Math 52
Midterm Review Solutions

(1) The reduced row-echelon form of the matrix representing this system is the identity, so:
  (a) Only $x = 0, y = 0, z = 0$.
  (b) The system always has solutions.

(2) First recall that "homogeneous" means that the values on the right side of the equals signs are all zero.
  (a) The rank of the matrix of this system is ≤ 2, so there is (at least one) free variable, so a 2×3 homogeneous system will always have nonzero solutions.
  (b) Rewrite the system as a linear combination of 3 vectors in $\mathbb{R}^2$, so a nonzero solution of this system gives a linear relation among the vectors.

(3) Use linearity. The answer is $\begin{bmatrix} 4 & -6 & 4 \\ 8 & -10 & 7 \\ 6 & -2 & 2 \end{bmatrix}$.

(4) $\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$,

$\text{im}(A) = \text{the span of any 2 of } \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

(5) (a) Solutions are of the form $\begin{bmatrix} 2 + s - t \\ 1 - \frac{3}{4}s + t \\ s \\ t \end{bmatrix}$.

  (b) $\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ -\frac{3}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$.

(6) $\ker(A) = \operatorname{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$,

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 13 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right).$$

(7) To find a vector orthogonal to the plane spanned by $\vec{v}_1$ and $\vec{v}_2$, take the cross product $\vec{v}_1 \times \vec{v}_2 = (2, 2, -1)$. Then the matrix of the projection is

$$\frac{1}{9}\begin{bmatrix} 5 & -4 & 2 \\ -4 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}.$$

(8) $\ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$, $\operatorname{im}(A) = \mathbb{R}^3$.

(9) (a) True.
(b) False. Note: If one is true, the other is false.

(10) $\operatorname{rank}(B) = 3$.

(11) Solutions are of the form $\begin{bmatrix} -1-s+t \\ s-t \\ s \\ -1+t \\ t \end{bmatrix}$.

(12) (a) $\frac{1}{26}\begin{bmatrix} 25 & -4 & 3 \\ -4 & 10 & 12 \\ 3 & 12 & 17 \end{bmatrix}$.

(b) This matrix has rank 2, or if you play with the numbers, you'll notice that

$$2 \cdot (\text{1st column}) = -3 \cdot (\text{2nd column})$$
$$(\text{1st column}) = -3 \cdot (\text{3rd + 4th columns}).$$

Thus $W = \operatorname{span}\left(\begin{bmatrix} 18 \\ -12 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 14 \\ -13 \end{bmatrix}\right)$.

(13) (a) True. $\vec{x} = \vec{0}$ is always a solution.
(b) False. $A\vec{x} = \vec{b}$ has at most one solution when there are no free variables, i.e. when $r = n$. So this statement is false unless $n = 2$.
(c) False. Again, this statement is true only if $r = n$.
(d) False. This is only true if both $r < n$ (free variables) and $\vec{b} \in \mathrm{im}(A)$.

(14) Solutions are of the form $\begin{bmatrix} 3 - y + z \\ 2 + \frac{1}{2}y - \frac{3}{2}z \\ y \\ z \end{bmatrix}$.

(15) $A^{-1} = \begin{bmatrix} -3 & 1 & -1 \\ 3 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

(16) (a) $\frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.
(b) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.
(c) Since $\mathrm{proj}_V(\vec{b}) \neq \vec{b}$, $\vec{b}$ is not in V. Therefore $W = \mathbb{R}^3$ and the matrix of the projection onto W is just the identity matrix I_3!

(17) $\ker(A) = \mathrm{span}\left(\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$,

$\mathrm{im}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right)$.

(18) $\frac{1}{4}\begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}+1 & \sqrt{3}-1 \end{bmatrix}$.

(19) Row-reducing the matrix, we find that the matrix has rank 3 if and only if $-a + b + c \neq 0$. Alternatively, the image of A is not $\mathbb{R}^3$ if and only if the vector (a, b, c) is in the plane spanned by the first two columns of A. To find the equation of this plane, take the cross product

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

so the equation of the plane is $-x + y + z = 0$, the same answer as before.

(20) (a) False.
(b) True. If AB is invertible, then $(AB)^{-1}A = B^{-1}$, so B would be invertible too.

(21) (a) $\begin{bmatrix} -6 & -6 & 2 \\ 24 & -20 & 25 \end{bmatrix}$.
(b) $\begin{bmatrix} 3 & 9 \\ -1 & 5 \\ 9 & 24 \end{bmatrix}$.
(c) Not possible.

(22) $x_1 = 3, x_2 = -1, x_3 = -2$.

(23) $A^{-1} = \frac{1}{16} \begin{bmatrix} 2\sqrt{2} & -2\sqrt{2} & 0 \\ \sqrt{6} & \sqrt{6} & -2 \\ \sqrt{2} & \sqrt{2} & 2\sqrt{3} \end{bmatrix}$.

(24) You don't need to know what a "diagonalizable" matrix is (yet).

From this point on, we'll only provide solutions if the problem is substantially different from earlier problems.

(25) Row-reduce.

(26) Find the rank of the matrix whose columns are these three vectors. If the rank is 3, then the answer is yes. Otherwise, no.

(27) Notice that the matrix whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is really just A^{-1}. So simply invert this matrix to find A.

(28) This is really an eigenvalue/eigenvector problem, so you don't really need to know how to do it. However, it is fairly straightforward to solve directly, and if you do, you should get

$$c = \frac{1 \pm \sqrt{5}}{4}, \quad \vec{x} = \begin{bmatrix} (1 \pm \sqrt{5})s \\ s \end{bmatrix}.$$

(29) Similar to (26).

(30) Row-reduce the augmented matrix.

(31) Similar to (25).

(32) First, notice that T can never be 1-to-1, because the domain has larger dimension (4) than the codomain (3), so there must be a nontrivial kernel. Second, T will be onto if and only if the image of $T = \mathbb{R}^3$, which happens if and only if the rank of $A = 3$. So, row-reducing the matrix, we find that its rank is 3 as long as $h \neq 3$.

(33) $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Computing A^{-1} is straightforward.

(34) This is similar to problems (7), (12) and (16). The only new wrinkle is that you have to realize that the image of A is precisely the plane spanned by its column vectors.

(35) If you try to row-reduce this matrix, you will find that you are forced to divide by $1 - t^2$ at some point, so clearly $t \neq \pm 1$. Otherwise, the row reduction works fine and the matrix is invertible. Notice that if you plug $t = 1$ or $t = -1$ into the matrix, you get problems, because then you get the same columns repeated, which always means a matrix is non-invertible.

(36) The homogeneous system $A\vec{x} = \vec{0}$ will have a nonzero solution iff $\text{rank}(A) < 3$. Row-reducing the matrix, this only occurs if $7 - 4k = 0$, i.e. $k = \frac{7}{4}$.

(37) Straightforward.

(38) (a) $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) $\ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$.

(c) $\text{rank}(A) = 3$.

(d) If we apply the same sequence of operations to this vector as we did to row-reduce A, we get the vector

$$\begin{bmatrix} 3 \\ 4 \\ -1 \\ d - 5 \end{bmatrix}.$$

In order for the system to have a solution, the bottom entry of this vector must be zero. Hence $d = 5$.

(39) Straightforward.

(40) (a) $\ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right)$,

$\operatorname{im}(A) =$ the span of the 1st, 3rd, and 5th columns of A, because these are the columns containing the leading 1's in $\operatorname{rref}(A)$.

(b) Any $\vec{b}$ in the image of A.

(c) This system never has a unique solution, because it has a nontrivial kernel (equivalently, there are free variables).

(d) Again, because there are free variables, for every $\vec{b}$ in the image of A the system will have infinitely many solutions.

(e) A is not invertible, because it is not a square matrix.

(f) You don't have to be able to do this (yet). The easiest way to see the answer may be as follows: if you had performed row-reduction on the augmented matrix $[A \mid \vec{b}]$ instead of on A (much like in part (d) of (38)), you would have found that the last row becomes

$$[\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \mid -2b_1 + b_2 - b_3 + b_4\,],$$

hence $\vec{b}$ is in the image of A iff $-2b_1 + b_2 - b_3 + b_4 = 0$. Therefore $(-2, 1, -1, 1)$ is a normal vector to this hyperplane and we can now find a unit vector $\vec{u}$ and apply the formula $\vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$.

(41) Any nilpotent matrix is non-invertible. If a nilpotent matrix A were invertible, then we could multiply the equation $A^m = 0$ by A^{-1} repeatedly until we got $A = 0$, a contradiction. An example of a nilpotent matrix is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(42) True. This follows immediately from the Rank-Nullity Theorem.