HOMEWORK 1

DUE WEDNESDAY, OCTOBER 16, 2019 IN CLASS

Part I: from the textbook

Chapter I, Section 3: **3, 5, 8, 9**

Part II

1. (a) Let A be an integrally closed integral domain, F its field of fractions, K a finite separable algebraic extension of F, and B the integral closure of A in K. Prove that there exists a basis $v_1, \ldots, v_n$ of K over F such that

$$B \subseteq Av_1 + Av_2 + \cdots + Av_n.$$

(b) Use the first part to prove that the ring of integers $\mathcal{O}_K$ of a number field K is a finitely generated $\mathbb{Z}$-module. Conclude that the ring of integers $\mathcal{O}_K$ of the number field K is a noetherian ring.

2. Find all the units in $\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{d})$ where $d < 0$ is a square-free integer. Are they all roots of unity?

3. Find all the units in $\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{2})$. Are they all roots of unity? (*Hint:* Reformulate the problem in terms of a Pell equation.)

4. Suppose α is an algebraic integer and f is a monic polynomial over $\mathbb{Z}$ (not necessarily irreducible) such that $f(\alpha) = 0$. Denote $n = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Show that the discriminant $\operatorname{disc}(1, \alpha, \ldots, \alpha^{n-1})$ divides $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}\left(f'(\alpha)\right)$.

5. Let I be the ideal generated by 2 and $1+\sqrt{-3}$ in the ring $\mathbb{Z}\left[\sqrt{-3}\right]$. Show that

(a) $I \neq (2)$ but $I^2 = 2I$;
(b) the ideals in $\mathbb{Z}\left[\sqrt{-3}\right]$ do not factor uniquely into prime ideals;
(c) I is the unique prime ideal containing (2);
(d) (2) is not a product of prime ideals.

Does this contradict Corollary 3.9 of the textbook? Why or why not?

Part III: optional exercises

These problems are optional. However they illustrate important facts and you should try them if you have not seen them before. They will not be graded, but you are welcome to come and discuss them with me.

1. Find the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer, and compute its discriminant.

2. Show that the prime elements of the ring $\mathbb{Z}[i]$ are exactly (up to multiplication by units):

- $1+i$
- p prime of $\mathbb{Z}$ with $p \equiv 3 \pmod 4$
- $a \pm bi$ where $a, b \in \mathbb{Z}$ with $a^2+b^2 = p$ a rational prime such that $p \equiv 1 \pmod 4$.

3. Show that for any prime $p \in \mathbb{Z}$,

$$p\mathbb{Z}[i] = \begin{cases} (1+i)^2, & \text{if } p = 2; \\ \mathfrak{p}, & \text{if } p \equiv 3 \pmod 4; \\ \mathfrak{p}_1\mathfrak{p}_2 \text{ where } \mathfrak{p}_1 = (a+bi), \mathfrak{p}_2 = (a-bi), a^2+b^2 = p, & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

Here $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$ denote prime ideals of $\mathbb{Z}[i]$.