HOMEWORK 2

DUE 16 SEPTEMBER 2008

1. **Poisson Summation:** Show that for a function $f(x)$ defined on $\mathbb{R}$, such that $f(x) = O(|x|^{-c})$ for some $c > 1$, then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$, where the Fourier coefficients of f are given by

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx.$$

Deduce that $\theta\left(\frac{1}{t}\right) = \sqrt{t}\,\theta(t)$, where $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$.

2. If χ is a primitive character modulo N and $\tau(\chi) = \sum_{n \bmod N} \chi(n)e^{2\pi in/N}$ is the Gauss sum associated to it, show that

   (a) $\sum_{n \bmod N} \chi(n)e^{2\pi inm/N} = \overline{\chi(m)}\tau(\chi)$;

   (b) $|\tau(\chi)| = \sqrt{N}$;

   (c) $\tau(\overline{\chi}) = \chi(-1)\tau(\chi)$;

   (d) $\chi(n) = \dfrac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)}e^{2\pi inm/N}$.

3. What is the connection between $\tau(\chi)$ and $G(n)$, where χ denotes the Legendre symbol modulo a prime number $q > 2$ and $G(n)$ is the Gauss sum defined in Davenport, page 7? What is $|G(n)|$?

4. **Twisted Poisson Summation:** Under the same conditions, prove that

$$\sum_{n\in\mathbb{Z}} \chi(n)f(n) = \frac{\tau(\chi)}{N} \sum_{n\in\mathbb{Z}} \overline{\chi(n)}\,\hat{f}\left(\frac{n}{N}\right),$$

where χ is a primitive Dirichlet character of modulus N.

5. Let χ be a primitive character modulo N.

   (a) If $\chi(-1) = 1$, define $\theta_\chi(t) = \sum_{n\in\mathbb{Z}} \chi(n)e^{-\pi n^2 t}$. Prove that

$$\theta_\chi(t) = \frac{\tau(\chi)}{N\sqrt{t}}\,\theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right).$$

(b) If $\chi(-1) = -1$, define $\theta_\chi(t) = \sum_{n\in\mathbb{Z}} n\chi(n)e^{-\pi n^2 t}$. Prove that

$$\theta_\chi(t) = -\frac{i\tau(\chi)}{N^2 t^{3/2}}\,\theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right).$$

6. Choose $a = 0, 1$ such that the primitive character χ modulo N satisfies $\chi(-1) = (-1)^a$. Show that the function

$$\Lambda(s,\chi) = \pi^{-\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi).$$

satisfies the functional equation

$$\Lambda(s,\chi) = (-i)^a\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi}),$$

and that, for $\chi \neq 1$, the function $\Lambda(s,\chi)$ has analytic continuation to the whole $\mathbb{C}$, while, for $\chi = 1$, it has analytic continuation to all s except for simple poles at $s = 0$ and $s = 1$.

7. Let k be a number field and $\mathcal{O}$ its ring of integers. We know that the ideals of $\mathcal{O}$ have unique factorization into prime ideals. Show that the following series is absolutely convergent for $\Re(s) > 1$, the convergence being uniform on compact subsets, and that it has the Euler product decomposition

$$\sum_{I \text{ ideal in } \mathcal{O}} \mathbb{N}I^{-s} = \prod_{\mathcal{P} \text{ prime ideal in } \mathcal{O}} (1-\mathbb{N}\mathcal{P}^{-s})^{-1}.$$

Here $\mathbb{N}I$ denotes the norm of the ideal I, namely the number of elements in the ring $\mathcal{O}/I$.

8. Let $\mathbb{F}_q$ be the finite field with q elements. Define the zeta function

$$Z(t) = (1-t)^{-1}\prod_p (1-t^{\deg p})^{-1},$$

where p ranges over all monic irreducible polynomials $p = p(X)$ in $\mathbb{F}_q[X]$. Prove that $Z(t)$ is a rational function and determine this rational function. What are its zeros? What are its poles? Write down the functional equation that $Z(t)$ satisfies as $t \to \frac{1}{qt}$.

9. Show that if $\omega \in \mathbb{C}\setminus\mathbb{R}$ with $L(1,\chi_\omega) = 1$, then $\lim_{s\to 1+} L(s,\chi_\omega)L(s,\chi_{\overline{\omega}})L(s,\chi_1) = 0$.

10. Prove that $-\log(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}$ for $|z| \leq 1$, $z \neq 1$, and log the principal branch of the logarithm.