FINAL EXAM

1. (20 points). Find the orthogonal trajectories of the family of curves

$$x^2 - y^2 = k$$

and sketch a rough picture of the two families.

$2x - 2y \frac{dy}{dx} = 0$

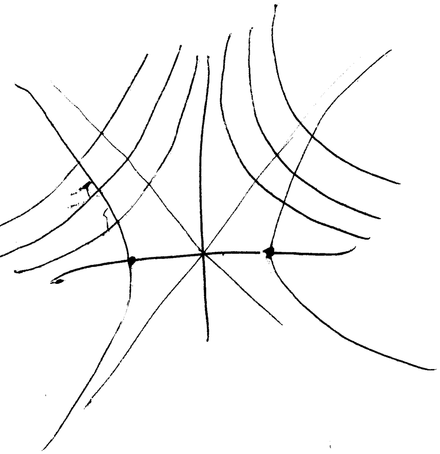
$\frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$

$\frac{dy}{dx} = -\frac{y}{x}$

$\frac{dy}{y} = -\frac{dx}{x}$

$\ln y = -\ln x + C.$

$= \ln \frac{1}{x} + c$



$y = \frac{C}{x}$

~~$\frac{dy}{dx} = Cy$~~

$xy = c$

$y = \frac{c}{x}$ orthogonal to $x^2 - y^2 = k.$

2. (15+15 points). After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) What is the half-life of radon-222?

(b) How long would it take the sample to decay to 10% of its original amount?

$\frac{dP}{dt} = kP$ $\quad k<0$

$\frac{1}{2} = e^{kt_*}$

~~$P_0 = P_0$~~ $P(0) = P_0$

$P(3) = \frac{58}{100} P_0$

$-\ln 2 = k t_*$

$P(t) = P_0 e^{kt}$

~~$k=$~~ $\frac{58}{100} \cancel{P_0} = \cancel{P_0} e^{3k}$

$\ln\left(\frac{58}{100}\right) = 3k$ $\quad k = \frac{1}{3} \ln\left(\frac{58}{100}\right)$

$$\boxed{t_{1/2} = \frac{-\ln 2}{\cancel{t}k} = \frac{-3\ln 2}{\ln\left(\frac{58}{100}\right)} = \frac{3\ln 2}{\ln\left(\frac{100}{58}\right)}}$$

$\frac{1}{10} = e^{kt}$

$-\ln 10 = kt$

$t = \left(\frac{3\ln 10}{\ln\left(\frac{100}{58}\right)}\right)$ to be 10% of original.

3. (30 points) Solve the initial value problem.

$$x^2\,y' = y + \frac{x^3\,e^{-1/x}}{x^2+1}, \qquad x > 0, \qquad y(1) = 0$$

$$p\,y' - \frac{1}{x^2}\,p\,y = \frac{x e^{-\frac{1}{x}}}{x^2+1}\,p$$

$$\frac{dp}{dx} = -\frac{p}{x^2} \qquad p = \exp\left(-\int \frac{1}{x^2}\,dx\right)$$

$$= \exp\left(+\frac{1}{x}\right).$$

$$\frac{d}{dx}(p\,y) = \frac{x}{x^2+1}\,e^{\cancel{-\frac{1}{x}}}\,\cancel{e}^{+\frac{1}{x}}$$

$$e^{\frac{1}{x}}\,y' - \frac{1}{x^2}\,e^{\frac{1}{x}}\,y = \frac{x}{x^2+1}$$

$$p\,y = \frac{1}{2}\int \frac{2x}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C$$

$$y(x) = \exp\left(-\frac{1}{x}\right)\left[\frac{1}{2}\ln(x^2+1) + C\right].$$

$$0 = y(1) = e^{-1}\left[\frac{1}{2}\ln 2 + C\right] \qquad C = -\ln\sqrt{2}$$

$$y(x) = \exp\left(-\frac{1}{x}\right)\left[\ln\sqrt{\frac{x^2+1}{2}}\right].$$

4. (20 points). Solve the boundary value problem, if possible.

$$y'' - 6y' + 9y = 0, \qquad y(0) = 1, \qquad y(1) = 0$$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0 \qquad r = 3$

$y(x) = C_1 e^{3x} + C_2 x e^{3x}$

$1 = y(0) = C_1 \qquad\qquad C_2 = -C_1$

$0 = y(1) = C_1 e^3 + C_2 e^3$

$$\boxed{\begin{aligned} y(x) &= e^{3x} - x e^{3x} \\ &= (1-x)e^{3x} \end{aligned}}$$

$y(0) = 1$

$y(1) = 0$ ✓

~~$-e^{3x}$~~

5. (25 points). Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \ln n}$$

$N^2 + \ln N \geqslant N^2 \quad \forall N$

$\frac{1}{N^2 + \ln N} \leq \frac{1}{N^2} \quad \forall N$

$\Rightarrow \sum_{N=1}^{\infty} \frac{1}{N^2 + \ln N} \leq \sum_{N=1}^{\infty} \frac{1}{N^2}$

RHS is convergent by p-test $(p = 2)$

So $\sum_{N=1}^{\infty} \frac{1}{N^2 + \ln N}$ converges. too.

6. (30 points). Determine whether the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^{3/2}}{n^{5/2}+n^2}$$

$b_n = \frac{n^{3/2}}{n^{5/2}+n^2}$

. $\lim_{n\to\infty} b_n = 0$

. $b_n \geq 0 \quad \forall n.$

. b_n decreasing.

$$f(x) = \frac{x^{3/2}}{x^{5/2}+x^2} = \frac{x^{3/2}}{x^{3/2}\left[x+\sqrt{x}\right]} = \frac{1}{x+\sqrt{x}}$$

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$$n+1+\sqrt{n+1} \geq n+\sqrt{n} \quad \forall n$$

$\Rightarrow \quad b_{n+1} \leq b_n \quad \forall n$ So b_n is decreasing

So by AST $\sum_{n=1}^{\infty}(-1)^n \frac{n^{3/2}}{n^{5/2}+n^2}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges by limit comparison (with $\frac{1}{n}$)

So $\sum_{n=1}^{\infty}(-1)^n \frac{n^{3/2}}{n^{5/2}+n^2}$ converges conditionally.

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7. (30 points). Determine the interval of convergence of the power series

$$\sum_{n=5}^{\infty} \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \cdots \cdot (2n-1)^2}{(2n+2)!} x^n. \quad ⊛$$

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$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+1)^2 \quad (2n+2)!}{(2n+4)!} |x|$$

$$= \frac{(2n+1)^2}{(2n+4)(2n+3)} |x| \xrightarrow[n\to\infty]{} |x|$$

If $|x|<1$ the ⊛ converges.

$2n+2 \quad 2n+1$

At $x=1$

$$\sum_{n=5}^{\infty} \frac{1^2\cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{(2n+2)!} = \sum_{n=5}^{\infty} \underbrace{\frac{1\cdot 3\cdot 5\cdots (2n-1)}{2\cdot 4\cdot 6\cdots \cdot (2n)}}_{<1} \cdot \frac{1}{2n+2} \cdot \frac{1}{(2n+1)}$$

$$\le \sum_{n=5}^{\infty} \frac{1}{(2n+2)} \cdot \frac{1}{2n+1} < +\infty$$

6 Converges.

$$x=-1 \qquad \sum_{n=5}^{\infty} \frac{1^2\cdot 3^2\cdot 5^2 \cdots \cdot (2n-1)^2}{(2n+2)!}(-1)^n$$

Convergent since absolutely convergent by $x=1$

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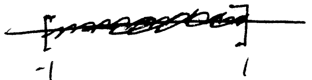

8. (15+10+15 points). (a) Find the Taylor series for $f(x)$ centered at the given value of a by **using the definition of a Taylor series.**

$$f(x) = \ln x, \qquad a = 1.$$

$f(x) = \ln x$

$f'(x) = +\frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = \frac{-3\cdot 2\cdot 1}{x^4}$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$ $\qquad n \geq 1$

$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$f(1) = 0$

5 diff

5 $f^{(n)}$

5 Taylor

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

(b) Find the same Taylor series by using the geometric power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

-2 $+C$

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(c) Determine the interval of convergence of the Taylor series you found.

(b) $\frac{1}{1-X} = \sum_{N=0}^{\infty} X^N$

$\ln(1-x) + C = \sum_{N=0}^{\infty} \frac{X^{N+1}}{N+1}$ $M = N+1$

$= \sum_{M=1}^{\infty} \frac{X^M}{M}$ $C = 0$ $X = 0$ $0 + C = 0$

$\ln(1-x) = \sum_{M=1}^{\infty} \frac{X^M}{M}$ $\ln u = \sum$

~~$u = 1-x$~~

~~$x = u-1$~~

$x = -y + 1$

$= -1(\overset{y-1}{\cancel{1-y}})$

$R = 1$

$[0, 2]$

~~[illegible]~~ $\ln y = \sum_{M=1}^{\infty} \frac{(-1)^M (y-1)}{M}$

(c) $|x| < 1$

9. (15+15+15 points). A curve is defined by the parametric equations

$$x = t^2 - 1, \qquad y = t^3 - 12t, \qquad -4 \leq t \leq 4.$$

(a) Find the equation of its tangent line at $t = -1$.

(b) Give a definite integral for the length of the curve (Do not evaluate!).

(c) For which values of t is the curve concave upward?

$x' = 2t$ $\qquad \frac{dy}{dx} = \frac{3t^2-12}{2t}$

$y' = 3t^2 - 12$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3-12}{-2} = -\frac{9}{-2} = \frac{9}{2}$$

$y(-1) = -1 + 12 = 11$ $\qquad X(-1) = 0$

$y - 11 = \frac{9}{2}(x-0) = \qquad \boxed{y = \frac{9}{2}x + 11}$

(b) $L = \int_{-4}^{4} \sqrt{4t^2 + (3t^2-12)^2}\, dt$

$= \int_{-4}^{4} \sqrt{4t^2 + 9t^4 - 72t^2 + 144}\, dt$

$$\boxed{L = \int_{-4}^{4} \sqrt{9t^4 - 68t^2 + 144}\, dt}$$

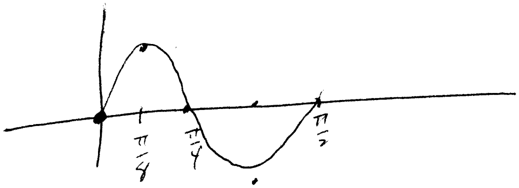
(c) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-12}{2t}\right)}{2t} = \frac{\frac{6t \cdot 2t - 2(3t^2-12)}{4t^2}}{2t} = \frac{24}{8t^3} = \frac{3}{t^3}$

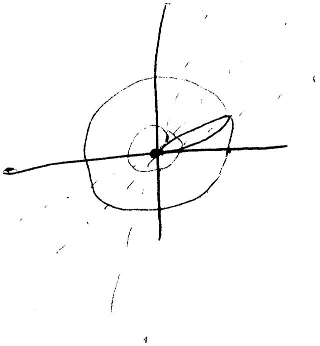
$\boxed{0 < t \leq 4 \quad \text{Concave upward.}}$

10. (30 points). Find the area enclosed by one loop of the curve given in polar coordinates by

$$r = 2\sin 4\theta$$

(Recall $\sin^2 x = \dfrac{1-\cos 2x}{2}$.)





$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} 4\sin^2 4\theta \, d\theta$$

$$= 2\int_0^{\pi/4} \frac{1-\cos 8\theta}{2} \, d\theta$$

$$= \int_0^{\pi/4} 1-\cos 8\theta \, d\theta$$

$$= \theta - \frac{1}{8}\sin 8\theta \Big|_0^{\pi/4}$$

$$= \boxed{\frac{\pi}{4}}$$