mathbb{Z}\Gamma$ module

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

where each $P_i$ is a finitely generated, projective $\mathbb{Z}\Gamma$ module.

In particular, Theorem 1 implies that there is no $K(\mathbf{SL_n}(\mathbb{Z}[t]), 1)$ with finite $(n-1)$-skeleton, where $K(G, 1)$ is the Eilenberg-Mac Lane space for $G$.

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## 1.1 Outline of paper

The general outline of this paper is modelled on the proofs in [Bu-Wo 1] and [Bu-Wo 2], though some important modifications have to be made to carry out the proof in this setting.

As in [Bu-Wo 1] and [Bu-Wo 2], our approach is to apply Brown's filtration criterion [Br 1]. Here we will examine the action of $\mathbf{SL_n}(\mathbb{Z}[t])$ on the locally infinite Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. In Section 2 we will show that the infinite groups that arise as cell stabilizers for this action are of type $FP_m$ for all $m$, which is a technical condition that is needed for our application of Brown's criterion.

In Section 3 we will demonstrate the existence of a family of diagonal matrices that will imply the existence of a "nice" isometrically embedded codimension 1 Euclidean space in the building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. In [Bu-Wo 1] analogous families of diagonal matrices were constructed using some standard results from the theory of algebraic groups over locally compact fields. Because $\mathbb{Q}((t^{-1}))$ is not locally compact, our treatment in Section 3 is quite a bit more hands on.

Section 4 contains the main body of our proof. We use translates of portions of the codimension 1 Euclidean subspace found in Section 3 to construct spheres in the Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ (also of codimension 1). These spheres will lie "near" an orbit of $\mathbf{SL_n}(\mathbb{Z}[t])$, but will be nonzero in the homology of cells "not as near" the same $\mathbf{SL_n}(\mathbb{Z}[t])$ orbit. Theorem 1 will then follow from Brown's criterion.

## 1.2 Background material

Our proof relies heavily on the geometry of the Euclidean and spherical buildings for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. A good source of information for the former topic is Chapter 6 of [Br 2]. For the latter, we recommend Chapter 5 of [Ti].

# 2 Stabilizers

**Lemma 2.** *If $X$ is the Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$, then the $\mathbf{SL_n}(\mathbb{Z}[t])$ stabilizers of cells in $X$ are $FP_m$ for all $m$.*

*Proof.* Let $x_0 \in X$ be the vertex stabilized by $\mathbf{SL_n}(\mathbb{Q}[[t^{-1}]])$. We denote a diagonal matrix in $\mathbf{GL_n}(\mathbb{Q}((t^{-1})))$ with entries $s_1, s_2, ..., s_n \in \mathbb{Q}((t^{-1}))^\times$ by

$D(s_1, s_2, ..., s_n)$, and we let $\mathfrak{S} \subseteq X$ be the sector based at $x_0$ and containing vertices of the form $D(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq ... \geq m_n$.

The sector $\mathfrak{S}$ is a fundamental domain for the action of $\mathbf{SL_n}(\mathbb{Q}[t])$ on $X$ (see [So]). In particular, for any vertex $z \in X$, there is some $h'_z \in \mathbf{SL_n}(\mathbb{Q}[t])$ and some integers $m_1 \geq m_2 \geq ... \geq m_n$ with $z = h'_z D_z(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$. We let $h_z = h'_z D_z(t^{m_1}, t^{m_2}, ..., t^{m_n})$.

For any $N \in \mathbb{N}$, let $W_N$ be the $(N+1)$-dimensional vector space

$$W_N = \{\, p(t) \in \mathbb{C}[t] \mid \deg\big(p(t)\big) \leq N\}$$

which is endowed with the obvious $\mathbb{Q}-$structure. If $N_1, \cdots, N_{n^2}$ in $\mathbb{N}$ are arbitrary then let

$$\mathbf{G}_{\{N_1,\cdots,N_{n^2}\}} = \{\mathbf{x} \in \prod_{i=1}^{n^2} W_{N_i} |\det(\mathbf{x}) = 1\}$$

where $\det(\mathbf{x})$ is a polynomial in the coordinates of $\mathbf{x}$. To be more precise this is obtained from the usual determinant function when one considers the usual $n\times n$ matrix presentation of $\mathbf{x}$, and calculates the determinant in $\mathbf{Mat}_n(\mathbb{C}[t])$.

For our choice of vertex $z \in X$ above, the stabilizer of $z$ in $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ equals $h_z \mathbf{SL_n}(\mathbb{Q}[[t^{-1}]])h_z^{-1}$. And with our fixed choice of $h_z$, there clearly exist some $N_i^z \in \mathbb{N}$ such that the stabilizer of the vertex $z$ in $\mathbf{SL_n}(\mathbb{Q}[t])$ is $\mathbf{G}_{\{N_1^z,\cdots,N_{n^2}^z\}}(\mathbb{Q})$. Furthermore, conditions on $N_i^z$ force a group structure on $\mathbf{G}_z = \mathbf{G}_{\{N_1^z,\cdots,N_{n^2}^z\}}$. Therefore, the stabilizer of $z$ in $\mathbf{SL_n}(\mathbb{Q}[t])$ is the $\mathbb{Q}-$points of the affine $\mathbb{Q}$-group $\mathbf{G}_z$, and the stabilizer of $z$ in $\mathbf{SL_n}(\mathbb{Z}[t])$ is $\mathbf{G}_z(\mathbb{Z})$.

The action of $\mathbf{SL_n}(\mathbb{Q}[t])$ on $X$ is type preserving, so if $\sigma \subset \mathfrak{S}$ is a simplex with vertices $z_1, z_2, ..., z_m$, then the stabilizer of $\sigma$ in $\mathbf{SL_n}(\mathbb{Z}[t])$ is simply

$$\big(\mathbf{G}_{z_1} \cap ... \cap \mathbf{G}_{z_m}\big)(\mathbb{Z})$$

That is, the stabilizer of $\sigma$ in $\mathbf{SL_n}(\mathbb{Z}[t])$ is an arithmetic group, and Borel-Serre proved that any such group is $FP_m$ for all $m$ [Bo-Se].

□

# 3 Polynomial points of tori

This section is devoted exclusively to a proof of the following

**Proposition 3.** *There is a group $A \leq \mathbf{SL_n}(\mathbb{Z}[t])$ such that*

*(i) $A \cong \mathbb{Z}^{n-1}$*
*(ii) There is some $g \in \mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ such that $gAg^{-1}$ is a group of diagonal matrices*
*(iii) No nontrivial element of $A$ fixes a point in the Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$.*

The proof of this proposition is modelled on a classical approach to finding diagonalizable subgroups of $\mathbf{SL_n}(\mathbb{Z})$. The proof will take a few steps.

## 3.1 A polynomial over $\mathbb{Z}[t]$ with roots in $\mathbb{Q}((t^{-1}))$

Let $\{p_1, p_2, p_3, ...\} = \{2, 3, 5, ...\}$ be the sequence of prime numbers. Let $q_1 = 1$. For $2 \leq i \leq n$, let $q_i = p_{i-1} + 1$.

Let $f(x) \in \mathbb{Z}[t][x]$ be the polynomial given by

$$f(x) = \Big[\prod_{i=1}^{n}(x + q_i t)\Big] - 1$$

It will be clear by the conclusion of our proof that $f(x)$ is irreducible over $\mathbb{Q}(t)$, but we will not need to use this directly.

**Lemma 4.** *There is some $\alpha \in \mathbb{Q}((t^{-1}))$ such that $f(\alpha) = 0$.*

*Proof.* We want to show that there are $c_i \in \mathbb{Q}$ such that if $\alpha = \sum_{i=0}^{\infty} c_i t^{1-in}$ then $f(\alpha) = 0$.

To begin let $c_0 = -1$. We will define the remaining $c_i$ recursively. Define $c_{i,k}$ by $\alpha + q_k t = \sum_{i=0}^{\infty} c_{i,k} t^{1-in}$. Thus, $c_{i,k} = c_i$ when $i \geq 1$, each $c_{0,k}$ is contained in $\mathbb{Q}$, and $c_{0,1} = 0$.

That $\alpha$ is a root of $f$ is equivalent to

$$\begin{aligned} 1 = \prod_{k=1}^{n}(\alpha + q_k t) &= \prod_{k=1}^{n}\Big(\sum_{i=0}^{\infty} c_{i,k} t^{1-in}\Big) \\ &= \sum_{i=0}^{\infty}\Big(\sum_{\sum_{k=1}^{n} i_k = i}\Big(\prod_{k=1}^{n} c_{i_k,k}\Big)\Big) t^{n(1-i)} \end{aligned}$$

Our task is to find $c_m$'s so that the above is satisfied.

Note that for the above equation to hold we must have

$$0 \cdot t^n = \sum_{\sum_{k=1}^{n} i_k = 0} \left(\prod_{k=1}^{n} c_{i_k,k}\right) t^{n(1-0)}$$

That is

$$0 = \prod_{k=1}^{n} c_{0,k}$$

which is an equation we know is satisfied because $c_{0,1} = 0$. Now assume that we have determined $c_0, c_1, ..., c_{m-1} \in \mathbb{Q}$. We will find $c_m \in \mathbb{Q}$.

Notice that the first coefficient in our Laurent series expansion above which involves $c_m$ is the coefficient for the $t^{-nm}$ term. This follows from the fact that each $i_k$ is nonnegative.

Since

$$\sum_{\sum_{k=1}^{n} i_k = m} \left(\prod_{k=1}^{n} c_{i_k,k}\right)$$

is the coefficient of the $t^{-nm}$ term in the expansion of 1, we have

$$0 = \sum_{\sum_{k=1}^{n} i_k = m} \left(\prod_{k=1}^{n} c_{i_k,k}\right)$$

The above equation is linear over $\mathbb{Q}$ in the single variable $c_m$ and the coefficient of $c_m$ is nonzero. Indeed, $\sum_{k=1}^{n} i_k = m$, each $i_k \geq 0$, and $c_0, ..., c_{m-1} \in \mathbb{Q}$ are assumed to be known quantities. Thus, $c_m \in \mathbb{Q}$.

□

## 3.2 Matrices representing ring multiplication

By Lemma 4 we have that the field $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$ is an extension of $\mathbb{Q}(t)$ of degree $d$ where $d \leq n$. It follows that $\mathbb{Z}[t][\alpha]$ is a free $\mathbb{Z}[t]$-module of rank $d$ with basis $\{1, \alpha, \alpha^2, ..., \alpha^{d-1}\}$.

For any $y \in \mathbb{Z}[t][\alpha]$, the action of $y$ on $\mathbb{Q}(t)(\alpha)$ by multiplication is a linear transformation that stabilizes $\mathbb{Z}[t][\alpha]$. Thus, we have a representation of $\mathbb{Z}[t][\alpha]$ into the ring of $d \times d$ matrices with entries in $\mathbb{Z}[t]$. We embed the ring of $d \times d$ matrices with entries in $\mathbb{Z}[t]$ into the upper left corner of the ring of $n \times n$ matrices with entries in $\mathbb{Z}[t]$.

By Lemma 4

$$\prod_{i=1}^{n}(\alpha + q_i t) = 1$$

so each of the following matrices are invertible:

$$\alpha + q_1 t,\ \alpha + q_2 t,\ ...,\ \alpha + q_n t$$

(We will be blurring the distinction between the elements of $\mathbb{Z}[t][\alpha]$ and the matrices that represent them.)

For $1 \leq i \leq n-1$, we let $a_i = \alpha + q_{i+1}t$. Since $a_i$ is invertible, it is an element of $\mathbf{GL_n}(\mathbb{Z}[t])$, and hence has determinant $\pm 1$. By replacing each $a_i$ with its square, we may assume that $a_i \in \mathbf{SL_n}(\mathbb{Z}[t])$ for all $i$. We let $A = \langle a_1, ...a_{n-1}\rangle$ so that $A$ is clearly abelian as it is a representation of multiplication in an integral domain. This group $A$ will satisfy Proposition 3.

## 3.3 $A$ is free abelian on the $a_i$

To prove part (i) of Proposition 3 we have to show that if there are $m_i \in \mathbb{Z}$ with

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

then each $m_i = 0$. But the first nonzero term in the Laurent series expansion for $\alpha$ is $-t$, which implies that the first nonzero term in the Laurent series expansion for each $a_i$ is $-t + q_{i+1}t = p_i t$. Hence, the first nonzero term of

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

is

$$\prod_{i=1}^{n-1} (p_i t)^{m_i} = t^0$$

Thus

$$\prod_{i=1}^{n-1} p_i^{m_i} = 1$$

and it follows by the uniqueness of prime factorization that $m_i = 0$ for all $i$ as desired.

Thus, part (i) of Proposition 3 is proved.

### 3.4 $A$ is diagonalizable

Recall that $\alpha$ is a $d \times d$ matrix with entries in $\mathbb{Z}[t]$ where $d$ is the degree of the minimal polynomial of $\alpha$ over $\mathbb{Q}(t)$. Let that minimal polynomial be $q(x)$. Because the characteristic of $\mathbb{Q}(t)$ equals 0, $q(x)$ has distinct roots in $\mathbb{Q}(t)(\alpha)$.

Let $Q(x)$ be the characteristic polynomial of the matrix $\alpha$. The polynomial $Q$ also has degree $d$ and leading coefficient $\pm 1$ with $Q(\alpha) = 0$. Therefore, $q = \pm Q$. Hence, $Q$ has distinct roots in $\mathbb{Q}(t)(\alpha)$ which implies that $\alpha$ is diagonalizable over $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$. That is to say that there is some $g \in \mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ such that $g\alpha g^{-1}$ is diagonal.

Because every element of $\mathbb{Z}[t][\alpha]$ is a linear combination of powers of $\alpha$, we have that $g(\mathbb{Z}[t][\alpha])g^{-1}$ is a set of diagonal matrices. In particular, we have proved part (ii) of Proposition 3.

### 3.5 $A$ has trivial stabilizers

To prove part (iii) of Proposition 3 we begin with the following

**Lemma 5.** *If $\Gamma \leq \mathbf{SL_n}(\mathbb{Q}[t])$ is bounded under the valuation for $\mathbb{Q}((t^{-1}))$, then the eigenvalues for any $\gamma \in \Gamma$ lie in $\overline{\mathbb{Q}}$.*

*Proof.* We let $X$ be the Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. By the Bruhat-Tits fixed point theorem, $\Gamma z = z$ for some $z \in X$.

Let $x_0 \in X$ be the vertex stabilized by $\mathbf{SL_n}(\mathbb{Q}[[t^{-1}]])$. We denote a diagonal matrix in $\mathbf{GL_n}(\mathbb{Q}((t^{-1})))$ with entries $s_1, s_2, ..., s_n \in \mathbb{Q}((t^{-1}))^\times$ by $D(s_1, s_2, ..., s_n)$, and we let $\mathfrak{S} \subseteq X$ be the sector based at $x_0$ and containing vertices of the form $D(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq ... \geq m_n$.

The sector $\mathfrak{S}$ is a fundamental domain for the action of $\mathbf{SL_n}(\mathbb{Q}[t])$ on $X$ [So] which implies that there is some $h \in \mathbf{SL_n}(\mathbb{Q}[t])$ with $hz \in \mathfrak{S}$.

Clearly we have $(h\Gamma h^{-1})hz = hz$, and since eigenvalues of $h\Gamma h^{-1}$ are the same as those for $\Gamma$, we may assume that $\Gamma$ fixes a vertex $z \in \mathfrak{S}$.

Fix $m_1, ..., m_n \in \mathbb{Z}$ with $m_1 \geq ... \geq m_n \geq 0$ and such that $z = D(t^{m_1}, ..., t^{m_n})x_0$. Without loss of generality, there is a partition of $n$ — say $\{k_1, ..., k_\ell\}$ — such that

$$\{m_1, ..., m_n\} = \{q_1, ..., q_1,\ q_2, ..., q_2,\ ...,\ q_\ell, ...q_\ell\}$$

where each $q_i$ occurs exactly $k_i$ times and

$$q_1 > q_2 > ... > q_\ell$$

We have that $D(t^{m_1}, ..., t^{m_n})^{-1}\Gamma D(t^{m_1}, ..., t^{m_n})x_0 = x_0$. That gives us, $D(t^{m_1}, ..., t^{m_n})^{-1}\Gamma D(t^{m_1}, ..., t^{m_n}) \subset \mathbf{SL_n}(\mathbb{Q}[[t^{-1}]])$. Furthermore, a trivial calculation of resulting valuation restrictions for the entries of $D(t^{m_1}, ..., t^{m_n})\mathbf{SL_n}(\mathbb{Q}[[t^{-1}]])D(t^{m_1}, ..., t^{m_n})^{-1}$ shows that $\Gamma$ is contained in a subgroup of $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ that is isomorphic to

$$\prod_{i=1}^{\ell} \mathbf{SL_{k_i}}(\mathbb{Q}) \ltimes U$$

where $U \leq \mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ is a group of upper-triangular unipotent matrices.

The lemma is proved.

□

Our proof of Proposition 3 will conclude by proving

**Lemma 6.** *No nontrivial element of $A$ fixes a point in the Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$.*

*Proof.* Suppose $a \in A$ fixes a point in the building. We will show that $a = 1$. Let $F(x) \in \mathbb{Z}[t][x]$ be the characteristic polynomial for $a \in \mathbf{SL_n}(\mathbb{Z}[t])$. Then

$$F(x) = \pm \prod_{i=1}^{n} (x - \beta_i)$$

where each $\beta_i \in \mathbb{Q}((t^{-1}))$ is an eigenvalue of $a$. By the previous lemma, each $\beta_i \in \overline{\mathbb{Q}}$. Hence, each $\beta_i \in \mathbb{Q} = \overline{\mathbb{Q}} \cap \mathbb{Q}((t^{-1}))$. It follows that $F(x) \in \mathbb{Z}[x]$ so that each $\beta_i$ is an algebraic integer contained in $\mathbb{Q}$. We conclude that each $\beta_i$ is contained in $\mathbb{Z}$.

Recall, that $a$ has determinant 1, and that the determinant of $a$ can be expressed as $\prod_{i=1}^{n} \beta_i$. Hence, each $\beta_i$ is a unit in $\mathbb{Z}$, so each eigenvalue $\beta_i = \pm 1$. It follows – by the diagonalizability of $a$ – that $a$ is a finite order element of $A \cong \mathbb{Z}^{n-1}$. That is, $a = 1$.

□

We have completed our proof of Proposition 3.

# 4 Body of the proof

Let $P \leq \mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ be the subgroup where each of the first $n-1$ entries along the bottom row equal 0. Let $R_u(P) \leq P$ be the subgroup of elements that contain a $(n-1) \times (n-1)$ copy of the identity matrix in the upper left corner. Thus $R_u(P) \cong \mathbb{Q}((t^{-1}))^{n-1}$ with the operation of vector addition.

Let $L \leq P$ be the copy of $\mathbf{SL_{n-1}}(\mathbb{Q}((t^{-1})))$ in the upper left corner of $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. We apply Proposition 3 to $L$ (notice that the $n$ in the proposition is now an $n-1$) to derive a subgroup $A \leq L$ that is isomorphic to $\mathbb{Z}^{n-2}$. By the same proposition, there is a matrix $g \in L$ such that $gAg^{-1}$ is diagonal.

Let $b \in \mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ be the diagonal matrix given in the notation from the proofs of Lemmas 2 and 5 as $D(t, t, ..., t, t^{-(n-1)})$. Note that $b \in P$ commutes with $L$, and therefore, with $A$. Thus the Zariski closure of the group generated by $b$ and $A$ determines an apartment in $X$, namely $g^{-1}\mathcal{A}$ where $\mathcal{A}$ is the apartment corresponding to the diagonal subgroup of $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$.

## 4.1 Actions on $g^{-1}\mathcal{A}$.

If $x_* \in g^{-1}\mathcal{A}$, then it follows from Proposition 3 that the convex hull of the orbit of $x_*$ under $A$ is an $(n-2)$-dimensional affine space that we will name $V_{x_*}$. Furthermore, the orbit $Ax_*$ forms a lattice in the space $V_{x_*}$.

We let $g^{-1}\mathcal{A}(\infty)$ be the visual boundary of $g^{-1}\mathcal{A}$ in the Tits boundary of $X$. Recall that the Tits boundary of $X$ is isomorphic to the spherical building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. The definition of visual boundary used above is the standard defintion from CAT(0) geometry.

The visual boundary of $V_{x_*}$ is clearly an equatorial sphere in $g^{-1}\mathcal{A}(\infty)$. Precisely, we let $P^-$ be the transpose of $P$. Then $P$ and $P^-$ are opposite vertices in $g^{-1}\mathcal{A}(\infty)$. It follows that there is a unique sphere in $g^{-1}\mathcal{A}(\infty)$ that is realized by all points equidistant to $P$ and $P^-$. We call this sphere $S_{P,P^-}$.

**Lemma 7.** *The visual boundary of $V_{x_*}$ equals $S_{P,P^-}$.*

*Proof.* Since $g \in P \cap P^-$, it suffices to prove that $gV_{x_*}$ is the sphere in the boundary of $\mathcal{A}$ that is determined by the vertices $P$ and $P^-$.

Note that $gV_{x_*}$ is a finite Hausdorff distance from any orbit of a point in $\mathcal{A}$ under the action of the diagonal subgroup of $L$. The result follows by observing that the inverse transpose map on $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ stabilizes diagonal matrices while interchanging $P$ and $P^-$.

□

We let $R_1, R_2, ..., R_{n-1}$ be the standard root subgroups of $R_u(P)$. Recall that associated to each $R_i$ there is a closed geodesic hemisphere $H_i \subseteq \mathcal{A}(\infty)$ such that any nontrivial element of $R_i$ fixes $H_i$ pointwise and translates any point in the open hemisphere $\mathcal{A}(\infty) - H_i$ outside of $\mathcal{A}(\infty)$. Note that $\partial H_i$ is a codimension 1 geodesic sphere in $\mathcal{A}(\infty)$.

We let $M \subseteq g^{-1}\mathcal{A}(\infty)$ be the union of chambers in $g^{-1}\mathcal{A}(\infty)$ that contain the vertex $P$. There is also an equivalent geometric description of $M$:

**Lemma 8.** *The union of chambers $M \subseteq g^{-1}\mathcal{A}(\infty)$ can be realized as an $(n-2)$-simplex. Furthermore,*

$$M = \bigcap_{i=1}^{n-1} g^{-1}H_i$$

*and, when $M$ is realized as a single simplex, each of the $n-1$ faces of $M$ is contained in a unique equatorial sphere $g^{-1}\partial H_i = \partial g^{-1}H_i$.*

*Proof.* Let $M' \subseteq \mathcal{A}(\infty)$ be the union of chambers in $\mathcal{A}(\infty)$ containing the vertex $P$. Since $M = g^{-1}M'$, it suffices to prove that $M'$ is an $(n-2)$-simplex with $M' = \cap_{i=1}^{n-1} H_i$ and with each face of $M'$ contained in a unique $\partial H_i$.

For any nonempty, proper subset $I \subseteq \{1, 2, ..., n\}$, we let $V_I$ be the $|I|$-dimensional vector subspace of $\mathbb{Q}((t^{-1}))^n$ spanned by the coordinates given by $I$, and we let $P_I$ be the stabilizer of $V_I$ in $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$. For example, $P = P_{\{1,2,...,n-1\}}$.

Recall that the vertices of $\mathcal{A}(\infty)$ are given by the parabolic groups $P_I$, that edges connect $P_I$ and $P_{I'}$ exactly when $I \subseteq I'$ or $I' \subseteq I$, and that the remaining simplicial description of $\mathcal{A}(\infty)$ is given by the condition that $\mathcal{A}(\infty)$ is a flag complex.

We let $\mathcal{V}$ be the set of vertices in $\mathcal{A}(\infty)$ of the form $P_J$ where $\emptyset \neq J \subseteq \{1, 2, ..., n-1\}$. Note that $M'$ is exactly the set of vertices $\mathcal{V}$ together with the simplices described by the incidence relations inherited from $\mathcal{A}(\infty)$. Thus, $M'$ is easily seen to be isomorphic to a barycentric subdivision of an abstract $(n-2)$-simplex. Indeed, if $\overline{M'}$ is the abstract simplex on vertices $P_{\{1\}}, P_{\{2\}}, ..., P_{\{n-1\}}$, then a simplex of dimension $k$ in $\overline{M'}$ corresponds to a unique $P_J \in \mathcal{V}$ with $|J| = k+1$. So we have that $M'$ can be topologically realized as an $(n-2)$-simplex.

Let $F_i$ be a face of the simplex $\overline{M'}$. Then there is some $1 \leq i \leq n-1$ such that the set of vertices of $F_i$ is exactly $\{P_{\{1\}}, P_{\{2\}}, ..., P_{\{n-1\}}\} - P_{\{i\}}$.

Note that $R_i V_I = V_I$ exactly when $n \in I$ implies $i \in I$. It follows that $R_i$ fixes $M'$ pointwise, and thus $M' \subseteq H_i$ for all $1 \leq i \leq n-1$. Furthermore, if $P_I \in H_i$ for all $1 \leq i \leq n-1$, then $R_i P_I = P_I$ for all $i$ so that $n \in I$ implies $i \in I$ for all $1 \leq i \leq n-1$. As $I$ must be a proper subset of $\{1, 2, ..., n\}$, we have $P_I \in \mathcal{V}$, so that $M' = \cap_{i=1}^{n-1} H_i$.

All that remains to be verified for this lemma is that $F_i \subseteq \partial H_i$. For this fact, recall that $F_i$ is comprised of $(n-3)$-simplices in $\mathcal{A}(\infty)$ whose vertices are given by $P_J$ where $J \subseteq \{1, 2, ..., n-1\} - \{i\}$. Hence, if $\sigma \subseteq \mathcal{A}(\infty)$ is an $(n-3)$ simplex of $\mathcal{A}(\infty)$ with $\sigma \subseteq F_i$, then $\sigma$ is a face of exactly 2 chambers in $\mathcal{A}(\infty)$: $\mathfrak{C}_P$ and $\mathfrak{C}_{P_{J'}}$ where $\mathfrak{C}_P$ contains $P$ and thus $\mathfrak{C}_P \subseteq M'$, and $\mathfrak{C}_{P_{J'}}$ contains $P_{J'}$ where $J' = \{1, 2, ..., n\} - \{i\}$ and thus $\mathfrak{C}_{P_{J'}} \nsubseteq M'$. Furthermore, $\sigma = \mathfrak{C}_P \cap \mathfrak{C}_{P_{J'}}$.

Since $R_i V_{J'} \neq V_{J'}$, it follows that $\mathfrak{C}_{P_{J'}}$ is not fixed by $R_i$. Since $\mathfrak{C}_{P_J}$ is fixed by $R_i$ we have that $\sigma = \mathfrak{C}_P \cap \mathfrak{C}_{P_{J'}} \subseteq \partial H_i$. Therefore, $F_i \subseteq \partial H_i$. □

For any vertex $y \in X$, we let $C_y \subseteq X$ be the union of sectors based at $y$ and limiting to a chamber in $M$. Thus, $C_y$ is a cone. Note also that because any chamber in $g^{-1}\mathcal{A}(\infty)$ has diameter less than $\pi/2$, it follows that $M \cap S_{P,P^-} = \emptyset$. Therefore, if we choose $x_*, y \in g^{-1}\mathcal{A}$ such that $x_*$ is closer to $P$ than $y$, then $C_y \subseteq g^{-1}\mathcal{A}$ and $V_{x_*} \cap C_y$ is a simplex of dimension $n-2$.

We will set on a fixed choice of $y$ before $x_*$, and we will choose $y$ to satisfy the below

**Lemma 9.** *There is some $y \in g^{-1}\mathcal{A}$ such that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $C_y$ pointwise.*

*Proof.* Let $x_0$ be the point in $X$ stabilized by $\mathbf{SL_n}(\mathbb{Q}[[t^{-1}]])$. Recall that $R_u(P)M = M$ so that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $C_{x_0}$ pointwise.

Because $M \subseteq g^{-1}\mathcal{A}(\infty)$, there is a $y \in C_{x_0} \cap g^{-1}\mathcal{A}$. Any such $y$ satisfies the lemma.

□

Choose $e$ such that with $x_* = e$ as above and with $y$ as in Lemma 9, there exists a fundamental domain $D_e$ for the action of $A$ on $V_e$ that is contained in $C_y$. The choice of $e$ can be made by travelling arbitrarily far from $y$ along a geodesic ray in $g^{-1}\mathcal{A}$ that limits to $P$.

By the choice of $D_e$ we have that

$$AD_e = V_e$$

and that the $\mathbb{Q}([[t^{-1}]])$-points of $R_u(P)$ fix $D_e$.

## 4.2 The filtration

We let

$$X_0 = \mathbf{SL_n}(\mathbb{Z}[t])D_e$$

and for any $i \in \mathbb{N}$ we choose an $\mathbf{SL_n}(\mathbb{Z}[t])$-invariant and cocompact space $X_i \subseteq X$ somewhat arbitrarily to satisfy the inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq \cup_{i=1}^{\infty} X_i = X$$

In our present context, Brown's criterion takes on the following form [Br 1]

**Brown's Filtration Criterion 10.** *By Lemma 2, the group $\mathbf{SL_n}(\mathbb{Z}[t])$ is not of type $FP_{n-1}$ if for any $i \in \mathbb{N}$, there exists some class in the homology group $\widetilde{H}_{n-2}(X_0\,, \mathbb{Z})$ which is nonzero in $\widetilde{H}_{n-2}(X_i\,, \mathbb{Z})$.*

## 4.3 Translation to $P$ moves away from filtration sets

The following is essentially Mahler's compactness criterion.

**Lemma 11.** *Given any $i \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $b^k e \notin X_i$.*

*Proof.* The lemma follows from showing that the sequence

$$\{\mathbf{SL_n}(\mathbb{Z}[t])b^k e\}_k \subseteq \mathbf{SL_n}(\mathbb{Z}[t])\backslash X$$

is unbounded.

Since stabilizers of points in $X$ are bounded subgroups of $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$, the claim above follows from showing that the sequence

$$\{\mathbf{SL_n}(\mathbb{Z}[t])b^k\}_k \subseteq \mathbf{SL_n}(\mathbb{Z}[t])\backslash\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$$

is unbounded.

But bounded sets in $\mathbf{SL_n}(\mathbb{Z}[t])\backslash\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$ do not contain sequences of elements $\{\mathbf{SL_n}(\mathbb{Z}[t])g_\ell\}_\ell$ such that $1 \in \overline{g_\ell^{-1}(\mathbf{SL_n}(\mathbb{Z}[t]) - \{1\})g_\ell}$. And clearly $b^k$'s contract some root groups to 1. Thus none of the sequences above is bounded. □

## 4.4 Applying Brown's criterion

As is described by Brown's criterion, we will prove Theorem 1 by fixing $X_i$ and finding an $(n-2)$-cycle in $X_0$ that is nontrivial in the homology of $X_i$.

Recall that we denote the standard root subgroups of $R_u(P)$ by $R_1, ..., R_{n-1}$. Each group $g^{-1}R_j g$ determines a family of parallel walls in $g^{-1}\mathcal{A}$. By Lemma 8, each face of the cone $C_y$ is contained in a wall of one of these families.

Choose $r_j \in g^{-1}R_j g$ for all $j$ such that $b^k e$ is contained in the wall determined by $r_j$ where $k$ is determined by $i$ as in Lemma 11. In particular, $r_j b^k e = b^k e$.

The intersection of the fixed point sets in $g^{-1}\mathcal{A}$ of the elements $r_1, ..., r_{n-1}$ determine a cone that we name $Z$. Note that $Z$ is contained in – and is a finite Hausdorff distance from – the cone $C_y$.

Let $Z^- \subseteq g^{-1}\mathcal{A}$ be the closure of the set of points in $g^{-1}\mathcal{A}$ that are fixed by none of the $r_j$. The set $Z^-$ is a cone based at $b^k e$, containing $y$, and asymptotically containing the vertex $P^-$.

As the walls of $Z^-$ are parallel to those of $Z$ – and hence of $C_y$, we have that $Z^- \cap V_e$ is an $(n-2)$-dimensional simplex. We will name this simplex $\sigma$.

The component of $Z^- - V_e$ that contains $b^k e$ is an $(n-1)$-simplex that has $\sigma$ as a face. Call this $(n-1)$ simplex $Y$.

For any $\ell \in \mathbb{N}$, there are exactly $2^{n-1}$ possible subsets of the set $\{r_1^\ell, ..., r_{n-1}^\ell\}$. For each such subset $S_\ell$, we let

$$Y_{S_\ell} = (\prod_{g \in S_\ell} g)Y$$

and

$$\sigma_{S_\ell} = (\prod_{g \in S_\ell} g)\sigma$$

Notice that the product of group elements in the equations above are well-defined regardless of the order of the multiplication since $R_u(P)$ is abelian. In the degenerate cases, $\prod_{g \in \emptyset} g = 1$, so $Y_\emptyset = Y$ and $\sigma_\emptyset = \sigma$.

For any $\ell \in \mathbb{N}$, we let $Y_\ell = \cup_{S_\ell} Y_{S_\ell}$. Because the wall in $g^{-1}\mathcal{A}$ determined by $r_j^\ell$ is the same as the wall determined by $r_j$, the space $Y_\ell$ is a closed ball containing $b^k e$ whose boundary sphere is $\cup_{S_\ell} \sigma_{S_\ell}$. Indeed the simplicial decomposition of $Y_\ell$ described above is isomorphic to the simplicial decomposition of the unit ball in $\mathbb{R}^{n-1}$ that is given by the $n-1$ hyperplanes defined by setting a coordinate equal to 0.

Let $\omega_\ell = \cup_{S_\ell}\sigma_{S_\ell}$. Thus $\omega_\ell = \partial Y_\ell$. Furthermore, the building $X$ is $(n-1)$-dimensional and contractible, so any $(n-1)$-chain with boundary equal to $\omega_\ell$ must contain $Y_\ell$ and thus $b^k e$. That is for all $\ell \in \mathbb{N}$

$$[\omega_\ell] \neq 0 \in \widetilde{H}_{n-2}(X - b^k e\,,\, \mathbb{Z})$$

If we can show that $\omega_\ell \subseteq X_0$ for some choice of $\ell$, then we will have proved our main theorem by application of Brown's criterion since we would have

$$[\omega_\ell] \neq 0 \in \widetilde{H}_{n-2}(X_i\,,\, \mathbb{Z})$$

by Lemma 11.

**Lemma 12.** *There exists some $\ell \in \mathbb{N}$ such that $\omega_\ell \subseteq X_0$.*

*Proof.* For any $u \in R_u(P)$ there is a decomposition $u = u'u''$ where the entries of $u' \in R_u(P)$ are contained in $\mathbb{Q}[t]$ and the entries of $u'' \in R_u(P)$ are contained in $\mathbb{Q}[[t^{-1}]]$.

For any $a \in A$ and $u \in R_u(P)$ there is a power $\ell(a,u) \in \mathbb{N}$ such that

$$(a^{-1}u^{\ell(a,u)}a)' = ((a^{-1}ua)')^{\ell(a,u)} \in \mathbf{SL_n}(\mathbb{Z}[t])$$

(For the above equality recall that $A \leq L$ normalizes $R_u(P)$ and the group operation on $R_u(P)$ is vector addition.)

There are only finitely many $a \in A$ such that $aD_e \cap \sigma \neq \emptyset$ (or equivalently, such that $aD_e \cap Z^- \neq \emptyset$). Call this finite set $\mathcal{D} \subseteq A$.

At this point we fix

$$\ell = \prod_{a \in \mathcal{D}} \prod_{i=1}^{n-1} \ell(a, r_i)$$

Thus,

$$[a^{-1}(\prod_{g \in S_\ell} g)a]' \in \mathbf{SL_n}(\mathbb{Z}[t])$$

for any $a \in \mathcal{D}$ and any $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$.

Because $\omega_\ell = \cup_{S_\ell}\sigma_{S_\ell}$ and $\sigma_{S_\ell} = (\prod_{g \in S_\ell} g)\sigma = (\prod_{g \in S_\ell} g)(AD_e \cap Z^-)$, we can finish our proof of this lemma by showing

$$(\prod_{g \in S_\ell} g)aD_e \subseteq X_0$$

for each $a \in \mathcal{D} \subseteq A \leq \mathbf{SL_n}(\mathbb{Z}[t])$ and each $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$. For this, recall that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $D_e$ and thus

$$\begin{aligned}
\Big(\prod_{g \in S_\ell} g\Big) a D_e &= a[a^{-1}\Big(\prod_{g \in S_\ell} g\Big)a] D_e \\
&= a[a^{-1}\Big(\prod_{g \in S_\ell} g\Big)a]'[a^{-1}\Big(\prod_{g \in S_\ell} g\Big)a]'' D_e \\
&= a[a^{-1}\Big(\prod_{g \in S_\ell} g\Big)a]' D_e \\
&\subseteq \mathbf{SL_n}(\mathbb{Z}[t]) D_e \\
&= X_0
\end{aligned}$$

□

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