# Final Review

This packet contains review problems for the whole course, including all the problems from the previous reviews. We also suggest below problems from the textbook for chapters 5, 6, and 7. (Problems from earlier chapters were suggested on earlier review packets.) The suggested problems are often higher-numbered, more conceptual problems. If you are having trouble with any of the more basic problems, you should pick some of those yourself to do.

Good luck studying!

Section 5.1: #17, 19, 29
Section 5.2: #32-35
Section 5.3: #33, 38, 52, 53, 56, 57
Chapter 5 True/False: #21, 23, 24
Section 6.1: #49, 50, 57
Section 6.2: #40, 41, 45, 48, 53
Chapter 6 True/False: #10-13, 18, 35
Section 7.2: #19, 38, 47, 48
Section 7.3: #23, 29, 31, 39
Section 7.4: #47, 51, 60, 64
Chapter 7 True/False: #4, 5, 6, 9, 11, 25, 29, 30, 32, 46, 53

1. Consider the system of linear equations

$$\begin{aligned} x + y - z &= a \\ 3x - y/2 &= b \\ 7x/2 - 3z &= c \end{aligned}$$

   (a) Find all the solutions of this system when $a = b = c = 0$.

(b) Find the values for $a$, $b$ and $c$ for which this system has no solutions (there may be no such values).

2. Let $S$ be the subspace of $P_2$ of polynomials $f(x)$ such that $f(2) = 0$.

   (a) Show that $S$ is in fact a linear subspace of $P_2$.

   (b) Find a basis for $S$; you have to show that what you found is indeed a basis.

3. (a) Show that a homogeneous system of equations with all non-zero coefficients with two equations and three variables always has a non-zero solution.

   (b) Use part (a) to show that three vectors in $\mathbb{R}^2$, all of whose coordinates are non-zero, are always linearly dependent.

4. (a) Express the matrix $A = \begin{bmatrix} 0.5 & 0 \\ 2 & 1.5 \end{bmatrix}$ as a product $SDS^{-1}$, where $D$ is a diagonal matrix.

   (b) Find a formula for $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

5. Consider the basis $B_1 = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Let $T$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{B_1}$ be the matrix representing $T$ in the basis $B_1$; that is

$$T \begin{bmatrix} z \\ w \end{bmatrix}_{B_1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{B_1} \begin{bmatrix} z \\ w \end{bmatrix}_{B_1}.$$

Find the matrix representation of $T$ in the standard basis.

6. Compute the determinant of the following matrix:

$$\begin{bmatrix} 1 & -1 & -2 & 6 \\ 3 & 1 & 2 & 4 \\ 2 & 0 & 5 & 1 \\ -2 & 3 & 2 & 3 \end{bmatrix}.$$

7. (a) Find the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that sends $(0,0,1)$ to $(4,7,2)$, $(0,4,5)$ to $(-4,-5,2)$, and $(1,3,4)$ to $(2,6,8)$.
   (b) Find at least one eigenvector of the linear transformation from part (a).

8. Prove or disprove and salvage if possible:

   (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the transpose of $A$ by $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then $A$ and $A^T$ have the same eigenvalues.

   (b) Every $3 \times 3$ matrix has at least one real eigenvalue.

   (c) A real number $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A^n$ for all positive integers $n$.

9. **Either** give an example exhibiting the stated properties **or** prove that no such example exists.

   (a) Square matrices $A$ and $B$ with the same characteristic polynomial so that $A$ is not similar to $B$.
   (b) A square matrix $A$ which is not diagonalizable.

10. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 6 \\ 0 & 2 & 0 & 1 & 1 & 4 \\ 0 & 1 & 0 & 2 & 0 & 2 \end{bmatrix}.$$

Find a basis for $\ker(A)$. Fully justify your answer.

11. Assume that

$$A = \begin{bmatrix} 3 & 4 & 3 \\ -1 & -4 & -5 \\ 1 & 8 & 9 \end{bmatrix}$$

has characteristic polynomial $16 - 20t + 8t^2 - t^3 = -(t-2)^2(t-4)$. Find the eigenvalues and eigenspaces of $A$.

12. Let $T : P_2 \to P_2$ be defined by $T(f) = f + f' + f''$. Find an eigenbasis for $T$.

13. Consider the system of linear equations:

$$\begin{aligned} x_1 - x_3 + x_4 &= 2 \\ 3x_1 + 4x_2 - x_4 &= 10 \\ 5x_1 + 4x_2 - 2x_3 + x_4 &= 14 \\ 9x_1 + 8x_2 - 3x_3 + x_4 &= 26 \end{aligned}$$

(a) Find the reduced row-echelon form of the augmented matrix associated to this linear system and use it to solve the system.

(b) Let $A$ be the coefficient matrix of the system above. Using your result in part (a), give a basis of $\ker(A)$.

14. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{bmatrix} 1 & 4 & 2 & -5 & 1 \\ -1 & -3 & -1 & 1 & 0 \\ 4 & 13 & 5 & -8 & 1 \\ 3 & 7 & 1 & 5 & -1 \end{bmatrix}.$$

(a) Find a basis for the image of $T$.

(b) What is the dimension of the kernel of $T$?

15. Let $P_2$ denote the linear space of all polynomials of degree $\leq 2$. Let $T : P_2 \to P_2$ be the linear transformation $T(f) = f + f'$. Is $T$ invertible? If yes, find its inverse.

16. Let $B = (\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

    (a) Show that $B$ is a basis for $\mathbb{R}^2$. What are the $B$-coordinates of the vector $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$?

    (b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with $B$-matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. What is the standard matrix representing $T$ in cartesian coordinates?

17. Let

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

    These vectors form a basis of $\mathbb{R}^3$. (Note: you do not have to show this.)

    (a) Use the Gram-Schmidt process on these vectors to produce an orthonormal basis of $\mathbb{R}^3$.

    (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection of $\mathbb{R}^3$ onto the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$. Write down a matrix representing $T$. *Hint: your work in part (a) might be useful.*

18. Let

$$A = \begin{bmatrix} -1 & -2 & 0 & 1 \\ 2 & 4 & -1 & 3 \\ -3 & -6 & 2 & -6 \end{bmatrix}.$$

    Find bases for the kernel and the image of $A$.

19. Let $P_2$ denote the linear space of polynomials of degree $\leq 2$. Let $T : P_2 \to \mathbb{R}^3$ be the linear transformation

$$T(f) = \begin{bmatrix} f(-2) \\ f(-1) \\ f(2) \end{bmatrix}.$$

Show that $T$ is invertible and find $f \in P_2$ such that $T(f) = \begin{bmatrix} -4 \\ 3 \\ 12 \end{bmatrix}$.

20. Suppose that $A$ and $B$ are $n \times n$ matrices such that $A$ and $B$ commute, i. e. $AB = BA$. Suppose further that $A$ has distinct real eigenvalues. Show that there is a matrix $S$ such that both $SAS^{-1}$ and $SBS^{-1}$ are diagonal matrices (i. e. $A$ and $B$ are simultaneously diagonalizable).

21. Let

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) Find the characteristic polynomial of $A$. What are the eigenvalues of $A$? *Hint: It factors!*

(b) Find an invertible matrix $S$ and a diagonal matrix $D$ so that $A = SDS^{-1}$. *Hint: If this is painful or impossible, you may have found the wrong eigenvalues!*

22. Let

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of $A$.

(b) Describe the image of the unit circle under the transformation $T(\vec{x}) = A\vec{x}$.

23. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$

True or false?

(a) $A$ is invertible.

(b) $A$ has rank 2.

(c) There exists a basis of eigenvectors for $A$.

24. Let

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

True or false?

(a) $B$ has rank 4.
(b) Each eigenvalue $\lambda$ of $B$ is real and positive.
(c) $\lambda = 0$ is an eigenvalue for $B$.
(d) $\lambda = 1$ is an eigenvalue for $B$.
(e) All eigenvalues of $B$ satisfy $|\lambda| = 1$.

25. Find the general solution to the following system of equations:

$$\begin{aligned} 3x_1 + 8x_2 - 5x_3 + 9x_4 - 4x_5 &= -12 \\ 3x_1 + x_2 + 2x_3 - 4x_4 + 2x_5 &= 1 \\ 2x_1 + 7x_2 - 5x_3 + 10x_4 - 5x_5 &= -12 \end{aligned}$$

*Hint: fractions can be avoided. The solution involves only small integers.*

26. (a) Let $V$ be a subspace of $\mathbb{R}^3$ with basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the projection matrix $P$ onto $V$.

(b) Let

$$P = \frac{1}{27}\begin{bmatrix} 18 & -12 & -3 & -3 \\ -12 & 8 & 2 & 2 \\ -3 & 2 & 14 & -13 \\ -3 & 2 & -13 & 14 \end{bmatrix}.$$

Assume that $P$ is the projection matrix onto a subspace $W$ of $\mathbb{R}^4$. Find a basis for $W$.

27. (a) Let $T$ be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_2 \\ x_1 + 7x_3 \end{bmatrix}$$

Find the matrix $A$ representing $T$ with respect to the basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Assume that

$$A = \frac{1}{420}\begin{bmatrix} 648 & -7306 & 6320 & -2154 \\ -360 & 2445 & -960 & 765 \\ 468 & -1446 & 1500 & -774 \\ 936 & -8317 & 6080 & -2493 \end{bmatrix}.$$

Let $\vec{v}_1, \ldots, \vec{v}_4$ be a basis for $\mathbb{R}^4$ with

$$\vec{v}_1 = \begin{bmatrix} 9 \\ 0 \\ 3 \\ 8 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ -1 \end{bmatrix},$$

and let $M = [\vec{v}_1|\vec{v}_2|\vec{v}_3|\vec{v}_4]$. Assume that

$$\begin{aligned} A\vec{v}_1 &= 2\vec{v}_1, \\ A\vec{v}_2 &= \vec{v}_1 + 3\vec{v}_2, \\ A\vec{v}_3 &= 5\vec{v}_1 - \vec{v}_3, \\ A\vec{v}_4 &= \vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 + \vec{v}_4. \end{aligned}$$

Find $B = M^{-1}AM$. *Hint: very little equation is required.*

28. Let $M_2$ denote the vector space of all $2 \times 2$ matrices and $B = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$.

(a) Let $T$ be the linear transformation from $M_2$ to $M_2$ defined by $T(C) = B^{-1}CB$. Consider the basis for $M_2$ consisting of

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The $4 \times 4$ matrix $A = \begin{bmatrix} 1 & & & 0 \\ 3 & & & -3 \\ 0 & & & 0 \\ 0 & & & 1 \end{bmatrix}$ represents $T$ with respect to the basis $E_1, \ldots, E_4$. Supply the entries in the 2nd and 3rd columns of $A$.

(b) Find all numbers $\lambda$ for which there exists a nonzero $2 \times 2$ matrix $C$ with $B^{-1}CB = \lambda C$. *Hint: use the results in part (a).*

29. Consider

$$A = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 5 & 1 \\ 3 & 1 & 1 & 4 \end{bmatrix}.$$

How many eigenvalues (counting multiplicities) are positive? Negative? Zero?

30. Here $A$ denotes an $m \times n$ matrix of rank $r$, $\vec{b}$ an $m \times 1$ vector and $\vec{x}$ an $n \times 1$ vector. True of false?

(a) If $\vec{b} = \vec{0}$, then the equation $A\vec{x} = \vec{b}$ **always** has a solution.

(b) If $\vec{b} \in \text{im}(A)$, then the equation $A\vec{x} = \vec{b}$ **always** has a solution.

(c) If $\vec{b} \in \ker(A)$, then the equation $A\vec{x} = \vec{b}$ **always** has a solution.

(d) If $\ker(A) = \{\vec{0}\}$, then the equation $A\vec{x} = \vec{b}$ has **at most one** solution.

(e) If $\vec{b} \in \text{im}(A)$, then the equation $A\vec{x} = \vec{b}$ has **at most one** solution.

(f) If $r = 2$, then the equation $A\vec{x} = \vec{b}$ has **at most one** solution.

(g) If $m > n$, then the equation $A\vec{x} = \vec{b}$ has **at most one** solution.

(h) If $m > n$, then the equation $A\vec{x} = \vec{b}$ has **infinitely many** solutions.

(i) If $m < n$ and $\vec{b} \in \text{im}(A)$, then the equation $A\vec{x} = \vec{b}$ has **infinitely many** solutions.

31. Find the general solution of the system of equations:

$$\begin{aligned} 2x - y + 3z &= 4 \\ w + y - z &= 3 \\ 3w - 2x + 4y - 6z &= 5 \end{aligned}$$

32. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$.

33. (a) Let $V$ be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Find the matrix of the projection onto $V$.

(b) Let $V$ be as in part (a) and $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. Find the projection of $\vec{b}$ onto $V$.

(c) Let $V$ and $\vec{b}$ be as in parts (a) and (b). Let $W$ be the subspace of $\mathbb{R}^3$ spanned by $\vec{b}$ and the vectors in $V$. Find the matrix of the projection onto $W$. *Hint: this requires almost no computation. Think!*

34. Let $V \subseteq \mathbb{R}^4$ be the subspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

Find an orthonormal basis for $V$.

35. (a) Let $V$ and $W$ be vector spaces and let $T : V \to W$ be a function. Under what conditions is $T$ a **linear transformation**?

(b) Define what it means for a matrix to be **orthogonal**.

36. Let $A$ be the matrix $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of $A$.

37. (a) $A$ is a certain $3 \times 3$ matrix, which is real and symmetric. Furthermore, two of its eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Using only this information, find a third eigenvector for $A$ which is not a linear combination of the above two.

(b) Let $B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$. Find the eigenvalues of $B$.

38. (a) Compute the determinant of the matrix

$$C = \begin{bmatrix} 2 & 0 & -6 \\ 0 & -3 & 2 \\ 0 & 0 & -4 \end{bmatrix}.$$

(b) Recall that a matrix $Q$ is called **skew-symmetric** if $Q^T = -Q$. Prove that if $Q$ is a $3 \times 3$ skew-symmetric matrix, then $\det Q = 0$.

(c) Prove that if $Q$ is any skew-symmetric matrix, than the trace of $Q$ is 0. *Hint: what are the diagonal entries?*

39. (a) Let $V$ be a vector space. Suppose $T : V \to V$ is a linear transformation with $T \circ T = \text{Identity}$. Prove that all the eigenvalues of $T$ are either 1 or $-1$.

(b) Let $V$ be the vector space of all $2 \times 2$ matrices. Let $T : V \to V$ be the linear map defined by $T(A) = A^T$. Find the eigenvalues and eigenmatrices of $T$. *Hint: use part (a)*

(c) Let $V$ and $T$ be as in part (b). Write down a basis for $V$ and find the matrix to describe $T$ with respect to that basis.

40. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & -1 & 1 \\ 2 & 4 & 1 & -2 & 1 \\ -3 & -6 & -2 & 3 & -2 \end{bmatrix}$. The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Find a basis for $\operatorname{im} A$. Determine the dimension of this space and the $\mathbb{R}^n$ in which it is contained.

   (b) Find a basis for $\ker A$. Determine the dimension of this space and the $\mathbb{R}^m$ in which it is contained.

41. The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$ are linearly independent. Find a basis for $\mathbb{R}^3$ which contains $\vec{v}_1$ and $\vec{v}_2$ as two of its three elements.

42. Let $V$ be the one-dimensional subspace of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first projects vectors onto $V$ and then rotates them by $30°$ counterclockwise. Find the matrix representation of $L$ in the standard basis coordinates.

43. Consider the two bases for $\mathbb{R}^2$, $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$. Write and expression for the change-of-basis matrix from $B_1$ to $B_2$.

44. Suppose $\vec{v}_1$ and $\vec{v}_2$ are two linearly independent vectors in $\mathbb{R}^2$. Find the change-of-basis matrix from $B_1 = \{\vec{v}_1, \vec{v}_2\}$ to $B_2 = \{\vec{v}_2, 2\vec{v}_1\}$.

45. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \\ x - y \end{bmatrix}.$$

Write an expression for the matrix representation of $T$ in the bases $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^2$ and $B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

46. Solve
$$\begin{bmatrix} 1 & 2 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

47. Consider the matrix $A = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{bmatrix}$.

    (a) Calculate the determinant of $A$.

    (b) Find $a$, $b$ and $c$ such that the image of $A$ is $\mathbb{R}^3$.

48. Find all the eigenvalues of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Use one of the eigenvalues you found to calculate the associated eigenvectors.

49. Consider the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}.$$
Suppose $M = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ is the matrix representation of $S$ in the coordinates of some basis $B$ of $\mathbb{R}^2$. Find the basis $B$.

50. True or false?

    (a) A linear system of 3 equations in 3 unknowns always has at least one solution.

    (b) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $A^{31} = A$.

(c) Suppose $A$ is an invertible $n \times n$ matrix and $B$ is a non-invertible $n \times n$ matrix. Then $AB$ is not invertible.

51. If possible, perform the multiplication. If not, justify.

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & -4 & 3 \\ 6 & -2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$

52. The following linear system has a unique solution. Find it.

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5 \\ x_2 - x_3 &= 1 \\ 2x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

53. Compute the determinant of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 6 & 4 \\ 0 & 1 & 13 & 1 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Is $A$ invertible or not? Why?

54. Let $A = \begin{bmatrix} 2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ -2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ 0 & -2 & 2\sqrt{3} \end{bmatrix}$.

(a) Compute $AA^T$.

(b) What is $A^{-1}$?

55. True or false? The product of any two orthogonal matrices is orthogonal.

56. Define each of the following terms and in each case give a $2 \times 2$ example:

    (a) diagonal matrix

    (b) upper triangular matrix

    (c) diagonalizable matrix

57. Let
$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 1 & 2 & 1 \end{bmatrix}.$$

    (a) Find a basis for $\ker A$. What is the dimension of the kernel?

    (b) Find a basis for the image of $A$. What is its dimension?

58. Let $A$ be an $n \times n$ matrix.

    (a) Suppose $\vec{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$. Show that $\vec{v}$ is also an eigenvector of $A^2$.

    (b) Prove that if $A$ is diagonalizable, then so is $A^2$.

59. Let $A$ and $B$ be two similar matrices.

    (a) Show that $A$ and $B$ have the same characteristic polynomial.

    (b) Prove that $A$ and $B$ have the same eigenvalues.

60. Decide if the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable. Justify your answer.

61. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by
$$\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \text{ and } \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

62. Find the eigenvalues of $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. *Hint: they are integers.*

63. The matrix $A = \begin{bmatrix} 4 & 3 & 3 \\ -12 & -8 & -6 \\ 6 & 3 & 1 \end{bmatrix}$ is diagonalizable and has eigenvalues $-2, -2, 1$. Find a matrix which diagonalizes $A$.

64. Does $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

65. Find a matrix $A$ which transforms the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ into the standard coordinate vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$. In other words, $A\vec{v}_i = \vec{e}_i$ for $i = 1, 2, 3$.

66. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

67. Let $V$ be the vector space consisting of all polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ of degree $\leq 3$. Let $T$ be the linear transformation

$$T\left(p(x)\right) = x^2 \frac{d^2p}{dx^2} + p(x).$$

   (a) Find the matrix $A$ associated to $T$ for some suitable basis of $V$.

(b) For which real numbers $\lambda$ does there exist a non-zero solution $p(x)$ to the equation

$$x^2 \frac{d^2p}{dx^2} + p(x) = \lambda p(x)?$$

For each such $\lambda$ find the corresponding $p(x)$.

68. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, find all numbers $c$ for which the equation $A\vec{x} = cB\vec{x}$ has a nonzero solution. For each such $c$, find the corresponding $\vec{x}$.

69. Let $\vec{v}$ and $\vec{w}$ be eigenvectors of $A$ with corresponding eigenvalues 2 and 3, respectively. Are $\vec{v}$ and $\vec{w}$ linearly dependent or linearly independent? Give a detailed explanation (or proof).

70. Determine whether the vectors

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 6 \\ 4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix}$$

span $\mathbb{R}^4$. Are they linearly independent?

71. Find the general solution for the following matrix equation:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ -2 & -4 & 2 & 10 \\ 3 & 6 & 4 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}.$$

72. Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 1 & -3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

(a) Find a basis for $\ker A$.

(b) Are the columns of $A$ linearly independent? If not, find the linear dependence relation.

73. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -2 \\ 2 & 4 & -1 & -1 \\ -3 & -9 & h & 6 \end{bmatrix}.$$

For which values of $h$, if any, is the transformation $T(\vec{x}) = A\vec{x}$ onto? For which values of $h$, if any, is $T$ $1-1$?

74. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ 3x_1 + 3x_2 + x_3 \\ x_1 - x_2 \end{bmatrix}.$$

Find the matrix $A$ of the transformation $T$. Compute $A^{-1}$.

75. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with $T(\vec{e}_1) = \vec{e}_1$ and $T(\vec{e}_2) = \vec{e}_1 + \vec{e}_2$. Give a possible value for $T(\vec{e}_3)$ so that $T$ is invertible. Explain why your answer works.

76. Check which of the following subsets are vector subspaces:

(a) The set of all vectors $\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that $x_1 = 2x_3 - x_2 + 1$ and $x_1 + 3x_2 = 0$.

(b) The set of all $2 \times 2$ matrices with determinant equal to 0.

77. Check which of the following functions are linear transformations:

(a) The function $F : \mathbb{R}^4 \to \mathbb{R}^3$ with $F \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_4 \\ 3x_1 + x_3 \\ x_4 - 1 \end{bmatrix}$

(b) The function $F(A) = A^T - 5A$ defined on the space of $4 \times 4$ matrices.

(c) The function $F : P_7 \to P_7$ with $F(p(t)) = 2p'(t) + p(0)$.

78. (a) Find bases for the subspaces $\ker A$ and $\operatorname{im} A$ associated to the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 15 \\ 3 & 14 & 25 & -3 \end{bmatrix}.$$

(b) What is the orthogonal complement of the kernel of the above matrix $A$? Verify orthogonality.

79. Find an orthonormal basis for the subspace $W$ of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

80. (a) Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto the image of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) Find a basis for $(\operatorname{im} A)^{\perp}$, i. e. the orthogonal complement of $\operatorname{im} A$ in $\mathbb{R}^3$.

81. (a) Compute the determinant of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$.

(b) Find all values of $t$ for which the matrix $A$ is invertible.
$$A = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

82. For which values of $k$ does the homogeneous system $A\vec{x} = \vec{0}$ with

$$A = \begin{bmatrix} 1 & 2 & k \\ 0 & -3 & 0 \\ 4 & k & 7 \end{bmatrix}$$

have a nonzero solution?

83. (a) Let $H$ be the subspace of $P_3$ spanned by

$$\{1 + 2t + t^3, t - t^2, -3 + 2t - 8t^2 - 3t^3, -1 - 2t - t^3\}.$$

Does the polynomial $2+4t+2t^2+2t^3$ belong to the above subspace $H$?

(b) Find a basis for $H$.

84. (a) Compute the characteristic polynomial, the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & -6 \\ 1 & 0 & -1 \end{bmatrix}.$$

(b) Is $A$ diagonalizable? If not, explain why. If it is, write it as $A = PDP^{-1}$.

85. (a) Show that $\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$, and find the matrix of the change of coordinates from $\mathcal{B}$ to the standard basis of $\mathbb{R}^3$.

(b) Find the matrix of the change of coordinates from the standard basis of $\mathbb{R}^3$ to $\mathcal{B}$. *Course moral: judicious choice of row operations can often save one many an ugly calculation!*

86. (a) Let $A$ be a $3 \times 3$ matrix having the following properties:

i. $\ker A$ contains the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

ii. $A^3\vec{v} = 8\vec{v}$ , where $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

iii. multiplication by $A$ leaves every vector on the line lying in the horizontal $(x_1, x_2)$-plane of equation $x_1 + x_2 = 0$ unchanged.

Find the eigenvalues and eigenvectors of $A$. Is $A$ diagonalizable? Is $A$ invertible?

(b) Find a basis for $\operatorname{im} A$.

87. Find the inverse of the matrix $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix}$.

88. Determine whether or not the following sets of vectors are linearly independent. If they are linearly dependent, express $v_3$ as a linear combination of $v_1$ and $v_2$.

(a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

(b) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$.

89. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \end{bmatrix}$.

(a) Find an orthonormal basis $q_1, q_2, q_3$ for the image of $A$.

(b) Find a vector $q_4$ such that $q_1, q_2, q_3, q_4$ is an orthonormal basis for $\mathbb{R}^4$.

90. Compute the following determinant:

$$\det\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ -1 & -6 & 1 & -1 \end{bmatrix}.$$

91. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -1 & 1 \\ 3 & -2 & 3 & -4 & 3 \end{bmatrix}$.

(a) Find rref($A$).

(b) Find a basis for the kernel of $A$.

(c) What is the rank of $A$?

(d) Show that $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ is in the kernel of $A^T$. Show that this vectors spans $\ker A^T$.

(e) Prove that a vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in the im($A$) if and only if its dot product with $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ is 0.

(f) Find the value of $d$ for which the following equation has a solution:

$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ d \end{bmatrix}.$$

For the value of $d$ found, find all the solutions of the equation.

92. Let $V$ be the plane in $\mathbb{R}^3$ defined by $x - 2y + z = 0$.

    (a) Find a basis for $V$.

    (b) Find a basis for the orthogonal complement $V^\perp$ of $V$.

    (c) Find the matrix $P$ of the projection onto $V$.

    (d) Find all the eigenvalues and eigenvectors of $P$. *Hint: $P$ is a projection matrix.*

93. Consider the matrix $A = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix}$.

    (a) Diagonalize $A$, i. e. find an invertible matrix $S$ and a diagonal matrix $\Lambda$ such that $A = S\Lambda S^{-1}$.

    (b) Calculate $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

94. Let $V$ be the vector space of all $2 \times 2$ matrices. Prove or disprove the following statements:

    (a) The set $S_1 = \{A \in V; \operatorname{trace}(A) = 0\}$ is a subspace of $V$.

    (b) The set $S_2 = \{A \in V; \det(A) = 0\}$ is a subspace of $V$.

    (c) The transformation $L : V \to V$ defined by $L(A) = A^T$ is a linear transformation.

95. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 7 & 9 & 11 & 13 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

    (a) Find a basis for the image and kernel of $A$.

    (b) For what values of $\vec{b}$, if any, is $A\vec{x} = \vec{b}$ consistent?

    (c) For what values of $\vec{b}$, if any, does $A\vec{x} = \vec{b}$ have a unique solution?

(d) For what values of $\vec{b}$, if any, does $A\vec{x} = \vec{b}$ have infinitely many solutions?

(e) Is $A$ invertible? Why or why not? Is $A^T A$ invertible? Why or why not?

(f) Write down a formula for the projection onto the subspace spanned by the columns of $A$. (Do not multiply out!)

96. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 9 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

97. Prove which of the following are subspaces under matrix addition and which are not. For those that are subspaces, find a basis.

(a) The set of invertible $2 \times 2$ matrices.

(b) The set of $2 \times 2$ matrices with 0 in the upper right corner.

(c) The set of $3 \times 3$ matrices with determinant 1.

(d) The set of $2 \times 2$ orthogonal matrices.

(e) The set of $2 \times 2$ real skew-symmetric matrices. (Recall that $A$ is skew-symmetric if $A^T = -A$.)

98. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find an orthonormal basis for the space spanned by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. Find the projection of $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ onto this space.

99. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ -3 & 0 \end{bmatrix}.$$

100. We say that a matrix $A$ is *nilpotent* if $A^m = 0$ for some positive integer $m$.

   (a) Find, or prove that they don't exist, a $2 \times 2$ invertible nilpotent matrix and a $2 \times 2$ singular nilpotent matrix.

   (b) Let $V$ be the set of $2 \times 2$ nilpotent matrices. Is $V$ a subspace? Why or why not?

101. (a) For any projection $P$ we have $P^2 = P$. What are the possible eigenvalues of $P$? Be as specific as possible. Prove your claim.

   (b) Let $A$ be a real symmetric $n \times n$ matrix which is also orthogonal. Assume further that $A^4 = A$. Being as specific as possible, what can you say about the eigenvalues of $A$? If $A$ has these properties **and** is $2004 \times 2004$, are there infinitely many distinct such $A$? Why or why not?

102. Two matrices $A$ and $B$ are similar if there is an invertible $S$ such that $A = S^{-1}BS$.

   (a) If $A$ and $B$ are similar matrices, prove that they have the same eigenvalues.

   (b) Define trace $A$, the trace of an $n \times n$ matrix, and define what it means for a matrix to be diagonalizable. Prove that if $A$ is diagonalizable, then $\operatorname{trace}(A^k) = \lambda_1^k + \ldots + \lambda_n^k$, where the $\lambda_i$ are the eigenvalues of $A$. *The result holds for all $A$, but you may assume that $A$ is diagonalizable.*

103. Let $S$ be the subspace of $\mathbb{R}^3$ given by $x + y + z = 0$. Verify that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ form a basis of $S$.

104. Let $V$ and $W$ be vector spaces. Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in $V$ and let $\vec{w}_1$ and $\vec{w}_2$ be vectors in $W$. Let $C$ be a scalar. Complete the following definitions:

   (a) A *vector subspace* is a subset of a vector subspace which is ...

   (b) A *linear transformation* $T : V \to W$ is a function from $V$ to $W$ such that ...

   (c) An *eigenvector* of a linear transformation $T : V \to V$ is a vector $\vec{v}$ such that ...

105. Prove or disprove and salvage if possible:

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $\ker T = \{0\}$, then the image of $T$ is all of $\mathbb{R}^2$.

106. Determine whether or not each of the following matrices is diagonalizable. In order to make your job easier, here are the characteristic polynomials.

   (a) $A = \begin{bmatrix} 5 & 6 & -6 \\ 0 & -1 & 0 \\ 3 & 3 & -4 \end{bmatrix}$ $\qquad p_A(x) = (x-2)(x+1)^2$

   (b) $B = \begin{bmatrix} 6 & 1 & 3 \\ -7 & -2 & -3 \\ -8 & -2 & -4 \end{bmatrix}$ $\qquad p_B(x) = (x-2)(x+1)^2$

107. For each of the following sets of vectors, find out whether they are linearly independent or linearly dependent:

   (a) $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix};$

   (b) $f_1 = e^x$, $f_2 = e^{3x}$, $f_3 = e^{-2x}$;

(c) $g_1 = \sin^2 x$, $g_2 = \cos^2 x$, $g_3 = 1$;

(d) $\vec{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 8 \\ -2 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 1 \end{bmatrix}, \vec{w}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$;

(e) $\vec{u}_1 = \begin{bmatrix} 3 \\ 8 \\ 11 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -8 \\ 4 \\ 10 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 7 \\ -2 \\ 18 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 6 \\ -8 \\ 19 \end{bmatrix}$.

108. True or false?

(a) The intersection of two linear subspaces is always a subspace.

(b) The union of two linear subspaces is always a subspace.

(c) If $\det A = 0$, then the columns of $A$ must be dependent.

(d) If the columns of $A$ are dependent, the $\det A$ must equal 0.

(e) If an $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

(f) Every linear transformation can be put into upper triangular form (using complex numbers).

(g) Every linear transformation can be diagonalized (using complex numbers).

109. Complete each of the following sentences with the *definition* of the italized word or phrase:

(a) The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are *dependent* if ...

(b) The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a *basis* for the linear space $V$ if ...

(c) Let $V$ and $W$ be linear spaces. A function $T : V \to W$ is a *linear transformation* if ...

(d) A linear transformation $T : V \to V$ is *diagonalizable* if ...

(e) Let $T : V \to V$ be a linear transformation. A vector $\vec{v}$ is an *eigenvector* for $T$ if it satisfies ...

(f) A set of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is *orthonormal* if ...

110. Let $A$ be an $n \times n$ *n*ilpotent matrix, that is for which there exists an integer $k$ so that $A^k = 0$

(a) Prove that 0 is the only eigenvalue of $A$.

(b) Prove that the characteristic polynomial of $A$ is $x^n$.

111. Let $A$ and $B$ be two $n \times n$ matrices with the property that $AB = BA$. Suppose that $\vec{v}$ is an eigenvector for $A$. Prove that $B\vec{v}$ is an eigenvector for $A$, provided it isn't the zero vector.

112. (a) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}.$$

(b) Solve the system

$$\begin{aligned} x - y + 2z &= 8 \\ 3x + 2y + 4z &= -16 \\ y - 2z &= 4 \end{aligned}$$

(c) What is the rank of the matrix $A$? Find the kernel and the image of $A$.

113. Which of the following sets of vectors are linearly independent?

(a) $S_1 = \{(-4, 1, 3), (1, 2, 5), (-3, 3, 8)\}$.

(b) $S_2 = \{1 + 2x + x^2, 1 + 4x - x^2, 1 + x + 2x^2\}$.

(c) $S_3 = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$.

114. Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_3 \\ -x_1 + 3x_2 + 4x_3 \\ x_1 + 3x_2 + 2x_3 \end{bmatrix}.$$

(a) Find bases for the kernel and the image of $L$.

(b) State and verify the Rank-Nullity Theorem for $L$. Find the rank of the matrix of $L$.

115. (a) Check whether the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

has an inverse, and , if it does, find it.

(b) Let $T : P_3 \to P_3$ be the linear transformation from the space of polynomials of degree at most 3 into itself given by

$$T(p(x)) = \frac{dp}{dx} + p(x).$$

Check whether $T$ has an inverse and, if it does, find it.

116. Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -7 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & 2 \\ 6 & 0 & 3 \\ 4 & -2 & -1 \end{bmatrix}.$$

Find $\det A$, $\det B$, $\det(A^{-4}B^2)$, $\det(B^T A^3 B^{-1})$ and $\det(A + B)$.

117. For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$.

118. Let $A$ be a square matrix. True or false: if the columns of $A$ are linearly independent, then so are the columns of $A^{10}$. Justify.

119. For what value of $b$ does the system

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \\ b \end{bmatrix}$$

have a solution? Find the general solution of the system for this value of $b$.

120. Find the matrix of the reflection about the line $y = -4x$. Justify your answer.

121. If $A$ is a $64 \times 17$ matrix of rank 11, how many linearly independent vectors $\vec{x}$ satisfy $A\vec{x} = 0$? How many satisfy $A^T\vec{x} = 0$? Justify.

122. If $V$ is the subspace spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, find an orthonormal basis for $V$ and the projection matrix $P$ onto $V$.

123. For the subspace $V$ from the previous problem, find

(a) a basis, not necessarily orthonormal, in the orthogonal complement $V^\perp$ of $V$;

(b) the projection matrix $Q$ onto $V^\perp$.

124. Are the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 6 & 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

similar or not? Justify. *Hint: recall diagonalization.*

125. Assume a matrix $A$ has eigenvalues $\lambda_1$ and $\lambda_2$ and that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the corresponding eigenvectors. Find the eigenvalues and the eigenvectors of $A^T$.

126. (a) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & -1 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}.$$

(b) Compute $\det(3A)$, $\det(-A)$, $\det B$ and $\det C$, where

$$B = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 6 & -1 & 2 & 1 \\ 6 & 0 & -1 & 0 \\ 6 & -1 & 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & -1 & 2 & 3 \\ 2 & -1 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}.$$

*Hint: how are $B$ and $C$ related to $A$?*

127. Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 5 \end{bmatrix}.$$

128. Is the matrix $A$ from the previous problem similar to a diagonal one? What about $A^{-1}$? Justify.

129. In the space $P_2$ of polynomials of degree at most 2 define the transformation $T$ by $T(p) = 3p - 2p'$. Find the matrix of $T$ with respect to the standard basis $1, t, t^2$. Find the inverse of $T$.

130. Diagonalize the matrix $A = \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$.

131. Consider the following pair of matrices:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}.$$

For each, determine the characteristic polynomial and the eigenvalues. Determine whether $A$, or $B$, or both are diagonalizable. If either of them is diagonalizable, find a matrix which diagonalizes it.

132. Let $V$ be an $n$ dimensional subspace of $\mathbb{R}^m$ and $T$ an orthogonal transformation from $V$ to $V$. Show that if $\vec{v} \in V$ and $T(\vec{v}) = \lambda\vec{v}$ with $\lambda \in \mathbb{R}$, then $\lambda$ is either 1 or $-1$.

133. Suppose $T$ and $S$ are linear transformations from $V$ to $V$ which commute with one another. Suppose $T$ has an eigenvalue $\lambda$ such that the corresponding eigenspace $W$ is one dimensional. Show that every nonzero element of $W$ is an eigenvector of $S$.

134. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Compute explicitly the matrix $A^{2006}$. Find a formula for $A^n$ in general.