HOMEWORK 1

DUE 9 SEPTEMBER 2008

This problem set concerns the Γ-function and its properties. It is inspired by a homework given by Noah Snyder at Harvard in 2002 to a bunch of high-school students.

Define $\Gamma(x) = \displaystyle\int_0^\infty e^{-t}\, t^x\, \frac{dt}{t}$.

1. Show that this definition makes sense for all real $x > 0$. Furthermore, show by integration by parts that $\Gamma(x+1) = x\Gamma(x)$. Compute $\Gamma(n)$ for a positive integer n.

2. Show that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{t^y}{(1+t)^{x+y}} \frac{dt}{t}.$$

Conclude that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and prove the "duplication formula"

$$\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right).$$

3. Show that , for $0 < a < 1$,

$$\int_0^\infty \frac{t^a}{1+t} \frac{dt}{t} = \frac{\pi}{\sin a\pi}\,.$$

Hint: Consider $\displaystyle\int \frac{z^{a-1}}{1-z} dz$, taken over a well-chosen path. For instance, it could consists of two circles of radii R and ρ respectively, joined along the negative real axis from $-R$ to $-\rho$.

4. Combine your earlier results to show that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad \text{for } 0 < x < 1.$$

5. Show that the definition of Γ makes sense for all complex numbers with positive real part. Show that Γ has meromorphic continuation to the whole complex plane. Show that all the formulas you have proved for Γ for real values must be true for all complex values. Find the poles of $\Gamma(z)$, their orders, and compute the residues at those points. Show that $\frac{1}{\Gamma(z)}$ has no poles. Conclude that Γ has no zeros.

6. Prove the product expansions

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

and

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where γ is Euler's constant given by $\gamma = \lim_n 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$.

7. Stirling's Asymptotic Formula

Show that for $|z| \to \infty$ and $-\pi + \delta < \arg z < \pi - \delta$, where δ is any fixed positive number, we have

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + O\left(|z|^{-1}\right)$$

and

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O\left(|z|^{-1}\right).$$

Here $\log z$ denotes the principal branch of the logarithm.