HOMEWORK 3

DUE 30 SEPTEMBER 2008

1. **The number of nontrivial zeros** Define $N(T)$ to be the number of zeros of $\zeta(s)$ in the rectangle $0 < \Re(s) < 1$, $0 < \Im(s) < T$. This denotes also the zeros of $\xi(s)$ in the same region, where

$$\xi(s) = (s-1)\pi^{-s/2}\Gamma\left(1+\frac{s}{2}\right)\zeta(s).$$

(a) From the argument principle we know that if T is not the imaginary part of some zero, then

$$2\pi N(T) = \Delta_R \arg \xi(s) \text{ the change in the argument of } \xi(s) \text{ along } R,$$

where R is the rectangle with vertices 2, $2+iT$, $-1+iT$, -1 in the counter-clockwise direction. Show that actually

$$\pi N(T) = \Delta_L \arg \xi(s),$$

where L is the path that consists of the line from 2 to $2+iT$ and the line from $2+iT$ to $\frac{1}{2}+iT$.

(b) Estimating the change in the argument of the factors of $\xi(s)$ along L, show that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}),$$

where $\pi S(T) = \Delta_L \arg \zeta(s) = .$

(c) Show that for large T we have

$$\sum_{\substack{\rho=\beta+i\gamma \\ \text{nontrivial zero}}} \frac{1}{1+(T-\gamma)^2} = O(\log T).$$

(d) Deduce that the number of nontrivial zeros ρ with $|\Im(\rho) - T| < 1$ is $O(\log T)$ and that

$$\sum_{\substack{\rho=\beta+i\gamma, |T-\gamma|>1 \\ \text{nontrivial zero}}} \frac{1}{(T-\gamma)^2} = O(\log T).$$

(e) Show that for large t not coinciding with the imaginary part of a zero and $-1 \leq \sigma \leq 2$

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = \sum_{\substack{\rho \text{ nontrivial zero} \\ |\Im(\rho)-t|<1}} \frac{1}{s-\rho} + O(\log t).$$

Deduce that $S(T) = O(\log T)$.

2. Show that $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll \log(2|s|)$.

3. Denote $\delta(y) = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$

We have seen that $\delta(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds$ for any $c > 0$. Now set $I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s} ds$.

Show that, for $y > 0$, $c > 0$, $T > 0$, we have the following estimate

$$|I(y,T) - \delta(y)| < \begin{cases} y^c \min\left(1, \frac{1}{T|\log y|}\right) & \text{if } y \neq 1, \\ \frac{c}{T} & \text{if } y = 1. \end{cases}$$

4. (a) Deduce that, with the convention $\Lambda(x) = 0$ for non-integer x,

$$\left|\psi(x) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s} ds\right| < \sum_{n \neq x} \Lambda(n) \left(\frac{x}{n}\right)^c \min\left(1, \frac{1}{T|\log(x/n)|}\right) + \frac{c}{T}\Lambda(x).$$

(b) Take $c = 1 + \frac{1}{\log x}$ in the above inequality. Show that the contribution on the right hand side of the terms corresponding to $|n - x| \leq \frac{x}{4}$ adds up to

$$\ll \frac{x}{T}\left(-\frac{\zeta'(c)}{\zeta(c)}\right) \ll \frac{x \log x}{T}.$$

(c) Under the same conditions show that the contribution of the terms with $\frac{3}{4}x < n < x$ except the closest prime power to x adds up to $\ll \frac{x \log^2 x}{T}$, while the contribution of the closest prime power x_1 is $\ll \Lambda(x_1) \min\left(1, \frac{x}{T(x-x_1)}\right) \ll \log x \min\left(1, \frac{x}{T(x-x_1)}\right)$.

(d) Write $< x >$ for the distance for x to the nearest prime power. Show that

$$\left|\psi(x) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s} ds\right| \ll \frac{x \log^2 x}{T} + \log x \min\left(1, \frac{x}{T < x >}\right).$$

(e) By using the Residue Theorem from complex analysis for the function

$$-\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s}$$

along the rectangle $\gamma_{c,T,k}$ with vertices $c - iT$, $c + iT$ $-(2k+1) + iT$ and $-(2k+1) - iT$ (here k is some positive integer) and then letting $k \to \infty$ show that

$$\psi(x) = x - \sum_{\substack{\rho \text{ nontrivial zero} \\ |\Im(\rho)| < T}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}) + R(x,T),$$

where

$$|R(x,T)| \ll \frac{x\log^2(xT)}{T} + (\log x)\min\left(1, \frac{x}{T < x >}\right).$$

Hint: You'll probably need to use the estimates in 1(e) and 2.

5. (a) Using the zero-free region for $\zeta(s)$ we proved in class show that there exists a constant C for which

$$\sum_{\substack{\rho \text{ nontrivial zero} \\ |\Im(\rho)|<T}} \left|\frac{x^\rho}{\rho}\right| < x(\log T)^2 e^{-C\log x/\log T}.$$

(b) Take T such that $(\log T)^2 = \log x$, with x some integer, in the estimates in 4(e) and 5(a) and show that there exists a constant C such that

$$|\psi(x) - x| \ll xe^{-C\sqrt{\log x}}.$$

(Recall that this proves the Prime Number Theorem!)