ON ERGODIC PROPERTIES OF THE BURGER-ROBLIN MEASURE

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1. Introduction

In this note we intend to describe some dynamical properties of one-parameter unipotent flows on the frame bundle of a convex cocompact hyperbolic 3-manifold.

Much effort and study have been done in the case of manifolds with finite volume, and quite a rich theory is developed in this case. The case of infinite volume manifolds, however, is far less understood. The goal here is to highlight some of the difficulties one faces, and possible modifications, in extending techniques developed in the finite volume case to the case of infinite volume manifolds.

Throughout, $G = \mathrm{PSL}_2(\mathbb{C})$, the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^3$. We let Γ be a Zariski dense discrete subgroup of G which is *convex cocompact*, that is, the convex hull of the limit set of Γ is compact modulo Γ. Equivalently, $\Gamma\backslash\mathbb{H}^3$ admits a finite sided fundamental domain with no cusps.

The frame bundle of the manifold $\Gamma\backslash\mathbb{H}^3$ is identified with the homogeneous space $X = \Gamma\backslash G$. Certain subgroups of G will be of particular importance in the sequel. Let $K = \mathrm{PSU}_2$, $A = \{a_s : s \in \mathbb{R}\}$, and $N = \{n_z : z \in \mathbb{C}\}$, where

$$a_s = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} \text{ and } n_z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

Any one-parameter unipotent subgroup of G is conjugate to

$$U = \{u_t := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R}\}. \tag{1}$$

The flow considered here is indeed the right action of U on X.

One possible starting point is to study ergodicity [1] of this action with respect to natural measures on X. If X has finite volume, then Moore's ergodicity theorem [10], implies that this flow is ergodic with respect to the volume measure, i.e. the G-invariant measure. If Γ is not a lattice, however, the volume measure is not ergodic for the action of N. It turns out, in this case the interesting measure to consider is the so-called BR measure. This

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[1] Any invariant Borel subset is either null or co-null.

is an infinite measure when Γ is not a lattice, and is the volume measure when Γ is a lattice.

We now describe a construction of the BR measure. Let Γ be as above, and let δ denote the critical exponent of Γ. Fix $o \in \mathbb{H}^3$ stabilized by K, and denote by ν_o the Patterson-Sullivan measure on the boundary $\partial(\mathbb{H}^3)$ associated to o ([13], [19]), we will refer to it as the PS measure. Using the transitive action of K on $\partial(\mathbb{H}^3)$, we lift ν_o to a measure on K trivially, and continue to denote this extension by ν_o. Using the Iwasawa decomposition, $G = KAN$, we define a measure on G:

Burger-Roblin (BR) measure. Define the measure $\tilde{m}^{\mathrm{BR}}$ on G as follows: for $\psi \in C_c(G)$,

$$\tilde{m}^{\mathrm{BR}}(\psi) = \int_G \psi(k a_s n_z) e^{-\delta s} d\nu_o(k) ds\, dz,$$

where ds and dz are Lebesgue measures on $\mathbb{R}$ and $\mathbb{C}$ respectively. It is left Γ-equivariant and right N-invariant. The BR measure, m^{BR}, is a locally finite measure on X induced by $\tilde{m}^{\mathrm{BR}}$. It is an infinite measure except when $\delta = 2$, in which case it is the G-invariant measure on X. It is shown in [1], for surfaces, and in [17], in general, that the BR measure is the unique N-invariant ergodic measure on X, not supported on a closed N-orbit [2].

The question we are interested in is: whether the BR measure is ergodic for the action of U. The answer to this question turns out to depend on δ. The following is the main result in [11].

Theorem 1.1. *Let Γ be a Zariski dense convex cocompact subgroup. The action of U on X is ergodic with respect to m^{BR} if and only if $\delta > 1$.* [3]

The rest of this note is devoted to describing ideas involved in the proof of Theorem 1.1.

We close this section by mentioning that since in the interesting case, i.e. $\delta < 2$, the BR measure is an infinite measure, the usual approach, using the study of $L^2(X, m^{\mathrm{BR}})$, falls short in proving ergodicity. It is also worth mentioning that a priori it is not even clear that the action of U is conservative, that is: for any subset B of positive measure the $\{u_t\}$-orbit of almost every point in B spends infinite amount of time in B. Indeed one of the ingredients in the proof is to show that when $\delta > 1$ the flow is conservative and in the case $\delta \leq 1$ the flow does not have certain recurrence properties which are necessary for ergodicity.

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[2] The result in [17] is much more general.

[3] Understanding all U-invariant ergodic Radon measures seems to be a very interesting and difficult problem.

2. Properties of the BR measure.

The proof of Theorem 1.1 is based on a careful study of conditional measures of m^{BR} and its push-forward, $(a_{-s})_* m^{\mathrm{BR}}$, along the N-leaves. It turns out that the a lot of the "interesting" dynamical properties of the action of A, and that of N, on (X, m^{BR}) are in fact governed by a probability measure on X, the so called BMS measure. In this section we describe local structure of the BR and the BMS measure, and state an important result, due to Roblin, relating these two measures.

Retain the notation from the introduction. Further, let M denote the centralizer of A in K. Indeed the unit tangent bundle, $\mathrm{T}^1(\mathbb{H}^3)$, is identified with G/M. We let $\pi : \mathrm{T}^1\,\mathbb{H}^3 \to \mathbb{H}^3$ denote the natural projection.

Recall that a collection of nonzero finite Borel measures, $\{\mu_x : x \in \partial\mathbb{H}^3\}$, on $\partial\mathbb{H}^3$ is called Γ-*invariant conformal density of dimension* δ_ν if for any $x, y \in \mathbb{H}^3$, $\gamma \in \Gamma$, and $\xi \in \partial\mathbb{H}^3$ we have

$$\gamma_*\mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_\mu \beta_\xi(y,x)}.$$

Fix two Γ-invariant conformal densities μ and μ' of dimension δ_μ and $\delta_{\mu'}$. Following Roblin, we define a left Γ-equivariant right M-invariant measure $\tilde{m}^{\mu,\mu'}$ on G as follows: fix $o \in \mathbb{H}^3$ and identify $\mathrm{T}^1(\mathbb{H}^3)$ with

$$(\partial(\mathbb{H}^3) \times \partial(\mathbb{H}^3) - \{(\xi,\xi) : \xi \in \partial(\mathbb{H}^3)\}) \times \mathbb{R},$$

via the map $x \mapsto (x^+, x^-, \beta_{x^-}(o,x))$. Here $x^\pm \in \partial\mathbb{H}^3$ denote the end points of the corresponding geodesic, and for any $\xi \in \partial\mathbb{H}^3$, β_ξ is the Buseman function based at ξ. Define

$$\tilde{m}^{\mu,\mu'}(x) = e^{\delta_\mu \beta_{x^+}(o,\pi(x))}\; e^{\delta_{\mu'} \beta_{x^-}(o,\pi(x))}\; d\mu_o(x^+) d\mu'_o(x^-) dt,$$ [4]

Since μ and μ' are Γ-invariant, this defines a left Γ-equivariant measure on $\mathrm{T}^1\,\mathbb{H}^3$. We lift this to an M-invariant measure on G. Let $m^{\mu,\mu'}$ denote the measure induced by this measure on $X = \Gamma\backslash G$. This measure is quasi-invariant under the action of a_s. Indeed

$$a_s m^{\mu,\mu'} = e^{(\delta_\mu - \delta_{\mu'})s}\; m^{\mu,\mu'}. \tag{2}$$

With this notation, the Haar measure on X corresponds to $\mu = \mu' = m$, the (2-dimensional) G-invariant density on $\partial\mathbb{H}^3$. Let ν denote the unique (δ-dimensional) Γ-invariant geometric density on $\partial\mathbb{H}^3$ supported on the limit set, $\Lambda(\Gamma)$, of Γ. This will be referred to as the Paterson-Sullivan (PS) density, see [13] and [19]. With our choice of Γ, the PS measure coincides with the δ-dimensional Hausdorff measure of the limit set of Γ. Put

$$m^{\mathrm{BR}} = \mu^{m,\nu}, \ \text{ and } \ m^{\mathrm{BMS}} = \mu^{\nu,\nu},$$

We will refer to these measures as the BMS and the BR measures. The above definition coincides with the definition of the BR measure given in the introduction. It follows from (2) that m^{BR} is a finite measure if and

[4] See Hee Oh's article in this proceeding for further discussion of this definition.

only if $\delta = 2$, i.e. if and only if Γ is a lattice, ([21],[19]). Similarly we get that the BMS measure is A-invariant.

For later use, let us record here that $\operatorname{supp}(m^{\mathrm{BR}})$ is $\Gamma\backslash\{g \in G : g^- \in \Lambda(\Gamma)\}$, and $\operatorname{supp}(m^{\mathrm{BMS}})$ is $\Gamma\backslash\{g \in G : g^\pm \in \Lambda(\Gamma)\}$. In particular when Γ is convex cocompact the BMS measure has compact support, and is a finite measure[5].

The important properties of the BR and the BMS measure that will be used here are listed in the following theorem. Of particular interest to us is (3) in this Theorem, which is [17, Theorem 3.4]. This relates the BR measure, which is an infinite measure and $\{a_s\}$ quasi invariant, to the BMS measure which is a finite measure and $\{a_s\}$ invariant. Indeed we loose the N-invariance in this transition, but since the conditional measures of the BMS measure along the N-leaves are δ-dimensional Hausdorff measures, they still carries quite a lot of information.

Theorem 2.1 ([3], [17]). *Let Γ be as before and normalize so that m^{BMS} is a probability measure.*

(1) *The action of $\{a_s\}$ on X is mixing with respect to m^{BMS}, that is, for any $\psi_1, \psi_2 \in L^2(X, m^{\mathrm{BMS}})$, as $s \to \pm\infty$,*

$$\int_X \psi_1(ga_s)\psi_2(g)\, dm^{\mathrm{BMS}}(g) \to m^{\mathrm{BMS}}(\psi_1)m^{\mathrm{BMS}}(\psi_2).$$

(2) *m^{BR} is the unique N-ergodic measure on X which is not supported on a closed N-orbit.*

(3) *for any $\psi \in C_c(X)$ or for $\psi_i = \chi_{E_i}$ for bounded Borel subsets $E_i \subset X$ with $m^{\mathrm{BMS}}(\partial(E_i)) = 0$,*

$$\int_X \psi_1(ga_{-s})\psi_2(g)\, dm^{\mathrm{BR}}(g) \to m^{\mathrm{BMS}}(\psi_1)m^{\mathrm{BR}}(\psi_2) \ \text{ as } s \to +\infty.$$

In the sequel, specially when applying this theorem, we will need to take “nice” local neighborhoods. The following is the definition which will be used: A subset $E \subseteq X$ is called a BMS box if $E = x_0 N_\rho^- A_\rho N_\rho M$ where $x_0 \in \operatorname{supp}(m^{\mathrm{BMS}})$, $\rho > 0$ is at most the injectivity radius at x_0 and S_ρ means the ρ-neighborhood of e in S for any subset S of G.

3. Recurrence properties of the action of U and Theorem 1.1

In this section we describe the proof of Theorem 1.1, module the “Window Theorem” which will be stated and used as a black box. The proof of window theorem will be explained in the next section.

As was mentioned in the introduction for an infinite measure, L^2 considerations are not enough in proving ergodicity. Our proof is based on

[5]The BMS measure is a finite measure for any geometrically finite group, [20].

the polynomial divergence of unipotent flows. We investigate the "intermediate range" for two orbits of the unipotent flow and produce extra invariance [6]. The study of unipotent orbits in the intermediate range is not new, by any means, and has been used successfully in several prior works, e.g. [7, 8], [15, 16], [6] and [2].

Let us explain the proof in the case of a finite measure. Suppose μ is an N-invariant and ergodic probability measure on X. Further, let us assume that μ is an "interesting measure", e.g. it is not supported on a closed N orbit. Actually for the sake of simplicity here, we will assume that we can find x_n and $y_n = x_n g_n$ which are generic for the action of U, in sense of the Birkhoff ergodic theorem, such that

(g-1) $g_n \to e$,
(g-2) the $(1, 2)$ entry of g_n, which will be denoted by $(g_n)_{12}$, is non-zero,
(g-3) the real and the imaginary parts of $(g_n)_{12}$ have comparable sizes. [7]

Now for any $f \in C_c(X)$ we have

$$\frac{1}{T}\int_0^T f(\bullet u_t) \to \int_X f d\mu_\bullet, \text{ for } \bullet = x_n, y_n \tag{3}$$

where $\mu_\bullet$ denotes the corresponding ergodic components of μ.

We now compare the two orbits $x_n u_t$ and $y_n u_t$. Properties (g-2) and (g-3) imply that the divergence, in the transversal direction to U, is given by $u_{-t} g_n u_t$. The matrix entries of $u_{-t} g_n u_t$ are polynomials of degree at most 2, and the fastest divergence is along the $(2, 1)$-entry, which is a polynomial of degree 2. This polynomial is (i) non-constant, (ii) has size roughly 1, certainly nonzero, if $t \in [(1-\epsilon)T, T]$ for small ϵ and T of size $1/\sqrt{|(g_n)12|}$, and (iii) $u_{-T} g_n u_T \in N - U$, when T is of size $1/\sqrt{|(g_n)12|}$.

This special behavior of unipotent orbits in the intermediate range, and the fact that (3) *holds not only for* $[0, T]$ *but also for short intervals like* $[(1-\epsilon)T, T]$, imply that: if x_n and y_n are chosen from a "good" set in the sense that

(1) the convergence in (3) is uniform on this set, and
(2) $\{x_n\}, \{y_n\}$ are in a compact set where the map, $\bullet \mapsto \mu_\bullet$, is continuous on this set [8],

and if we suppose $x_n \to x$, then μ_x is invariant under an element in $N - U$. This and N-ergodicity of μ, using commutativity of N and standard arguments, will imply that μ is U-ergodic.

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[6]The fact that polynomial divergence can be used to prove ergodicity of unipotent flows is due to Margulis.

[7]This last condition is admittedly stating quite precise information about the structure of the "generic set" and in general is not easy to guarantee without having some information about the A-action on this measure space.

[8]Indeed by Egorov and Lusin's theorems, these hold on a set of "almost" full measure.

This argument is indeed an important step in Ratner's proof of measure classification theorem. Arguments of this kind, in topological context, are crucial in Margulis' proof of the Oppenheim conjecture.

The proof of Theorem 1.1 is along the lines of the above argument. However there are some rather serious difficulties in carrying out this idea, which we now discuss. The first difficulty is that in order for this argument to even start, one needs to have an ergodic theorem for the action of U. The suitable ergodic theorem is the Hopf ratio ergodic theorem. The form which we need, see [11, Theorem 7.4] and references there, is as follows: let μ be a U-invariant and conservative measure on X, and let $\psi \in L^1(X,\mu)$ be non-negative. Then for μ-a.e. $\{x : \int_o^T \psi(xu_t)dt \to \infty\}$, we have

$$\frac{\int_0^T f(xu_t)}{\int_0^T \psi(xu_t)dt} \to \frac{\mu_x(f)}{\mu_x(\psi)}, \quad \text{for any} \;\; f \in L^1(X,\mu).$$

Note that the Hopf ergodic theorem only kicks in if we have already established that the action of U is conservative, which, by Poincaré recurrence, is immediate in the case of a probability measure. We have the following, see [11, Theorem 6.6] and [11, Theorem 9.4].

Theorem 3.1. *Retain the above notation.*

1. *If $\delta > 1$, then the action of U is conservative for m^{BR}.*
2. *Suppose either that $0 < \delta < 1$ or that $\Lambda(\Gamma)$ is a purely unrectifiable 1-set. Then the action of U on X is not strongly recurrent for m^{BR}. In particular m^{BR} is not U-ergodic.*

Let us recall from [11, Definition 9.3] that a measure preserving flow $\{u_t\}$, on a σ-finite measure space (X,μ), is called *strongly recurrent* if for any two non-null measurable subsets B_1, B_2, we have $\{t : xu_t \in B_2\}$ is unbounded for μ-a.e. $x \in B_1$. It follows from the Hopf ratio ergodic theorem that any non-transitive ergodic action of U is strongly recurrent.

The proofs of the above statements are based on the general philosophy that the support of the BMS measure controls "a lot of" the dynamics of the action of U, and the action of A. The proofs also use some standard facts from geometric measure theory regarding dimension of projections, slices of certain measures, and some properties of *condensation points*.

Thanks to this theorem, we need to prove ergodicity in the case of $\delta > 1$. Thus we assume $\delta > 1$ for the rest of the argument. The algebraic part of the above argument goes through without change. That is: if we can find x_n, and $y_n = x_n g_n$, where g_n's satisfy properties g-1,2,3, then we can construct a non-trivial polynomial of degree 2 in the (2,1) matrix entry which governs the fastest divergence of the two orbits $x_n U$ and $y_n U$.

Let us continue to denote points by x_n, and $y_n = x_n g_n$. In order to get an element in $N - U$, which is the goal of the argument, it is essential to consider times in the intermediate range, i.e. when the the two orbits are roughly size one apart from each other. Indeed a simple matrix multiplication, considering $u_{-t}g_nu_t$, implies: this holds for $t \in [rT, T]$, where $0 < r < 1$ and T of order of $1/\sqrt{|(g_n)_{12}|}$.

In order to be able to use this algebraic fact, however, one needs to have some information regarding dynamics of this piece of the orbit. In the finite measure case it follows from the Birkhoff ergodic Theorem that for a typical point x, the piece of the orbit, $\{xu_t : t \in [rT, T]\}$, is equidistributed. In the case of an infinite measure, on the other hands, not only is this not free of charge, it seems to even be wrong in general [9]. This is the main technical difficulty in carrying out the above outline to the case in hand.

As we will see later our treatment of the main difficulty above makes it crucial that we are able to find g_n's satisfying g-1,2,3 in "all scales." The precise formulation is given in Proposition 3.2, see [11, Proposition 4.4].

Proposition 3.2. *Let $\delta > 1$. Fix some* BMS *box E and some $0 < r < 1$. There exist positive numbers $d_0 = d_0(r) > 1$ and $s_0 \gg 1$ such that for any Borel subset $F \subset E$ with $m^{\mathrm{BR}}(F) > r \cdot m^{\mathrm{BR}}(E)$ and any $s \geq s_0$, there exists a pair of elements $x_s, y_s \in F$ satisfying*

(1) $x_s = y_s n^-_{w_s}$ *for* $n^-_{w_s} \in N^-$,
(2) $\frac{1}{d_0 s} \leq |w_s| \leq \frac{d_0}{s}$ *and*
(3) $|\Im(w_s)| \geq \frac{|\Re(w_s)|}{d_0}$.

The above proposition gives a rather precise description of how two generic points approach each other along contracting leaves. In the case of a probability measure, conditions similar to this proposition, but weaker, can be proved using positive entropy. Our proof uses similar tools: we are able to guarantee this, because the conditional measures of BR along N^-, transpose of N, carry quite a lot of information. In particular, the conditional measure of a ball of radius r in N^-, centered at a point in the support of this conditional, is of order r^δ. This fact in view of $\delta > 1$, and a covering argument implies the proposition.

As we mentioned above the major difficulty in the proof is to get equidistribution of the orbits in the intermediate range of time. The following is a partial resolution of this problem and is the main technical result used in the proof of Theorem 1.1, see [11, Theorem 1.4].

Theorem 3.3 (Window Theorem)**.** *Retain the above notation and assumptions, in particular $\delta > 1$. Let $E \subset X$ be a* BMS *box, and suppose $\psi \in C_c(X)$ be a non-negative function with $\psi|_E > 0$. Then there exist $0 < r < 1$ and $T_0 > 1$ such that for any $T \geq T_0$,*

$$(4) \quad m^{\mathrm{BR}}\{x \in E : \int_{-rT}^{rT} \psi(xu_t)dt \leq (1-r)\int_{-T}^{T} \psi(xu_t)dt\} > \tfrac{r}{2} \cdot m^{\mathrm{BR}}(E).$$

[9] The author however does not know of an example.

This theorem is used to control the dynamics when two orbits have moderately diverged. Let us however mention two difficulties one faces in applying this theorem. First (and more importantly) is that the set in (4), where one has a "doubling" property of return times for the action of U, depends on the time parameter T. Hence in order to be able to apply this theorem successfully, one needs to be able to find "good points", as above, which are close to each other in all scales. This is why Proposition 3.2 is essential to our analysis. In working with an unknown measure a statement of this form will be difficult to utilize.

Secondly, as in the case of probability measure, we really need equidistribution of $[(1-\epsilon)T, T]$, for small ϵ. In order to achieve this; for any fixed n, we use the window theorem above and a simple covering argument to find a subinterval, I say, with length ϵT, where we have the equidistribution for x_n. We then use the fact that the two pieces of orbits $\{\bullet u_t : t \in I\}$, for $\bullet = x_n, y_n$, stay $O(1)$ of each other for $t \in [0, T]$, to show that $\{y_n u_t : t \in I\}$ is also equidistributed.

4. Proof of the Window Theorem

In this section we describe ingredients involved in the proof of Theorem 3.3. The proof has elements similar to the low entropy method introduced in [6].

Roughly speaking the idea is the following: The BMS measure is an A-invariant probability measure, and the leafwise measures of BMS along N are δ-dimensional Hausdorff measure. This, and the M-invariance of the BMS measure, imply: when $\delta > 1$ the entropy of a_s along U is nontrivial. Then, facts relating entropy and leafwise measures, see [6, 2], imply that the leafwise measures of BMS measure along U restricted to $[-1, 1]$ satisfy a doubling property on a set of "almost" full measure, see [6, 2] this is restated in [11, Theorem 6.11]. Now by (3) in Theorem 2.1 we have, after renormalization, $a_{-s}m^{\mathrm{BR}}|_E$ weakly converges to the BMS measure, as $s \to \infty$. Here E is a BMS box fixed once and for all. The goal now is to use these to prove such doubling properties for the return times of the action of U with respect to the BR measure.

It is worth mentioning, however, that usually such information cannot be extracted, e.g. due to discontinuity of the entropy with respect to weakstar topology. We succeed, essentially, for two reasons:

- (i) The BR and BMS measures have the same transversal measures, i.e. locally, up to normalization, they only differ along N-leaves. This follows from the definition of these two measure.
- (ii) The measures in question, i.e. m^{BMS} and $a_{-s}m^{\mathrm{BR}}|_E$, are quite regular, see Propositions 4.1, 4.2 below.

Let us now fix some notation to be used in the course of the proof. For each $s > 0$, define a Borel measure $\mu^{\mathrm{BR}}_{E,s}$ on X to be the normalization of the

$a_{-s}m^{\rm BR}|_E$: for $\psi \in C_c(X)$,

$$\mu^{\rm BR}_{E,s}(\psi) := \frac{1}{m^{\rm BR}(E)}\int_E \psi(ga_{-s})\, dm^{\rm BR}(g).$$

It follows from (3) in Theorem 2.1 that $\mu_{E,s} \to m^{\rm BMS}$ in the weak star topology. We have the following, see [11, Theorem 3.3].

Proposition 4.1. *Suppose that $x^- \in \Lambda(\Gamma)$. For all small enough $\rho > 0$, let $\lambda_{E,x,s}$ denote the conditional measure of $\mu_{E,s}$ along xN_ρ. We have,*

$$\lim_{s\to\infty} \lambda_{E,x,s}(\psi) = \mu^{\rm PS}_x(\psi), \quad \textit{for any } \ \psi \in C_c(xN_\rho),$$

The proof of the above proposition draws from the convergence of $\mu_{E,s}$ to $m^{\rm BMS}$, and a rather special feature of $\mu_{E,s}$: the conditionals of $\mu_{E,s}$ along N-leaves are obtained from a measure on the boundary, and in particular they change regularly as we move in the transversal direction.

We will now fix a some x with $x^\pm \in \Lambda(\Gamma)$, and some small $\rho > 0$, and investigate $\lambda_{E,x,s}$. Given some $\theta \in [0, 2\pi]$ we let $m_\theta \in M$ denote the corresponding element. Also we will put

$$U^\rho_\theta := \{t \exp(2\pi i\theta) : t \in [-\rho, \rho]\} \text{ and } \ V^\rho_\theta := \{it \exp(2\pi\theta) : t \in [-\rho, \rho]\}.$$

For any $0 < \tau \le \rho$, and $\theta \in [0, 2\pi]$ we let $\sigma^\tau_{x,\theta,s}$, (resp. $\sigma^\tau_{x,\theta}$) denote the projection to V^τ_θ of $\lambda_{E,x,s}|_{xU^\tau_\theta V^\tau_\theta}$, (resp. $\mu^{\rm PS}_x|_{xU^\tau_\theta V^\tau_\theta}$). Furthermore, for any measure $\bullet$ on $\mathbb{R}$, we let $D(\bullet)$ denote the Radon-Nikodym derivative of $\bullet$ with respect to the Lebesgue measure.

We unfortunately cannot quite show statement as strong as Proposition 4.1 for the disintegration of $\lambda_{E,x,s}$ along U-directions. The following is our replacement, which is [11, Proposition 5.10].

Proposition 4.2. *Let $s_i \to +\infty$ be a fixed sequence. For every $\epsilon > 0$ and every finite subset $\{\tau_1, \dots, \tau_n\}$ of $(0, \rho]$, there exists a Borel subset $\Theta(x) \subset M$ of measure at least $1 - \epsilon$ such that for any $\theta \in \Theta(x)$ we have*

(i) *for all $1 \le \ell \le n$, the measure $\sigma^{\tau_\ell}_{x,\theta}$ is absolutely continuous with respect to the Lebesgue measure on $xV^{\tau_\ell}_\theta$. Furthermore*

$$D(\sigma^{\tau_\ell}_{x,\theta}) \in H^r(xV^{\tau_\ell}_\theta), \quad \textit{for } \ r = \tfrac{\delta-1}{4},$$

(ii) *there is a subsequence $\{s_{i_j}\}$, depending on (x, θ), such that*

$$D(\sigma^{\tau_\ell}_{x,\theta,s_{i_j}}) \xrightarrow{L^2(xV_\theta)} D(\sigma^{\tau_\ell}_{x,\theta}), \ \textit{ for each } \ 1 \le \ell \le n.$$

The proof of this proposition uses some techniques from geometric measure theory. Indeed an essential ingredient in the proof is a uniform bound for the α-dimensional energy of $\lambda_{E,x,s}$'s, for some $1 < \alpha$, see [11, Theorem 5.7]. This gives a uniform control on fractional derivatives of the Radon-Nikodym derivative of the projection of these measures with respect to the Lebesgue measure, [14, Proposition 2.2] or [9, Theorem 4.5]. The

L^2-convergence in the above then follows from compact embedding theorem of Sobolev spaces.

The desired energy estimate is proved using: (i) the non-focusing property of the PS measure, which was also used in the proof of Proposition 3.2, (ii) the fact that support of μ_s is contained in $O(e^{-s})$-thickening of the support of BMS, see [11, Lemma 5.6], and (iii) some covering argument.

We can now complete the proof of the Window Theorem. As we mentioned above, for the BMS measure the contribution of U to the entropy of a_s is non-trivial. Thus there is a set, Ω', of almost full BMS-measure and some $\beta > 0$, such that

$$(m^{\rm BMS})^U_x[-\beta,\beta] < \tfrac{1}{2}(m^{\rm BMS})^U_x[-1,1], \quad \text{for all } x \in \Omega', \tag{5}$$

where $(m^{\rm BMS})^U_x$ denotes the U-leafwise measure. Now suppose the Window Theorem fails, then there is a sequence $r_i \to 0$ and a sequence $T_i \to \infty$ such that

$$m^{\rm BR}\{x \in E : \int_{-r_iT_i}^{r_iT_i} \psi(xu_t)dt \geq (1-r_i)\int_{-T_i}^{T_i}\psi(xu_t)dt\} > (1-r_i)m^{\rm BR}(E).$$

Let $s_i = \log T_i$ and denote the set on the left side of the above by $E(s_i, r_i)$. If we flow this by a_{-s_i}, and use the definition of μ_{E,s_i}, we get

$$(\mu_{E,s_i})^U_x([-r_i, r_i]) > 1 - r_i,$$

for all x in a subset $E_{s_i}(r_i)$ of E, with $\mu_{E,x,s_i}(E_{s_i}(r_i)) > 1 - r_i$.

Now by Fubini's theorem there is a subset in the transversal direction to N of measure $1 - r_i$, such that if the transversal component of x is in this set, then $\lambda_{E,x,s_i}(xN \cap E_{s_i}(r_i)) > 1 - \sqrt{r_i}$. This set, in the transversal direction, however depends on i. To find a set which works for all i, we pass to a subsequence and assume that $\sum \sqrt{r_i} < \epsilon$, and replace these sets by their intersection. Altogether: there is some x so that, simultaneously for all i, we have $\lambda_{E,x,s_i}(xN \cap E_{s_i}(r_i)) > 1 - \sqrt{r_i}$.

Indeed this condition is essentially to say that the disintegration of $\lambda_i = \lambda_{E,x,s_i}$, as a measure on N, along the direction of U is like a dirac mass. Note also that since $\mu_{E,s}$ is M-invariant the above could be done so that the same holds not only for x, but also for xm_θ for "most" θ's. Hence we have the slices of λ_i along U_θ is almost a dirac measure for many directions θ.

It is more convenient for us, however, to work with the projections. Fix some small $\tau \ll \beta$. We take projection of the set $\mathrm{Bad}_{\bullet,i} = xN_\bullet \cap E_{s_i}(r_i)$, for $\bullet = 1, \tau$, onto xV_θ. We have $\sigma_i^1(\mathrm{Bad}_{\tau,i}) > 1 - 2r_i\sigma_i^1(\mathrm{Bad}_{1,i})$, where σ denotes the projection of λ_i onto xV_θ. One needs to be cautions, however, since these sets change as i changes. We need to show that $\mathrm{Bad}_{\bullet,\mathrm{i}}$ has almost full PS measure in $xN_\bullet$. If this is established then (5) would give a contradiction. This thankfully follows from the L^2-convergence statement in Proposition 4.2, see [11, Lemma 5.11]. This completes the proof.

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