Chapter 3 & 4 Review
Solutions

① a)

$$S = \{ f(x) : f(2) = 0 \}$$
$$= \{ a + bx + cx^2 : a + 2b + 4c = 0 \}$$
$$= \{ -2b - 4c + bx + cx^2 \}$$
$$= \{ b(-2+x) + c(-4+x^2) \}$$
$$= \text{span} \{ -2+x, -4+x^2 \}$$

$\therefore$ S is a span, so it is a subspace.

b) By the above, $\mathcal{B} : -2+x, -4+x^2$ spans S. To show it is a basis, we note that $-2+x$ and $-4+x^2$ are linearly independent (since one is not a multiple of the other).

② The change of basis matrix

$$S_{B_1 \to std} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

has inverse $S_{std \to B_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

So $[T]_{std} = S_{B_1 \to std} [T]_{B_1} S_{std \to B_1}$

$$= \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix}$$

<u>Check:</u> $\begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $\begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(T\vec{v}_1 = \vec{v}_1)$ $\qquad$ $(T\vec{v}_2 = 2\vec{v}_1 - \vec{v}_2)$

good ✓

(3) $T: \mathbb{R}^5 \to \mathbb{R}^4$ $\quad A = \begin{bmatrix} 1 & 4 & 2 & -5 & 1 \\ -1 & -3 & -1 & 1 & 0 \\ 4 & 13 & 5 & -8 & 1 \\ 3 & 7 & 1 & 5 & -1 \end{bmatrix}$

row reduce $\longrightarrow$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 11 & 0 \\ 0 & 1 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} L & L & F & F & L \end{matrix} \end{array}$$

$\operatorname{Im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 13 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ (leading columns)

and these three vectors automatically form a basis

$$\begin{aligned} \dim(\ker(A)) &= \dim(\mathbb{R}^5) - \dim(\operatorname{im}(A)) \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

(alternatively, $\dim(\ker(A))$ = # free variables = 2)

(4) $T: P_2 \to P_2$ $\quad T(f) = f + f'$

$$\begin{aligned} T(a + bt + ct^2) &= a + bt + ct^2 + b + 2ct \\ &= \underset{a}{(a+b)} + \underset{b}{(b+2c)}t + \underset{c}{c}t^2 \end{aligned}$$

$\mathcal{B}: 1, t, t^2$ $\quad [T]_{\mathcal{B}} = \begin{array}{c} 1 \\ t \\ t^2 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \left(\text{since } T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ b+2c \\ c \end{pmatrix} \right)$

Find inverse by row reducing $\longrightarrow$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$

So $[T^{-1}]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

and $T^{-1}: P_2 \to P_2$ given by
$$\begin{aligned} T(a + bt + ct^2) &= a - b + 2c \\ &\quad + (b - 2c)t + ct^2 \\ &= f - f' + f'' \end{aligned}$$

cool!

(5) $\mathcal{B}: \vec{v}_1 = \begin{pmatrix}2\\1\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}-1\\3\end{pmatrix}$

a) There are 2 vectors, so it suffices to check independence

Method 1: $\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \Rightarrow$ invertible

Method 2: Note that $a\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$ has no solution in a.

* To find coordinates of $\begin{pmatrix}3\\5\end{pmatrix}$ with respect to $\mathcal{B}$, solve

$$\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\vec{x} = \begin{pmatrix}3\\5\end{pmatrix}$$

i.e. $$\left(\begin{array}{cc|c}2 & -1 & 3\\1 & 3 & 5\end{array}\right) \xrightarrow[\text{reduce}]{\text{row}} \left(\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & 1\end{array}\right)$$

So $$\left[\begin{pmatrix}3\\5\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}2\\1\end{pmatrix}$$

b) $S_{\mathcal{B}\to std} = \begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}$ $\qquad S_{std\to\mathcal{B}} = S_{\mathcal{B}\to std}^{-1} = \frac{1}{7}\begin{pmatrix}3 & 1\\-1 & 2\end{pmatrix}$

So
$$\begin{aligned}[T]_{std} &= S_{\mathcal{B}\to std}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}S_{std\to\mathcal{B}}\\
&= \frac{1}{7}\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}3 & 1\\-1 & 2\end{pmatrix}\\
&= \frac{1}{7}\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}3 & 1\\1 & -2\end{pmatrix}\\
&= \frac{1}{7}\begin{pmatrix}5 & 4\\6 & -5\end{pmatrix} = \begin{pmatrix}5/7 & 4/7\\6/7 & -5/7\end{pmatrix}\end{aligned}$$

<u>Check</u>: $\frac{1}{7}\begin{pmatrix}5 & 4\\6 & -5\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{7}\begin{pmatrix}14\\7\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ $\quad (T(\vec{v}_1) = \vec{v}_1)$ ✓

$\frac{1}{7}\begin{pmatrix}5 & 4\\6 & -5\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix} = \frac{1}{7}\begin{pmatrix}7\\-21\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$ $\quad (T(\vec{v}_2) = -\vec{v}_2)$

(6) $$A = \begin{bmatrix} -1 & -2 & 0 & 1 \\ 2 & 4 & -1 & 3 \\ -3 & -6 & 2 & -6 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} L & F & L & L \end{matrix} \end{array}$$

$$\mathrm{Im}(A) = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \right\}$$

And these three vectors form a basis (3 leading columns)

$$\mathrm{Ker}(A) = \left\{ \begin{pmatrix} -2t \\ t \\ 0 \\ 0 \end{pmatrix} \right\} = \mathrm{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and this one vector forms a basis (1 free column)

(7) $T: P_2 \to \mathbb{R}^3$

$$T(a+bt+ct^2) = \begin{pmatrix} a-2b+4c \\ a-b+c \\ a+2b+4c \end{pmatrix}$$

To find $\ker(T)$, solve

$$\begin{aligned} a-2b+4c &= 0 \\ a-b+c &= 0 \\ a+2b+4c &= 0 \end{aligned}$$

i.e. $A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$; $A\vec{x} = \vec{0}$

But A has rank 3 (by row reduction)

So $\ker(T) = \{0\}$.

$\dim(\mathbb{R}^3) = \dim(P_2) = 3$ and $\ker(T) = \{0\}$

So T is invertible.

To find $f \in P_2$ s.t. $T(f) = \begin{pmatrix} -4 \\ 3 \\ 12 \end{pmatrix}$, solve

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -4 \\ 1 & -1 & 1 & 3 \\ 1 & 2 & 4 & 12 \end{array}\right)$$

$a = 8,\ b = 4,\ c = -1$

So $f = 8 + 4t - t^2$.

(8) a) If P is the matrix for a projection onto W, then $\mathrm{Im}(P) = W$. So the question asks us for a basis for $\mathrm{Im}(P)$.

$$P \xrightarrow[\text{reduce}]{\text{row}} \begin{array}{c} \begin{bmatrix} 1 & -2/3 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} L & F & L & F \end{array} \end{array}$$

$W = \mathrm{Im}(P)$ has basis 1^{st} & 3^{rd} columns, or

$$\begin{pmatrix} 18 \\ -12 \\ -3 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 14 \\ -13 \end{pmatrix}$$

(9) a) This was <u>supposed</u> to read

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 - x_3 \\ -x_2 \\ x_1 + 7x_3 \end{pmatrix} \qquad \text{(sorry!)}$$

T has matrix $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix}$

$$S_{B\to std} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad S_{std\to B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

since

1^{st} col: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2^{nd} col: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

3^{rd} col: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Check:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

So $[T]_{\mathcal{B}} = S_{std\to\mathcal{B}}\,[T]_{std}\,S_{\mathcal{B}\to std}$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{pmatrix}$$

Check: (partial check)

$$\overbrace{\begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}^{\mathcal{B}\text{ basis}} = v_1 - v_2 + v_3$$

$$\Vert$$

$$T(\vec{v}_1) = \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = v_1 - v_2 + v_3}_{\text{standard basis}} \quad \checkmark$$

b) Here M is a change of basis matrix from $\mathcal{B}: \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ to the standard basis

And the last four equations tell us that $[T]_{\mathcal{B}}$ is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 0 & -1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

But $M^{-1}AM = S_{std\to\mathcal{B}}\,[T]_{std}\,S_{\mathcal{B}\to std} = [T]_{\mathcal{B}}$

So $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 0 & -1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$

This is a <u>concept-testing</u> "trick" question. You don't need to use the matrix A.

$$B = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -6 \\ 0 & 2 \end{bmatrix}$$

(10)

2^{nd} column:

$$T(E_2) = \frac{1}{6}\begin{bmatrix} 3 & -6 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 3 & -6 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3/2 \\ 0 & 0 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{3}{2}E_2$$

So the 2^{nd} column is $\begin{pmatrix} 0 \\ 3/2 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \\ \leftarrow E_2 \\ \\ \\ \end{matrix}$

3^{rd} column:

$$T(E_3) = \frac{1}{6}\begin{bmatrix} 3 & -6 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 3 & -6 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 6 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} -12 & -36 \\ 4 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -6 \\ 2/3 & 2 \end{bmatrix} = -2E_1 - 6E_2 + \tfrac{2}{3}E_3 + 2E_4$$

So the 3^{rd} column is $\begin{pmatrix} -2 \\ -6 \\ 2/3 \\ 2 \end{pmatrix}$.