Review

First and foremost, *review the concepts covered so far and your homework sets!* What follows is a list of suggested problems. The exam problems might look very different, but they use the same notions and techniques. Working on the problems in this list will help you deal with whatever may come up on the test.

Good luck! ☺

From textbook:

- section 1.1: 24, 27, 30–36, 46
- section 1.2: 44, 45, 61, 63, 67
- section 1.3: 46, 54, 58
- true/false at the end of chapter 1: 39, 40, 43
- section 2.1: 43, 49
- section 2.2: 38, 45
- section 2.3: 46, 51, 52, 53
- section 2.4: 32, 35
- true/false at the end of chapter 2: 4–10, 21, 36–41, 50, 52
- section 3.1: 5, 6, 21–33, 39, 46–50.

The following problems are inspired by tests from previous years.

1. Consider the system of linear equations

$$\begin{aligned} x + y - z &= a \\ 3x - y/2 &= b \\ 7x/2 - 3z &= c \end{aligned}$$

   (a) Find all the solutions of this system when $a = b = c = 0$.

   (b) Find the values for a, b and c for which this system has no solutions (there may be no such values).

2. (a) Show that a homogeneous system of equations with all non-zero coefficients with two equations and three variables always has a non-zero solution.

   (b) Use part (a) to show that three vectors in $\mathbb{R}^2$, all of whose coordinates are non-zero, are always linearly dependent.

3. Find the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that sends $(0, 0, 1)$ to $(4, 7, 2)$, $(0, 4, 5)$ to $(-4, -5, 2)$, and $(1, 3, 4)$ to $(2, 6, 8)$.

4. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 6 \\ 0 & 2 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 2 & 0 & 2 \end{bmatrix}.$$

   Find the kernel and the image of A.

5. Consider the system of linear equations:

$$\begin{aligned} x_1 - x_3 + x_4 &= 2 \\ 3x_1 + 4x_2 - x_4 &= 10 \\ 5x_1 + 4x_2 - 2x_3 + x_4 &= 14 \\ 9x_1 + 8x_2 - 3x_3 + x_4 &= 26 \end{aligned}$$

(a) Find the reduced row-echelon form of the augmented matrix associated to this linear system and use it to solve the system.

(b) Let A be the coefficient matrix of the system above. Determine $\ker(A)$.

6. Find the image and the kernel of the transformation $T : \mathbb{R}^5 \to \mathbb{R}^4$ given by $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{bmatrix} 1 & 4 & 2 & -5 & 1 \\ -1 & -3 & -1 & 1 & 0 \\ 4 & 13 & 5 & -8 & 1 \\ 3 & 7 & 1 & 5 & -1 \end{bmatrix}.$$

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection of $\mathbb{R}^3$ onto the subspace spanned by $\vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Write down a matrix representing T.

8. Let

$$A = \begin{bmatrix} -1 & -2 & 0 & 1 \\ 2 & 4 & -1 & 3 \\ -3 & -6 & 2 & -6 \end{bmatrix}.$$

Find the kernel and the image of A.

9. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$

True or false?

(a) A is invertible.

(b) A has rank 2.

10. Let

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Find the rank of B.

11. Find the general solution to the following system of equations:

$$\begin{aligned} 3x_1 + 8x_2 - 5x_3 + 9x_4 - 4x_5 &= -12 \\ 3x_1 + x_2 + 2x_3 - 4x_4 + 2x_5 &= 1 \\ 2x_1 + 7x_2 - 5x_3 + 10x_4 - 5x_5 &= -12 \end{aligned}$$

Hint: fractions can be avoided. The solution involves only small integers.

12. (a) Let V be a subspace of $\mathbb{R}^3$ spanned by

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the projection matrix P onto V.

(b) Let

$$P = \frac{1}{27}\begin{bmatrix} 18 & -12 & -3 & -3 \\ -12 & 8 & 2 & 2 \\ -3 & 2 & 14 & -13 \\ -3 & 2 & -13 & 14 \end{bmatrix}.$$

Assume that P is the projection matrix onto a subspace W of $\mathbb{R}^4$. Find W.

13. Here A denotes an $m \times n$ matrix of rank r, $\vec{b}$ an $m \times 1$ vector and $\vec{x}$ an $n \times 1$ vector. True of false?

(a) If $\vec{b} = \vec{0}$, then the equation $A\vec{x} = \vec{b}$ **always** has a solution.

(b) If $r = 2$, then the equation $A\vec{x} = \vec{b}$ has **at most one** solution.

(c) If $m > n$, then the equation $A\vec{x} = \vec{b}$ has **at most one** solution.

(d) If $m > n$, then the equation $A\vec{x} = \vec{b}$ has **infinitely many** solutions.

14. Find the general solution of the system of equations:

$$\begin{aligned} 2x - y + 3z &= 4 \\ w + y - z &= 3 \\ 3w - 2x + 4y - 6z &= 5 \end{aligned}$$

15. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$.

16. (a) Let V be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Find the matrix of the projection onto V.

(b) Let V be as in part (a) and $\vec{(b)} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. Find the projection of $\vec{b}$ onto V.

(c) Let V and $\vec{b}$ be as in parts (a) and (b). Let W be the subspace of $\mathbb{R}^3$ spanned by $\vec{b}$ and the vectors in V. Find the matrix of the projection onto W. *Hint: this requires almost no computation. Think!*

17. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & -1 & 1 \\ 2 & 4 & 1 & -2 & 1 \\ -3 & -6 & -2 & 3 & -2 \end{bmatrix}$. The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Find the kernel and the image of A.

18. Let V be the one-dimensional subspace of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first projects vectors onto V and then rotates them by $30°$ counterclockwise. Find the matrix representation of L.

19. Consider the matrix $A = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{bmatrix}$. Find a, b and c such that the image of A is $\mathbb{R}^3$.

20. True or false?

    (a) A linear system of 3 equations in 3 unknowns always has at least one solution.

    (b) Suppose A is an invertible $n \times n$ matrix and B is a non-invertible $n \times n$ matrix. Then AB is a noninvertible matrix.

21. If possible, perform the multiplication. If not, justify.

    (a) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & -4 & 3 \\ 6 & -2 & 4 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}$

    (c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$

22. The following linear system has a unique solution. Find it.

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5 \\ x_2 - x_3 &= 1 \\ 2x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

23. Let $A = \begin{bmatrix} 2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ -2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ 0 & -2 & 2\sqrt{3} \end{bmatrix}$. What is A^{-1}?

24. Define each of the following terms and in each case give a 2×2 example:

    (a) diagonal matrix
    (b) upper triangular matrix
    (c) invertible matrix
    (d) diagonalizable matrix

25. Let
$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 1 & 2 & 1 \end{bmatrix}.$$
Find the kernel and the image of A.

26. Does $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

27. Find a matrix A which transforms the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ into the standard coordinate vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$. In other words, $A\vec{v}_i = \vec{e}_i$ for $i = 1, 2, 3$.

28. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, find all number c for which the equation $A\vec{x} = cB\vec{x}$ has a nonzero solution. For each such c, find the corresponding $\vec{x}$.

29. Determine whether the vectors

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 6 \\ 4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix}$$

span $\mathbb{R}^4$.

30. Find the general solution for the following matrix equation:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ -2 & -4 & 2 & 10 \\ 3 & 6 & 4 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}.$$

31. Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 1 & -3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

Find the kernel and the image of A.

32. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -2 \\ 2 & 4 & -1 & -1 \\ -3 & -9 & h & 6 \end{bmatrix}.$$

For which values of h, if any, is the transformation $T(\vec{x}) = A\vec{x}$ onto? For which values of h, if any, is T $1-1$?

33. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ 3x_1 + 3x_2 + x_3 \\ x_1 - x_2 \end{bmatrix}.$$

Find the matrix A of the transformation T. Compute A^{-1}.

34. Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto the image of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$.

35. Find all values of t for which the matrix A is invertible.
$$A = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

36. For which values of k does the homogeneous system $A\vec{x} = \vec{0}$ with
$$A = \begin{bmatrix} 1 & 2 & k \\ 0 & -3 & 0 \\ 4 & k & 7 \end{bmatrix}$$
have a nonzero solution?

37. Find the inverse of the matrix $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix}$.

38. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -1 & 1 \\ 3 & -2 & 3 & -4 & 3 \end{bmatrix}$.

(a) Find rref(A).

(b) Find the kernel of A.

(c) What is the rank of A?

(d) Find the value of d for which the following equation has a solution:
$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ d \end{bmatrix}.$$

For the value of d found, find all the solutions of the equation.

39. Let V be the plane in $\mathbb{R}^3$ defined by $x - 2y + z = 0$. Find the matrix P of the projection onto V.

40. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 7 & 9 & 11 & 13 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

   (a) Find the image and the kernel of A.

   (b) For what values of $\vec{b}$, if any, is $A\vec{x} = \vec{b}$ consistent?

   (c) For what values of $\vec{b}$, if any, does $A\vec{x} = \vec{b}$ have a unique solution?

   (d) For what values of $\vec{b}$, if any, does $A\vec{x} = \vec{b}$ have infinitely many solutions?

   (e) Is A invertible? Why or why not?

   (f) Write down a formula for the projection onto the image of A. (Do not multiply out!)

41. We say that a matrix A is *nilpotent* if $A^m = 0$ for some positive integer m. Find, or prove that they don't exist, a 2×2 invertible nilpotent matrix and a 2×2 non-invertible nilpotent matrix.

42. Prove or disprove and salvage if possible:

    Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $\ker T = \{0\}$, then the image of T is all of $\mathbb{R}^2$.