Chapters 5,6,7 Review
SOLUTIONS PROBLEMS 1-20
Math 52 Spring 2006

1. (a) Express the matrix $A = \begin{bmatrix} 0.5 & 0 \\ 2 & 1.5 \end{bmatrix}$ as a product SDS^{-1}, where D is a diagonal matrix.

   (b) Find a formula for $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (a) By inspection, the eigenvalues are $\lambda_1 = 0.5$ and $\lambda_2 = 1.5$ (since A is lower triangular). Therefore D should be

$$D = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

However, to find S requires more work. The eigenspaces are

$$E_{\lambda_1} = \ker \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

$$E_{\lambda_1} = \ker \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} = span \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Therefore an eigenbasis is

$$\mathcal{B} : \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$S_{\mathcal{B}\to std} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$S_{std\to\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Note that a different choice of basis for the eigenspaces will give a different (but still correct) set of change of basis matrices. That's fine!

We obtain

$$A = S_{\mathcal{B}\to std} D S_{std\to\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

which multiplies out correctly.

(b) The fast way to see this is to note that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda_2 = 1.5$. Therefore

$$A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5^k \end{bmatrix}$$

Alternatively, a more general approach is as follows. We have a representation $A = SDS^{-1}$, so we see that

$$\begin{aligned} A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= (SDS^{-1})^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= (SDS^{-1})(SDS^{-1}) \dots (SDS^{-1}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= SD^k S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0.5^k & 0 \\ 0 & 1.5^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1.5^k \end{bmatrix} \end{aligned}$$

2. Compute the determinant of the following matrix:

$$\begin{bmatrix} 1 & -1 & -2 & 6 \\ 3 & 1 & 2 & 4 \\ 2 & 0 & 5 & 1 \\ -2 & 3 & 2 & 3 \end{bmatrix}.$$

The determinant is 306. Try using row operations if you find the Laplace expansion tedious.

3. Prove or disprove and salvage if possible:

   (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the transpose of A by $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then A and A^T have the same eigenvalues.

   (b) Every 3×3 matrix has at least one real eigenvalue.

   (c) A real number λ is an eigenvalue of A if and only if λ is an eigenvalue of A^n for all positive integers n.

   (a) Proof: We have

$$(A - \lambda I)^T = \begin{bmatrix} a - \lambda & c \\ b & d - \lambda \end{bmatrix} = A^T - \lambda I$$

   Therefore

$$\det((A - \lambda I)) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$$

   and so the characteristic polynomials satisfy $f_A(\lambda) = f_{A^T}(\lambda)$ and hence both A and A^T have the same eigenvalues.

   (b) Proof: The characteristic polynomial of a 3×3 matrix is a degree three polynomial. Therefore it has at least one real root (by calculus – e.g. look at limit as $\lambda \to \infty$ and $\lambda \to -\infty$). And hence the matrix has at least one real eigenvalue.

   (c) Not true! Counterexample:
   Consider the scaling

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

   This has eigenvalue $\lambda = 2$. But

$$A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

   has eigenvalue $\lambda = 4$. Therefore the “only if” part is false.

Salvage (corrected statement): A real number λ is an eigenvalue of A if and only if λ^n is an eigenvalue of A^n for all positive integers n.

Proof of Salvage: If λ^n is an eigenvalue of A^n for all positive integers n, then in particular, $\lambda^1 = \lambda$ is an eigenvalue of $A^1 = A$. This proves the "if" part.

For the "only if", assume that λ is an eigenvalue of A. Then by definition there is some vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. We proceed by induction on the claim that "$A^k\vec{v} = \lambda^k\vec{v}$".

Base case: The case $n = 1$ is exactly the statement of our assumption.

Inductive step: Suppose we have proven the claim for $n = k$. Then

$$\begin{aligned}
A^{k+1}\vec{v} &= A^k A\vec{v} \\
&= A^k\lambda\vec{v} && \text{by the initial assumption} \\
&= \lambda A^k\vec{v} \\
&= \lambda\lambda^k\vec{v} && \text{by the inductive hypothesis} \\
&= \lambda^{k+1}\vec{v}
\end{aligned}$$

And we have completed the proof.

4. **Either** give an example exhibiting the stated properties **or** prove that no such example exists.

   (a) Square matrices A and B with the same characteristic polynomial so that A is not similar to B.

   (b) A square matrix A which is not diagonalizable.

   (a) Let

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

   These two matrices have the same characteristic polynomial, i.e. $(1-\lambda)^2$. They both have eigenvalue $\lambda = 1$ with algebraic multiplicity 2. However, λ has geometric multiplicity 2 for A and only

1 for B. To see this geometrically, note that A is the identity, so the whole space is the eigenspace for $\lambda = 1$. However, B is a shear and shears are not diagonalizable. (If in doubt, verify the geometric multiplicity by a calculation.)

Note that if A and B are diagonalizable and have the same characteristic polynomial, then they have the same eigenvalues and are similar to the same diagonal matrix D. Therefore they are similar to each other (by transitivity of the equivalence relation "similar"). So to look for an example, we need matrices which are not both diagonalizable.

(b) The matrix B above is a good example of a nondiagonalizable square matrix.

5. Assume that

$$A = \begin{bmatrix} 3 & 4 & 3 \\ -1 & -4 & -5 \\ 1 & 8 & 9 \end{bmatrix}$$

has characteristic polynomial $16 - 20t + 8t^2 - t^3 = -(t-2)^2(t-4)$. Find the eigenvalues and eigenspaces of A.

The eigenvalues are $\lambda = 2$ and $\lambda = 4$ of algebraic multiplicity 2 and 1 respectively.

The eigenspace E_2 can have geometric multiplicity $1 \le g \le 2$.

$$E_2 = \ker \begin{bmatrix} 1 & 4 & 3 \\ -1 & -6 & -5 \\ 1 & 8 & 7 \end{bmatrix} = span \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Note that this is clearly the kernel since the rank of the matrix is 1 (the first two columns are obviously independent since one is not a multiple of the other). So the geometric multiplicity of E_2 is $g = 1$.

The eigenspace E_4 must have geometric multiplicity 1.

$$E_4 = \ker \begin{bmatrix} -1 & 4 & 3 \\ -1 & -8 & -5 \\ 1 & 8 & 5 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

This matrix is not diagonalizable since the geometric multiplicities only add up to 2.

6. Let $T : P_2 \to P_2$ be defined by $T(f) = f + f' + f''$. Find an eigenbasis for T.

First we must find the matrix for the transformation. We will use the basis

$$\mathcal{B} : 1, t, t^2$$

The transformation acts as follows:

$$T(a+bt+ct^2) = a+bt+ct^2+b+2ct+2c = (a+b+2c)+(b+2c)t+ct^2$$

Therefore its matrix is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

There is only one eigenvalue: $\lambda = 1$. The associated eigenspace is

$$E_1 = \ker \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

This transformation does not have an eigenbasis, since there are not enough eigenvectors to form one.

7. Let

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These vectors form a basis of $\mathbb{R}^3$. (Note: you do not have to show this.)

(a) Use the Gram-Schmidt process on these vectors to produce an orthonormal basis of $\mathbb{R}^3$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection of $\mathbb{R}^3$ onto the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$. Write down a matrix representing T. *Hint: your work in part (a) might be useful.*

(a) The Gram-Schmidt process gives the result

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \vec{u}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \vec{u}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Remember to check the result by quickly dotting the vectors pairwise in your head to make sure you get 0 or 1 where appropriate.

(b) Recall that the Gram-Schmidt orthogonalization process tells us that

$$span\,\{\vec{v}_1, \vec{v}_2\} = span\,\{\vec{u}_1, \vec{u}_2\}$$

Let

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Then

$$QQ^T = \frac{1}{9}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}$$

8. Let

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) Find the characteristic polynomial of A. What are the eigenvalues of A? *Hint: It factors!*

(b) Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$. *Hint: If this is painful or impossible, you may have found the wrong eigenvalues!*

(a) The characteristic polynomial is

$$f_A(\lambda) = -\lambda^3 - 2\lambda^2 + 5\lambda + 6 = -(\lambda+1)(\lambda+3)(\lambda-2)$$

You can factor it by testing and seeing that $\lambda = -1$ is a root, and proceeding from there. The eigenvalues of A are -1, -3, and 2.

(b) The eigenspaces will all have dimension one and are

$$E_{-1} = \ker \begin{bmatrix} -1 & 5 & 6 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \mathit{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$E_{-3} = \ker \begin{bmatrix} 1 & 5 & 6 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \mathit{span} \left\{ \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix} \right\}$$

$$E_{2} = \ker \begin{bmatrix} -4 & 5 & 6 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \mathit{span} \left\{ \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} \right\}$$

We have

$$S = \begin{bmatrix} -1 & 9 & -4 \\ 1 & -3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -5 & -6 \\ 1 & 3 & 2 \\ 2 & 8 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So

$$A = \begin{bmatrix} -1 & 9 & -4 \\ 1 & -3 & 2 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} -1 & -5 & -6 \\ 1 & 3 & 2 \\ 2 & 8 & 6 \end{bmatrix}\right)$$

9. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$

True or false?

(a) A is invertible.

(b) A has rank 2.

(c) There exists a basis of eigenvectors for A.

The determinant is an easy calculation expanding along the third row or column:

$$\det(A) = (-1)(3) + 4(-1) = -7$$

Hence we know it is invertible and has rank 3.

(a) True

(b) False

(c) True (However, this relies on the observation that *symmetric matrices are always diagonalizable.* This was not covered in the course. It shouldn't be on the review, because the characteristic polynomial isn't easily factored, so you can't see this directly.)

10. Let

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

True or false? (Do these without calculating the characteristic polynomial.)

(a) B has rank 4.

(b) $\lambda = 0$ is an eigenvalue for B.

(c) $\lambda = 1$ is an eigenvalue for B.

(d) All eigenvalues of B satisfy $|\lambda| = 1$.

(a) False. The second and fourth columns are the same.

(b) True. To see if a particular value is an eigenvalue, we consider the nullity of $B - \lambda I = B$. Since B does not have rank 4 (by (a)), the nullity is nonzero and therefore 0 is an eigenvalue.

(c) True. We have

$$B - 1(I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

(Don't forget the $\frac{1}{2}$ when you calculate this!) This matrix has determinant $\frac{1}{16}[(-1)((-2)(3) - 1(-2)) - 1(1(1) + 1(3))] = \frac{1}{16}[4 - 4] = 0$ and so it has nonzero nullity. This means $\lambda = 1$ is an eigenvalue.

(d) False. Since (b) is true.

11. Let M_2 denote the vector space of all 2×2 matrices and $B = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$.

(a) Let T be the linear transformation from M_2 to M_2 defined by $T(C) = B^{-1}CB$. Consider the basis for M_2 consisting of

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 4×4 matrix $A = \begin{bmatrix} 1 & & & 0 \\ 3 & & & -3 \\ 0 & & & 0 \\ 0 & & & 1 \end{bmatrix}$ represents T with respect to the basis $E_1, \ldots, E_4$. Supply the entries in the 2nd and 3rd columns of A.

(b) Find all numbers λ for which there exists a nonzero 2×2 matrix C with $B^{-1}CB = \lambda C$. *Hint: use the results in part (a). This is a chapter 7 problem!*

(a) This question was on a previous review.

$$T(E_2) = \frac{3}{2}E_2$$

$$T(E_3) = -2E_1 - 6E_2 + \frac{2}{3}E_3 + 2E_4$$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 3 & \frac{3}{2} & -6 & -3 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(b) Note that the question is asking you to find all eigenvalues of the transformation T. This can be done using the matrix found in part (a). We calculate the characteristic polynomial of A expanding down the first column:

$$f_A(\lambda) = (1-\lambda)^2(\frac{3}{2} - \lambda)(\frac{2}{3} - \lambda)$$

and see that the eigenvalues are $\lambda = 1, \frac{3}{2}, \frac{2}{3}$.

12. Let $V \subseteq \mathbb{R}^4$ be the subspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

Find an orthonormal basis for V.

Using the Gram-Schmidt process, we obtain
$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ and $\vec{u}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$.

13. Define what it means for a matrix to be **orthogonal**.

An orthogonal matrix is one whose columns form an orthonormal basis.

14. Let A be the matrix $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

The characteristic polynomial is

$$f_A(\lambda) = (3-\lambda)(-3-\lambda) + 8 = \lambda^2 - 1$$

Therefore the eigenvalues are $\lambda = 1, -1$. The eigenspaces are

$$E_1 = \ker \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$E_{-1} = \ker \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

15. (a) A is a certain 3×3 matrix, which has three distinct real eigenvalues. Furthermore, two of its eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Using only this information, find a third eigenvector for A which is not a linear combination of the above two.

(b) Let $B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$. Find the eigenvalues of B.

(a) Recall that the eigenvectors of different eigenspaces are orthogonal. Since A has three distinct eigenvalues, it has three orthogonal eigenspaces of dimension one. Since two are given, the third eigenvector must be perpendicular to the first two. One way to find it is by cross product. Remember that any scalar multiple will do. For example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) The characteristic polynomial is

$$\begin{aligned} f_B(\lambda) &= (3-\lambda)((3-\lambda)(1-\lambda)-3) - 1(1-\lambda-3) + 1(1-3+\lambda) \\ &= -\lambda(\lambda-1)(\lambda-4) \end{aligned}$$

Therefore the eigenvalues are 0, 1, and 4.

16. (a) Compute the determinant of the matrix

$$C = \begin{bmatrix} 2 & 0 & -6 \\ 0 & -3 & 2 \\ 0 & 0 & -4 \end{bmatrix}.$$

(b) Recall that a matrix Q is called **skew-symmetric** if $Q^T = -Q$. Prove that if Q is a 3×3 skew-symmetric matrix, then $\det Q = 0$.

(c) Prove that if Q is any skew-symmetric matrix, than the trace of Q is 0. *Hint: what are the diagonal entries?*

(a) This is an upper triangular matrix: $\det(C) = 2(-3)(-4) = 24$.

(b) Suppose that Q is skew-symmetric. Recall that matrices have the same determinant as their transposes. We have $\det(Q) = \det(Q^T) = \det(-Q) = -\det(Q)$. The only number equal to its own negative is zero, so $\det(Q) = 0$.

(c) Suppose that Q is skew-symmetric. Then since its transpose is its negative, we have the following relation on the entries: $Q_{ij} = -Q_{ji}$. In particular, for the diagonal entries (where $i = j$), we have $Q_{ii} = -Q_{ii}$ and so $Q_{ii} = 0$. Since all diagonal entries are zero, the trace must be zero.

17. (a) Let V be a vector space. Suppose $T : V \to V$ is a linear transformation with $T \circ T = \text{Identity}$. Prove that all the eigenvalues of T are either 1 or -1.

(b) Let V be the vector space of all 2×2 matrices. Let $T : V \to V$ be the linear map defined by $T(A) = A^T$. Find the eigenvalues and eigenmatrices of T. *Hint: use part (a)*

(c) Let V and T be as in part (b). Write down a basis for V and find the matrix to describe T with respect to that basis.

(a) Let $T : V \to V$ be a transformation on a vector space V. Suppose that $T \circ T = I$. Suppose that λ is an eigenvalue of T. Then for some $\vec{v}$, $T(\vec{v}) = \lambda\vec{v}$. But then

$$\begin{aligned} T \circ T(\vec{v}) &= T(\lambda\vec{v}) \\ &= \lambda T(\vec{v}) \qquad \text{by linearity of the transformation} \\ &= \lambda^2\vec{v} \end{aligned}$$

Since $T \circ T = I$, this tells us λ^2 is an eigenvalue of the identity transformation. But the identity transformation has only one eigenvalue, the eigenvalue 1. So $\lambda^2 = 1$ and so $\lambda = 1$ or -1.

(b) Note that for this transformation $T(A) = A^T$, we have $T \circ T = I$. Therefore the eigenvalues are either 1 or -1 by (a). The eigenspace for $\lambda = 1$ are all matrices such that $T(A) = A$ which is $A^T = A$. The 2×2 matrices which are self-transpose are the symmetric matrices. These have a basis of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The eigenspace for $\lambda = -1$ are all the matrices such that $T(A) = -A$ or $A^T = -A$, which is to say, all skew-symmetric matrices. These have a basis of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore we have found two eigenvalues $\lambda = 1$ and $\lambda = -1$, with geometric multiplicities of 3 and 1 respectively and the bases given above. (Since this adds up to $4 = \dim(V)$, T is in fact diagonalizable.)

(c) Choose as a basis for V the eigenbasis given above, i.e.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

. Then T has matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(You could also do this the hard way by choosing the standard basis and working it all out... but that would be silly.)

18. Consider the matrix $A = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{bmatrix}$.

 (a) Calculate the determinant of A.

 (b) Find a, b and c such that the image of A is $\mathbb{R}^3$.

 (a) The determinant is $1(c-b) + 1(2b-a) = c+b-a$. (Use the first column to expand.)

 (b) The matrix has image $\mathbb{R}^3$ exactly when it is invertible, which happens exactly when the determinant is nonzero. Therefore a, b, c must satisfy $a \neq b + c$.

19. Find all the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Use one of the eigenvalues you found to calculate the associated eigenvectors.

$$f_A(\lambda) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

The eigenspaces are

$$E_1 = \ker \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$E_3 = \ker \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

20. True or false? Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $A^{31} = A$.

    True. Note that $A^2 = I$. Therefore, $A^{31} = AA^{30} = A(A^2)^{15} = AI^{15} = A$.