Section 4.2

1. (Problem 22b) Show $f(x) = 4x^5 + 28x^4 + 7x^3 - 28x^2 + 14$ is irreducible over $\mathbf{Q}[x]$.

Solution: Notice that $f(x) \in \mathbf{Z}[x]$. Furthermore notice 7 divides every coefficient except the leading term, and that 7^2 does not divide the constant term. Thus we can apply Eisenstein's criterion with $p = 7$ to deduce that $f(x)$ is irreducible in $\mathbf{Q}[x]$.

2. (Problem 43c) If $\sigma : F[x] \to F[x]$ is a ring automorphism that fixes F, show there exist $a \in F \setminus \{0\}$ and $b \in F$ such that $\sigma(f) = f(ax + b)$ for all $f \in F[x]$.

Solution: The key point is that an automorphism of $F[x]$ which fixes F is determined by the image of x. To see what we mean, let $f \in F[x]$, say $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Using the fact that σ is a ring homomorphism, and that $\sigma(a_i) = a_i$ for all i (because σ fixes F), we calculate

$$\begin{aligned}\sigma(f) &= \sigma(a_0 + a_1 x + \cdots + a_n x^n) \\ &= \sigma(a_0) + \sigma(a_1)\sigma(x) + \cdots + \sigma(a_n)\sigma(x)^n \\ &= a_0 + a_1 \sigma(x) + \cdots + a_n \sigma(x)^n \\ &= f(\sigma(x)).\end{aligned}$$

Because this holds for any $f \in F[x]$, to complete the problem it suffices to show that there exist $a, b \in F$, $a \neq 0$, such that $\sigma(x) = ax + b$, or in other words it suffices to show that $\deg(\sigma(x)) = 1$. To see this, let $p(x) = \sigma(x)$ and let $q(x) = \sigma^{-1}(x)$. Then we calculate using the result of the calculation above (with f replaced by q)

$$x = \sigma(\sigma^{-1}(x)) = \sigma(q(x)) = q(\sigma(x)) = q(p(x)).$$

Now we claim that $\deg(q(p(x))) = \deg(q)\deg(p)$; this will imply that $\deg(q)\deg(p) = 1$, which lets us conclude $\deg(q) = \deg(p) = 1$, which completes the proof because we were supposed to show that $\deg(\sigma(x)) = 1$.

To prove the claim, let $m = \deg(q)$ and $n = \deg(p)$, so we can write $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{j=0}^{m} b_j x^i$ where $a_n, b_m \neq 0$. Then using the multinomial theorem we have

$$\begin{aligned} q(p(x)) &= \sum_{j=0}^{m} b_j (p(x))^j = \sum_{j=0}^{m} b_j \Big(\sum_{i=0}^{n} a_i x^i \Big)^j \\ &= \sum_{j=0}^{m} b_j \left(\sum_{k_0 + \cdots + k_n = j} \binom{j}{k_0, \ldots, k_n} \prod_{i=0}^{n} (a_i x^i)^{k_i} \right). \end{aligned}$$

By inspection we see the largest power of x occuring in this expression is x^{nm}, and the coefficient is $a_n^m b_m \neq 0$, which shows $\deg(q(p(x))) = nm = \deg(q)\deg(p)$.

Section 4.3

3. (Problem 1b) In each case find a monic polynomial h in $F[x]$ such that $I = \langle h \rangle$, where

$$I = \left\{ f \in F[x] \mid \text{the sum of the coefficients of } f \text{ is zero} \right\}.$$

Solution: We know there exists some monic polynomial h such that $I = \langle h \rangle$. Notice that $h \neq 0$ because $I \neq \{0\}$ (for instance $x - 1 \in I$), and also notice that h cannot be a constant because I does not contain any nonzero constant polynomials (this easily follows from the definition of I). Thus $\deg(h) \geqslant 1$. Now because $x - 1 \in I = \langle h \rangle$, we can write $x - 1 = h(x)q(x)$ for some $q \in F[x]$. Then we see $\deg(x - 1) = \deg(h) + \deg(q)$, so because $\deg(x - 1) = 1$ and $\deg(h) \geqslant 1$ we conclude $\deg(h) = 1$ and $\deg(q) = 0$. But because $x - 1$ and h are both monic we conclude that $q = 1$, and thus $h(x) = x - 1$, so $I = \langle x - 1 \rangle$.

4. (Problem 29) Let F be a field and $h = pq$ in $F[x]$, all polynomials monic. If p and q are relatively prime in $F[x]$, show that $F[x]/\langle h \rangle \cong F[x]/\langle q \rangle \times F[x]/\langle q \rangle$.

Solution: Because p and q are relatively prime, we have $\operatorname{lcm}(p, q) = pq = h$ as well as $\gcd(p, q) = 1$. Then using Problem 25 we see that

$$\langle p \rangle \cap \langle q \rangle = \langle \operatorname{lcm}(p, q) \rangle = \langle h \rangle \quad \text{and} \quad \langle p \rangle + \langle q \rangle = \langle \gcd(p, q) \rangle = \langle 1 \rangle = F[x].$$

The latter equality shows we can invoke the Chinese Remainder Theorem, and doing so we find

$$F[x]/\langle h \rangle = F[x]/(\langle p \rangle \cap \langle q \rangle) \cong F[x]/\langle p \rangle \times F[x]/\langle q \rangle.$$