A. Let \( f : [0, \infty) \to \mathbb{R} \) be continuous. Assume there exist positive numbers \( M \) and \( R \) so that the following hold.

- \( f \) is differentiable on \([M, \infty)\).
- \( |f'(x)| \leq R \) for all \( x \geq M \).

1. Prove that \( f \) is uniformly continuous on \([0, \infty)\).
2. Use part (1), to show that \( f(x) = x^p \) is uniformly continuous on \([0, \infty)\) for all \( 0 \leq p < 1 \).

B. Let
\[
f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2n+1} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise}
\end{cases}
\]
Find, with justification, all points where \( f \) is differentiable.

C. Let
\[
f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0
\end{cases}
\]
Find (with justification), \( f^{(n)}(0) \) for all \( n \).

D. Suppose \( f \) is a differentiable function defined on \( \mathbb{R} \) and assume that \( f' \) is strictly increasing. Prove that every tangent line of \( f \) intersects the graph of \( f \) only once.

E. Rudin, Chapter 5 (page 114), problems #1, 7, 9, 17, 26.

The following problems are for your practice, and will not be graded.

1. Rudin, Chapter 5 (page 114), problems #2, 3, 4, 5, 22.

2. Let \( P = \{p_1, p_2, \ldots\} \) denote the set of odd prime numbers listed in increasing order, i.e., \( p_1 < p_2 < \cdots \).

(a) Define functions \( f_1, f_2, \ldots \) and \( f \) on \((-1, 1)\) as follows. For every \( n \in \mathbb{N} \), define
\[
f_n(x) = x^{1 + \frac{1}{p_n}}
\]
and put
\[
f(x) = \begin{cases} f_n(x) & \text{if } x = \frac{1}{p_m} \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise}
\end{cases}
\]
Is \( f \) differentiable at 0?

(b) For all \( n \in \mathbb{N} \) let \( g_n : (-1, 1) \to \mathbb{R} \) be a twice differentiable function. Further, assume that

(i) \( g_n(0) = g_n'(0) = 0 \) for all \( n \), and

(ii) There exists some \( M \in \mathbb{R} \) so that for all \( n \in \mathbb{N} \) and all \( x \in (-1, 1) \) we have \( |g_n''(x)| \leq M \).

Define
\[
g(x) = \begin{cases} 
  g_n(x) & x = \frac{1}{p_n} \text{ for some } m \in \mathbb{N} \\
  0 & \text{otherwise}
\end{cases}
\]

Prove that \( g \) is differentiable at 0.

(Hint: Using (a) and (b) above prove that for every \( \epsilon > 0 \) there exists some \( \delta > 0 \) so that if \( |x| < \delta \), then \( |g_n'(x)| < \epsilon \) simultaneously for all \( n \).)