HOMEWORK 5

DUE FRIDAY, NOVEMBER 15, 2019 IN CLASS

Part I: from the textbook

Chapter I, Section 10: **3, 4**

Part II

Note: Some of the exercises below come from Atiyah–Macdonald.

1. Let p be an odd prime and $\zeta = \zeta_p$ a primitive p-th root of unity. Set $I = p\mathbb{Z}[\zeta]$ the principal ideal generated by p in $K = \mathbb{Q}(\zeta)$. Suppose u is a unit in $\mathbb{Z}[\zeta]$ and set $\alpha = u/\bar{u}$.

(a) Show that for any $\beta \in \mathbb{Z}[\zeta]$ we have $\beta^p \equiv b \pmod{I}$ for some $b \in \mathbb{Z}$.

(b) Show that α is a root of 1.
Hint: What are the absolute values of its Galois conjugates?

(c) Show that $\alpha = \pm\zeta^m$ for some integer m.

(d) Show that in fact $\alpha = \zeta^m$ for some integer m.
Hint: If $\alpha = -\zeta^m$ that would force $u^p = -\bar{u}^p$. Use part (a) to show that in this case u^p would be divisible by p in $\mathbb{Z}[\zeta]$.

2. Let A be a ring and S a multiplicatively closed subset of A that does not contain 0.

(a) Show that all the ideals of $S^{-1}A$ of the form $S^{-1}I$ for some ideal I of A.

(b) Show that the correspondence $I \mapsto S^{-1}I$ from part (a) takes prime ideals of A that do not intersect S to prime ideals of $S^{-1}A$ and that $S^{-1}A$ has no other prime ideals.

(c) What about maximal ideals? Prove the similar statement or find a counterexample.

3. Let A be a ring and P a prime ideal in A.

(a) Show that A_P is local ring with unique maximal ideal $M_P = PA_P$.

(b) Show that the map the inclusion map $A \to A_P$ induces an injective homomorphism $A/P \to A_P/M_P$.

(c) If P is maximal, show that the map from (b) induces an isomorphism between A/P^n and A_P/M_P^n for all $n \in \mathbb{Z}_{>0}$.

4. Let A be an integral domain, noetherian local ring of dimension 1 (i.e., every nonzero prime ideal is maximal). Let M be its maximal ideal and $k = A/M$ its residue field. Prove that the following are equivalent.

(a) A is a discrete valuation ring.

(b) A is integrally closed.

(c) M is a principal ideal.

(d) $\dim_k(M/M^2) = 1$

(e) Every nonzero ideal is a power of M.

(f) There exists $\pi \in A$ such that every nonzero ideal is of the form $\pi^k A$ for some $k \in \mathbb{Z}_{\geq 0}$.

5. Let A be an integral domain and I a fractional ideal of A. Prove that the following are equivalent.

(a) I is invertible, i.e. there exists a factional ideal J of A such that $IJ = A$.

(b) I is a finitely generated A-module and $I_{\mathfrak{p}}$ is invertible for every prime ideal $\mathfrak{p}$ of A.

(c) I is a finitely generated A-module and I_M is invertible for every maximal ideal M of A.

6. Let A be an integral domain and local ring. Prove that A is a discrete valuation ring if and only if every nonzero fractional ideal of A is invertible.

Part III: optional

These problems are optional. However they illustrate important facts and you should try them if you have not seen them before. They will not be graded, but you are welcome to come and discuss them with me.

1. Prove that every finite abelian group appears as the Galois group of a number field. (This is exercise 2, section 10 in Neukirch. There is also a discussion at `https://math.stackexchange.com/questions/131376/every-finite-abelian-group-is-the-galois-group-of-so`