

HOMEWORK 2

DUE FRIDAY, OCTOBER 18, 2024 AT 11:59PM

The homework is due in Gradescope at the stated time. You may submit handwritten text (scanned from paper or written on a tablet) or a typed PDF (e.g., created using LaTeX). If you choose the handwritten option, you have to also submit a hardcopy in class or put it in my mailbox.

Collaboration and research is fine, as long as you do the following.

- Try the problems yourself first!
- Write the solutions in your own words.
- Acknowledge all sources and collaborators.

For the Chinese remainder theorem for Dedekind domains, see for example Neukirch Theorem I.3.6.

1. Let R be a Dedekind domain. Prove that for every nonzero ideal I of R , R/I is a principal ideal ring. (Hint: use the Chinese remainder theorem to reduce to the case where I is a power of a prime $\mathfrak{p}$. In that case, choose $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ and consider the powers of π .)
2. Let R be a Dedekind domain. Prove that every ideal of R can be generated by at most two elements.
3. Let R be the ring $\mathbb{Z}[\alpha]/(\alpha^3 - \alpha - 1)$. Show that the prime factorization of the principal ideal $23R$ is given by

$$23R = (23, \alpha - 10)^2(23, \alpha - 3).$$

(In particular, you should show that the factors are indeed prime.)

4. Prove that the constant 2^n in Minkowski's lattice point theorem cannot be improved.
5. Let K be a number field. Using the finiteness of the class group of K , prove that there exists a finite extension L of K such that every ideal of $\mathcal{O}_K$ generates a principal ideal of $\mathcal{O}_L$.

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- 6.** Let $p > 2$ be a prime number and $K = \mathbb{Q}(\zeta_p)$.
- (a) Compute $\text{Trace}_{K/\mathbb{Q}}(\zeta_p^j)$ for $j = 0, \dots, p-1$.
 - (b) Compute $\text{Norm}_{K/\mathbb{Q}}(1 - \zeta_p)$.
 - (c) Show that $(1 - \zeta_p)\mathcal{O}_K \cap \mathbb{Z} = p\mathbb{Z}$. (Note: we will use this later to show that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$, so don't assume this here.)