Math 52
Midterm Solutions

3/8/06

1. (15 points) Find all solutions to the system

$$\begin{array}{rcrcrcrcrcl} x_1 & - & 3x_2 & + & 2x_3 & + & 2x_4 & + & 2x_5 & = & 1 \\ -x_1 & + & 3x_2 & + & & & 4x_4 & - & 6x_5 & = & 1 \\ 2x_1 & - & 6x_2 & + & 3x_3 & + & 6x_4 & + & x_5 & = & 1. \end{array}$$

Solution. The row-reduced echelon form of this system is

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right],$$

so solutions are of the form

$$\begin{bmatrix} 1+3s-2t \\ s \\ 1-t \\ t \\ t \end{bmatrix}.$$

2. (15 points) Consider an $n \times m$ matrix A with more columns than rows ($m > n$) and with $\text{rank}(A) = n$. How many solutions does the system $A\vec{x} = \vec{b}$ have? Justify your answer.

Solution. This system has infinitely many solutions. There are three steps to the argument.

1. Because the number of columns is more than the number of rows, there must be free variables. This means there are either infinitely many solutions or none (never a unique solution).
2. Since the rank equals the number if rows, there is a leading 1 for each row of the matrix. This means the row-reduced echelon form does not contain a row of zeroes, so the system must have solutions (it is consistent).
3. Combining 1 and 2, the only possibility is that there are infinitely many solutions.

3. (15 points total)

(a) (5 points) What geometric transformation does the matrix $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ represent?

Solution. A is projection onto the line $y = x$.

(b) (5 points) Find the matrix B of the counterclockwise rotation by 45 degrees.

Solution. $B = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(c) (5 points) Do you expect A and B to commute? Don't calculate; give a geometric argument to support your answer.

Solution. No. If we apply AB, points will be rotated then projected, so that the image will be the line $y = x$. On the other hand, if we apply BA, points will be projected onto the line $y = x$ and then that line will be rotated 45 degrees, so that the image will be the y-axis.

4. (20 points total)

(a) (15 points) Find the matrix of the projection onto the plane $2x - y + 2z = 0$.

Solution. The vector $(2, -1, 2)$ is orthogonal to this plane, and this vector has length 3, so the corresponding unit vector is $\vec{u} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.

Applying the formula $\text{proj}_V(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$, we find that the answer is

$$\frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}.$$

(b) (5 points) Describe the kernel and image of this projection.

Solution. The image is simply the plane $2x - y + 2z = 0$. That's where all the points end up.

The kernel is the set of vectors which project to the origin, which is the line through the origin orthogonal to the plane, namely

$$\text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right).$$

5. (20 points total)

(a) (15 points) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution. $A^{-1} = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 0 & -1 \\ -5 & -1 & 5 \end{bmatrix}$.

(b) (5 points) Solve the system $A\vec{x} = \vec{b}$ for $\vec{b} = \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix}$.

Solution. Notice that $A\vec{x} = \vec{b}$ is the same thing as $\vec{x} = A^{-1}\vec{b}$, as long as A is invertible. Therefore

$$\vec{x} = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 0 & -1 \\ -5 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

6. (15 points) Find the kernel and image of

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -2 & -6 & 1 & -1 \\ 1 & 3 & 0 & 1 \end{bmatrix}.$$

Solution. The row-reduced form of this matrix is

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel, then, is

$$\text{span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

The image is the span of the columns of A, and the 2nd and 4th columns are redundant, so the image is $\text{span}\left((1, -2, 1), (1, 1, 0)\right)$.