1. (15 points). Find the limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}.$$

$$e^{\lim_{x\to\infty} \ln(1+1/x)^{2x}} = e^{\lim_{x\to\infty} 2x \ln(1+1/x)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{\ln(1+1/x)}{1/2x}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{\ln(1+x) - \ln x}{1/2x}\right)}$$

$$\overset{L'H}{=} e^{\lim_{x\to\infty}\left(\frac{\frac{1}{x+1} - 1/x}{1/2x}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\left(\frac{-1}{x^2+x}\right)\Big/ -1/2x^2\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{2x^2}{x^2+x}\right)}$$

$$= e^{2}$$

2. (15 points). Check whether the integral test applies to the following series, and if it does, determine whether the series is convergent or divergent.

$$\sum_{n=4}^{\infty} \frac{1}{n \ln^3 n}.$$

$$f(x) = \frac{1}{x \ln^3 x}$$

is clearly decreasing because x & $\ln x$ are both increasing

furthermore $\frac{1}{x \ln^3 x} > 0$

Thus the integral test applies

$$\int_4^\infty \frac{1}{x(\ln x)^3}\,dx = \int_4^\infty \frac{1}{x}(\ln x)^{-3}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}(\ln x)^{-2}\right]_4^t$$

$$= \lim_{t\to\infty} -\frac{1}{2}(\ln t)^{-2} + \frac{1}{2}\frac{1}{\ln 4}$$

$$= \frac{1}{2}\frac{1}{\ln 4}$$

Thus the series is convergent.

3. (15 points). Use the limit comparison test, if appropriate, to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^{3n} + 3^{n+1}}{3^{2n} + 5^n}.$$

Use limit comparison test to the series

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}} = \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n \quad \left(\text{convergent geom. series } r = \tfrac{8}{9} < 1\right)$$

$$\lim_{n\to\infty} \left| \frac{2^{3n}/3^{2n}}{\dfrac{2^{3n}+3^{n+1}}{3^{2n}+5^n}} \right| = \lim_{n\to\infty} \left| \frac{2^{3n}}{3^{2n}} \cdot \frac{3^{2n}+5^n}{2^{3n}+3^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1 + 5^n/3^{2n}}{1 + 3^{n+1}/2^{3n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1 + (5/9)^n}{1 + 3(3/8)^n} \right|$$

$$= 1$$

Thus the series converges

4. (15 points). Check whether the alternating series test applies to the following series, and if it does, determine whether the series converges of diverges:

$$\sum_{n=6}^{\infty} \frac{(-1)^{n+1}}{\ln(n + n^{3/2})}.$$

n & $n^{3/2}$ are increasing for $n \geq 6$

so $\ln(n + n^{3/2})$ is increasing for $n \geq 6$

so $\frac{1}{\ln(n+n^{3/2})}$ is decreasing for $n \geq 6$

Furthermore $\lim_{n \to \infty} \frac{1}{\ln(n+n^{3/2})} = 0$

Thus the alternating series test applies, and the series converges.

5. (25 points). Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}}.$$

Ratio Test: $\displaystyle \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{\sqrt{n+1}}}{\frac{(x-3)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| (x-3) \frac{\sqrt{n}}{\sqrt{n+1}} \right|$

$= |x-3|$

$\Longrightarrow$ Radius of convergence is 1

endpts: $x=2 \Longrightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ convergent by alt. series test ($1/\sqrt{n}$ decreases to 0)

$x=4 \Longrightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent by comparison to harmonic series ($1/\sqrt{n} > 1/n$)

Interval of convergence $I = [2, 4)$

6. (15 points). Evaluate the indefinite integral as a power series, and determine the radius and center of convergence:

$$\int \frac{(x-1)}{1+(x-1)^3}\,dx.$$

$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$, so letting $u = -(x-1)^3$ we have

$$\frac{1}{1+(x-1)^3} = \sum_{n=0}^{\infty} (-1)^n (x-1)^{3n}$$

so
$$\frac{(x-1)}{1+(x-1)^3} = \sum_{n=0}^{\infty} (-1)^n (x-1)^{3n+1}$$

Thus
$$\int \frac{(x-1)}{1+(x-1)^3}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^{3n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n (x-1)^{3n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{3n+2}}{3n+2} + C$$

Method #1

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{3n+5}}{3n+5}}{\frac{(x-1)^{3n+2}}{3n+2}} \right| = (x-1)^3 \lim_{n\to\infty} \left|\frac{3n+2}{3n+5}\right|$$

$$= (x-1)^3$$

$\Rightarrow$ radius of conv is 1

Method #2

$\sum_{n=0}^{\infty} u^n$ converges for $|u|<1$, $u = -(x-1)^3$

$\Rightarrow$ series converges for $|x-1|^3 < 1$

$\Rightarrow$ radius of convergence is 1