HOMEWORK 6

DUE FRIDAY, NOVEMBER 22, 2019 IN CLASS

Part I: from the textbook

Chapter I, Section 12: **1, 3**

Part II

In what follows, *a ring of rank n* is a commutative ring with unit that is also a free $\mathbb{Z}$-module of rank n.

1. Prove that a finite integral domain is a field.

2. Prove that (up to isomorphism) the only ring of rank 1 is $\mathbb{Z}$ itself.

3. Prove that an order in a number field of degree n is a ring of rank n.

4. Let S be ring of rank 2, also called a *quadratic ring.*

(a) Prove that S admits a $\mathbb{Z}$-basis of the form $1, \tau$.

(b) For any basis $1, \tau$ of S, specifying the ring structure on S is equivalent to specifying $b, c \in \mathbb{Z}$ such that

$$\tau^2 = b\tau + c.$$

Prove that the quantity $D = b^2 + 4c$ does not depend on the choice of τ. The quantity D is called the *discriminant* of S and is denoted $\mathrm{disc}(S)$.

(c) Show that S admits a basis $1, \tau$ such that

$$\tau^2 = c \text{ or } \tau^2 = \tau + c$$

with $c \in \mathbb{Z}$.

(d) Conclude that the discriminant of S is congruent to either 0 or 1 modulo 4.

5. Prove that if K is a quadratic number field, the discriminant of K defined in lecture and the discriminant of $\mathcal{O}_K$ as a quadratic ring are the same.

6. Prove that isomorphism classes of quadratic rings S are in canonical bijection with elements of the set

$$\mathcal{D} = \{D \in \mathbb{Z}; D \equiv 0, 1 \pmod 4\}$$

of discriminants. Under this bijection, a quadratic ring S corresponds to $\operatorname{disc}(S) \in \mathcal{D}$, and an element $D \in \mathcal{D}$ corresponds to the quadratic ring

$$S(D) = \mathbb{Z}\left[\frac{D + \sqrt{D}}{2}\right].$$

7. Prove or disprove and salvage if possible the following statement.

In a noetherian integral domain, only a finite number of prime ideals contain a given nonzero ideal.